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Chern forms and the Riemann tensor for the moduli space of curves

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Two vector bundles associated to the moduli space of compact Riemann surfaces have a Hermitian metric derived from the hyperbolic geometry of Riemann surfaces. Briefly our purpose is to determine the connection and curvature forms for these metrics.

The first bundle is the holomorphic tangent bundle of the Teichmüller space of genus g, $g \geqq 2$, Riemann surfaces; the metric is the Weil-Petersson metric. Weil introduced a Kähler metric for the Teichmüller space T_g based on Petersson's Hermitian pairing for automorphic forms. Ahlfors considered the differential geometry of this metric; in particular he obtained integral formulas for the associated Riemann curvature tensor, [1, 2]. As an application he found that the Ricci, holomorphic sectional, and scalar curvatures are all negative. Royden latter showed that the holomorphic sectional curvature is bounded away from zero, [16]. More recently Tromba found that the sectional curvature is also negative, [32]. After this result Royden and then the author also found proofs of the negative sectional curvature, [17]. In the present work we develop a formalism for computing second variations of a hyperbolic structure and consider as the first application a formula for the Riemann tensor. Before presenting the formula recall that the holomorphic tangent space of Teichmüller space at the marked Riemann surface $\langle S \rangle$ is naturally isomorphic to $\mathscr{B}(S)$, the space of harmonic Beltrami differentials ($(-1,1)$ tensors) on S. Now denoting by dA the area element of the hyperbolic metric on S and by D the Laplacian of the hyperbolic metric then the Riemann tensor is given as

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -2\int_S (D-2)^{-1}(\mu_\alpha \bar{\mu}_\beta)(\mu_\gamma \bar{\mu}_\delta)\, dA \qquad (0.1)$$
$$-2\int_S (D-2)^{-1}(\mu_\alpha \bar{\mu}_\delta)(\mu_\gamma \bar{\mu}_\beta)\, dA$$

for μ_α, μ_β, μ_γ, $\mu_\delta \in \mathscr{B}(S)$ representing tangent vectors to T_g. Recall that the Laplacian D acting on L^2 functions is a self adjoint operator with non-positive

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spectrum; the inverse $(D-2)^{-1}$ exists and is a compact integral operator. Also note that given $\mu, \nu \in \mathscr{B}(S)$ the product $(\mu\bar{\nu})$ defines a function on S. Now the reader may check that (0.1) defines a 4-tensor on T_g with the appropriate symmetries. As an application of the formula we find explicit bounds for the curvatures: the holomorphic sectional curvature is bounded above by $\frac{-1}{2\pi(g-1)}$, a result conjectured by Royden [16]. Furthermore we find that all sectional curvatures are indeed negative (Theorem 4.5). In fact our considerations show that the curvatures are governed by the spectrum of the Laplacian: the negative curvature is a manifestation of the non-positivity of the Laplacian. We note that the above formula and bounds have also been obtained by Royden.

In [28] we combine the negative curvature result and the observation that the geodesic length functions are convex along Weil-Petersson geodesics to study the geometry of T_g. The main result is that Teichmüller space is geodesically convex. Each pair of points is joined by a unique Weil-Petersson geodesic. Given the negative curvature it follows that the exponential maps are homeomorphisms of their domains to T_g.

The second bundle under consideration, a line bundle, is the vertical bundle of the fibration $\pi\colon \mathscr{T}_g \to T_g$ of the Teichmüller curve over Teichmüller space. Briefly the fibre of the projection π above a marked surface $\langle S \rangle$ is a compact submanifold isomorphic to S. The kernel of the differential $d\pi\colon T^{1,0}\,\mathscr{T}_g \to T^{1,0}\,T_g$ defines a line bundle (v) on $\mathscr{T}_g$, the vertical bundle of the fibration. The restriction of (v) to a fibre of π is isomorphic to the tangent bundle of the fibre. Consequently the Uniformisation Theorem with parameters provides that the hyperbolic metrics of the individual fibres piece together to define a smooth metric on the line bundle (v). We compute the connection 1-form θ and curvature 2-form Θ for this metric. As the first application of the formula we find that the Chern form $c_1(v) = \frac{i}{2\pi}\,\Theta$ is negative, a differential-geometric analogue of a result of Arakelov, [5]. Once again we find that the curvature is governed by the spectrum of the Laplacian. As the second application we investigate the characteristic classes

$$\tilde{\kappa}_n(p) = \int_{\pi^{-1}(p)} c_1(v)^{n+1}, \quad n \in \mathbb{Z}^+, \quad p \in T_g,$$

$c_1(v)$ the Chern form of (v), originally considered by the algebraic geometers, [5, 9, 10, 13, 14]. We derive the formula for $\tilde{\kappa}_n$ as computed from the hyperbolic metric. In particular we find the pointwise equality of characteristic forms $\tilde{\kappa}_1 = \frac{1}{2\pi^2}\,\omega_{WP}$, where ω_{WP} is the Kähler form of the Weil-Petersson metric. This result was foreshadowed by our previous result: $\tilde{\kappa}_1$ and $\frac{1}{2\pi^2}\,\omega_{WP}$ represent the same cohomology class on $\bar{M}_g$, the moduli space of stable curves, [26].

Our approach for the calculations is formal in nature and involves the SL $(2;\mathbb{R})$ invariant first order differential operators L and K as well as the invariant Laplacian D introduced by Maass. If κ is the canonical bundle of the upper half plane H and $S(k)$ the space of smooth section of $\kappa^{k/2} \otimes \bar{\kappa}^{-k/2}$, $k \in \mathbb{Z}$ then Maass introduced invariant differential operators $K_k: S(k) \to S(k+1)$, $L_k: S(k) \to S(k-1)$ and $D_k: S(k) \to S(k)$, [11]. Given $f \in S(k)$ and $\gamma \in \mathrm{SL}(2;\mathbb{R})$ then γ acts by pullback on the tensor f, $f \xrightarrow{\gamma} \gamma * f$. The Maass operators commute with this action. A heuristic principle due to Selberg provides that the *general* $\mathrm{SL}(2;\mathbb{R})$ invariant linear operator should be thought of as a combination of: the L, K and D operators, their inverses (when defined) and projections onto the eigenspaces of the Laplacians D, [18]. Furthermore the inverse of L and K can be given in terms of that for the Laplacian, when appropriate. For instance to solve the equation $K_k f = g$ for the tensor f first note that $D_k - k(k+1) = L_{k+1} K_k$ and hence the equation may be replaced by $(D_k - k(k+l))f = L_{k+1} g$ which admits the formal solution $f = (D_k - k(k+1))^{-1} L_{k+1} g$. This example suggests an explanation for the appearance of the operator $(D_0 - 2)^{-1}$ in our formulas.

A basic observation is that the fundamental operators of Teichmüller theory are $\mathrm{SL}(2;\mathbb{R})$ invariant. The above approach yields very nice results when considering first variations, the case of linear operators. On the other hand second variations are necessarily given by quadratic operators, a case not formally covered by the Selberg approach. *Nevertheless* we find that many of the techniques are still valid when applied to the specific operators of deformation theory. For example our basic concern is the variation of the hyperbolic area element dA under pullback by a quasiconformal homeomorphism f^μ, $\mu = f_{\bar{z}}/f_z$ of the upper half plane H. We find for $\mu \in \mathscr{B}(S)$, harmonic, the expansion (Theorem 3.3)

$$(f^{\varepsilon\mu})^* dA = (1 - \varepsilon^2(\mu\bar{\mu} + 2(D_0 - 2)^{-1}\mu\bar{\mu}) + 0(\varepsilon^3))\, dA \tag{0.2}$$

for the pullback of the area element, where $0(\varepsilon^3)$ is uniform on compact subsets of H. The vanishing of the term linear in ε is an earlier result of Ahlfors [1, 2]; we note that Royden has also obtained an expansion similiar to the above.

The manuscript is divided into five chapters. In the first we review Maass' calculus of differential operators and use this to derive Ahlfors' result on the first variation of area. The second chapter is devoted to obtaining variational formulas for the operators characterizing the complex structure of Teichmüller space as well as the projection operator of Beltrami differentials onto the harmonic differentials. A brief review of Teichmüller theory and the Weil-Petersson metric is contained in sections 2.3, 2.4 and 2.5. In the following chapter we apply the preceeding techniques and derive the above formula for the second variation of area (Theorem 3.3). The fourth chapter is devoted to a discussion of the Riemann tensor of the Weil-Petersson metric. The main formula may be found in Theorem 4.2 and the estimates appear in Theorem 4.5 and Lemma 4.6. We start the final chapter with a review of the Teichmüller curve and its universal cover, the Bers fibre space, and then we consider the characterestic classes of the Teichmüller curve.

1. The Maass calculus and the first variation of area

1.1. Maass introduced a calculus of $\mathrm{SL}(2;\mathbb{R})$ translation invariant differential operators. These operators are essential to the organization of our calculations. With this in mind we start by reviewing Maass' approach. As the first application we give a new proof of the Ahlfors' result; the first variation of hyperbolic area vanishes for the harmonic Beltrami differentials, [1, 2].

1.2. Start by considering the space $S(k)$ of smooth sections of $\kappa^{k/2} \otimes \bar{\kappa}^{-k/2}$, where κ is the canonical bundle of the upper half plane H and k is integral. Classically $S(k)$ is the space of tensors $f = f(z)\left(\frac{dz}{d\bar{z}}\right)^{k/2}$. An element $\gamma \in \mathrm{SL}(2;\mathbb{R})$ acts naturally on $S(k)$, $\gamma_k^* f = f(\gamma z)\, \gamma'(z)^{k/2}\, \overline{\gamma'(z)}^{-k/2}$ where $\gamma'(z)$ is the complex derivative $\frac{dw}{dz}$ for $w = \gamma(z)$. We shall write γ^* instead of γ_k^*; the subscript will be given by the context. A more general smooth section $g \in S(2p, 2q)$ of the bundle $\kappa^p \otimes \bar{\kappa}^q$, $2p$ and $2q$ integral, will be studied by considering $(z-\bar{z})^{p+q} g \in S(p-q)$. Note that $\lambda \in S(2,2)$ where $\lambda = \frac{-4}{(z-\bar{z})^2}$ is the hyperbolic volume element.

Maass introduced the differential operators

$$K_k = (z-\bar{z})\frac{\partial}{\partial z} + k : S(k) \to S(k+1)$$

$$L_k = (\bar{z}-z)\frac{\partial}{\partial \bar{z}} - k : S(k) \to S(k-1)$$

and the Laplacians D_k: $S(k) \to S(k)$

$$D_k = L_{k+1} K_k + k(k+1) = K_{k-1} L_k + k(k-1), \quad [11].$$

The operators satisfy the identities $L_{k+1}K_k = K_{k-1}L_k - 2k$, $D_{k+1}K_k = K_k D_k$, $D_k L_{k+1} = L_{k+1} D_{k+1}$ and $K_k = \overline{L_{-k}}$: The $\mathrm{SL}(2;\mathbb{R})$ invariance is as follows

$$K_k \gamma^* f = \gamma^* K_k f$$
$$L_k \gamma^* f = \gamma^* L_k f$$

for $f \in S(k)$ and $\gamma \in \mathrm{SL}(2;\mathbb{R})$. An immediate consequence is the invariance of the Laplacian, $D_k \gamma^* = \gamma^* D_k$. The derivative of the product fg, $f \in S(l)$, $g \in S(k-l)$ also satisfies a simple rule

$$K_k(fg) = gK_l f + fK_{k-l} g$$
$$L_k(fg) = gL_l f + fL_{k-l} g\,.$$

We assume now and for the remainder of the manuscript that $\Gamma \subset \mathrm{SL}(2;\mathbb{R})$ is the uniformisation group of a compact Riemann surface. The hypothesis guarantees that a square root $\kappa^{1/2}$ of the canonical bundle is Γ invariant. Furthermore it will vastly simplify the convergence considerations for our variational formulas as well as the spectral theory of the Laplacian. Now given f and g measurable Γ invariant sections of $\kappa^{k/2} \otimes \bar{\kappa}^{-k/2}$ define their Hermitian product

$$\langle f, g \rangle_k = \int_{H/\Gamma} f\bar{g}\, dA \tag{1.1}$$

for dA the hyperbolic area element. As above we shall write $\langle\ ,\ \rangle$ in place of $\langle\ ,\ \rangle_k$.

Definition 1.1. $\mathscr{H}_k$ is the Hilbert space of measurable Γ invariant sections f of $\kappa^{k/2} \otimes \bar{\kappa}^{-k/2}$ with $\langle f, f \rangle$ finite.

Roelcke studied the Laplacian D_k acting on $\mathscr{H}_k$; the operator is self adjoint on the dense subspace $\mathscr{H}_k \cap S(k)$, [*15*]. In particular D_0 is the classical Laplace-Beltrami operator, having non-positive discrete spectrum on $\mathscr{H}_0$. The inverse $(D_0-2)^{-1}$ exists and is a compact integral operator from $\mathscr{H}_0$ to $\mathscr{H}_0$. Many of our formulas will involve the operator $\Delta = -2(D_0-2)^{-1}$. Finally for $f \in \mathscr{H}_{k+1} \cap C^1$ and $g \in \mathscr{H}_k \cap C^1$ we may integrate by parts

$$\langle f, K_k g\rangle = -\langle L_{k+1} f, g\rangle .$$

1.3. Our goal is to use the Maass calculus to obtain variational formulas for the hyperbolic metric as well as those operators defining the complex structure of Teichmüller space. The idea of Selberg is simple enough. A $\mathrm{SL}(2;\mathbb{R})$ invariant linear operator can be thought of as a combination of the differential operators L, K and D, the inverses $(D-c)^{-1}$, c a constant, and the projections onto eigenspaces of the Laplacians. Many of the operators of Teichmüller theory are in fact $\mathrm{SL}(2;\mathbb{R})$ invariant.

Let f^μ be a μ quasiconformal self homeomorphism of the upper half plane H and dA the area element of the hyperbolic metric. The f^μ induced deformation of the conformal structure is characterized by the Beltrami equation $f_{\bar z} = \mu f_z$. Recall that in one complex dimension a metric is determined modulo scaling by its conformal structure; consequently the f^μ deformation of the hyperbolic metric is completely determined by the Beltrami differential μ and by the pullback $(f^\mu)^* dA$ of the area form. Our goal is to obtain formulas valid to second order in ε for $(f^{\varepsilon\mu})^* dA$. The main result is Theorem 3.3. This question was considered previously by Ahlfors; his formulas are in terms of iterated singular integrals, [2]. Our formula is given in terms of the operator $\Delta = -2(D_0-2)^{-1}$.

Obtaining a first order expansion for $(f^{\varepsilon\mu})^* dA$ is equivalent to studying the action by the Lie derivative of vector fields on the tensor dA. Starting with X a smooth vector field on H associate the section $\Phi = \frac{1}{(z-\bar z)} X \in S(-1)$ and indicate the X Lie derivative by $L(X)$.

Lemma 1.2. $L(X)\, dA = 2\,\mathrm{Re}\,(K_{-1}\Phi)\, dA$.

Proof. For Ω open, relatively compact in H we may integrate X to obtain a flow F^ε defined for ε small. By definition of the Lie derivative

$$L(X)\, dA = \frac{d}{d\varepsilon}\,(F^{\varepsilon *} dA)\,|_{\varepsilon=0} = \frac{d}{d\varepsilon}\left(\frac{|F_z^\varepsilon|^2 - |F_{\bar z}^\varepsilon|^2}{(\mathrm{Im}\, F^\varepsilon)^2}\,\frac{i}{2}\, dz \wedge d\bar z\right)\Bigg|_{\varepsilon=0}$$

and using that $X = \frac{d}{d\varepsilon} F^\varepsilon|_{\varepsilon=0}$ we obtain $\mathrm{Re}\left(2X_z - \frac{4}{(z-\bar z)} X\right) dA$. Finally the reader will check that $K_{-1}\Phi = X_z - \frac{2}{(z-\bar z)} X$.

1.4. In the study of Teichmüller space deformations are parametrized by the $(-1,1)$ tensors $X_{\bar z}$ rather than vector fields X on the upper half plane. The advantage of this approach is twofold: first the hypothesis that X be an infinitesimal deformation of a group $\Gamma \subset \mathrm{SL}(2;\mathbb{R})$ is replaced by the elementary condition

$\mu = X_{\bar{z}}$ is a Γ invariant tensor and secondly it is straight–forward to characterize the infinitesimally trivial deformations. But this approach requires an analysis of the potential equation $F_{\bar{z}} = \mu$. We now review the elementary theory of this equation.

A Beltrami differential is a bounded measurable section of $\kappa^{-1} \otimes \bar{\kappa}$. Given μ, a Beltrami differential, its absolute value $|\mu|$ is independent of coordinates; consequently the L^∞ norm of μ is well defined.

Definition 1.3. B is the complex Banach space of Beltrami differentials of finite L^∞ norm. $B(\Gamma) \subset B$ is the subspace of Γ invariant differentials.

The quotient H/Γ has finite area; the inclusion $B \hookrightarrow \mathscr{H}_2$ is continuous.

Lemma 1.4. *Let $\mu \in B$ be a Beltrami differential.*

i) *If $F_{\bar{z}} = \mu$ then* $L_{-1}\left(\frac{1}{(z-\bar{z})} F\right) = -\mu$.

ii) *Solutions of $F_{\bar{z}} = \mu$ are unique modulo holomorphic functions.*

iii) *There exists a unique solution $F[\mu]$, a continuous function vanishing at* 0, 1 *and* $0(|z|^2)$ *at ∞ satisfying*

$$F_{\bar{z}} = \mu \quad \text{in } H$$
$$F_{\bar{z}} = \bar{\mu}(\bar{z}) \quad \text{in } \mathbb{C} - H$$

in the sense of weak L^2 derivatives.

Proof. Remarks i) and ii) are left to the reader. Remark iii) will be found in a standard reference on Teichmüller theory, [3, 4].

The potential $F[\mu]$ is a section of κ^{-1}, a vector field.

Definition 1.5. Given $\mu \in B$, define $\Phi[\mu] = \frac{1}{(z-\bar{z})} F[\mu]$, $\Phi[\mu] \in S(-1)$.

In the study of Teichmüller space the harmonic Beltrami differentials play a central role. A Beltrami differential μ is harmonic provided $(D_{-2}-2)\mu = 0$. Now $(D_{-2}-2) = L_{-1}K_{-2}$ and for H/Γ compact the operator L_{-1} has trivial kernel. A Γ invariant harmonic Beltrami differential is a solution of the equation $K_{-2}\mu = 0$ or equivalently μ is harmonic if it can be written in the form $\mu = (z-\bar{z})^2 \bar{\phi}$, ϕ a holomorphic quadratic differential.

Definition 1.6. $\mathscr{B} \subset B$ is the subspace of harmonic Beltrami differentials. $\mathscr{B}(\Gamma) \subset \mathscr{B}$ is the subspace of Γ invariant differentials.

We shall see latter that $\mathscr{B}(\Gamma)$ is naturally isomorphic to the holomorphic tangent space of Teichmüller space at a point representing H/Γ.

We are ready to consider the first variation of $(f^{\varepsilon\mu})^* dA$. By Lemma 1.2 this is equivalent to evaluating $\operatorname{Re} K_{-1}\Phi[\mu]$; the argument is also the prototype for our latter calculations. Alternate proofs of the following appear in [1, 2].

Lemma 1.7. *Given $\mu \in \mathscr{B}(\Gamma)$ then* $\operatorname{Re} K_{-1}\Phi[\mu] = 0$.

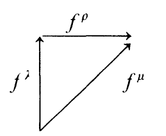
Proof. The first step is to verify that $\operatorname{Re} K_{-1}\Phi[\mu]$ is a Γ invariant function. For $\mu \in \mathscr{B}(\Gamma)$ consider the potential $F[\mu]$; $F[\mu]$ induces an infinitesimal deformation of $\Gamma \subset \mathrm{SL}(2;\mathbb{R})$. The Lie algebra $\mathscr{SL}(2;\mathbb{R})$ of $\mathrm{SL}(2;\mathbb{R})$ is represented by vector fields on H with coefficients quadratic polynomials having real coefficients. Accordingly for all $\gamma \in \Gamma$, $F[\mu] \circ \gamma \gamma'^{-1} - F[\mu] = p_\gamma(z)$, p_γ a quadratic polynomial.

Now the hyperbolic area element dA is SL $(2;\mathbb{R})$ hence $\mathcal{SL}(2;\mathbb{R})$ invariant. Thus forming Lie derivatives we find that $L(F\circ\gamma\gamma'^{-1})\,dA = L(F)\,dA$ or equivalently by Lemma 1.2. we have that $\operatorname{Re}K_{-1}\frac{1}{(z-\bar{z})}F = \operatorname{Re}K_{-1}\Phi$ is Γ invariant.

Now in order to determine the function $\operatorname{Re}K_{-1}\Phi$ we compute its Laplacian (for $\mu\in\mathscr{B}$, μ and thus $\Phi[\mu]$ are smooth). By definition $D_0 = K_{-1}L_0$ and thus $K_{-1}L_0\operatorname{Re}K_{-1}\Phi = \operatorname{Re}K_{-1}L_0K_{-1}\Phi \;= \operatorname{Re}K_{-1}(K_{-2}L_{-1}+2)\Phi \;= 2\operatorname{Re}K_{-1}\Phi$, where we have used the hypothesis $\mu\in\mathscr{B}(\Gamma)$. In particular $K_{-2}L_{-1}\Phi[\mu] = -K_{-2}\mu = 0$ by Lemma 1.4 and the definition of harmonic. The Laplacian D_0 has non-positive spectrum in particular $D_0\operatorname{Re}K_{-1}\Phi \;= 2\operatorname{Re}K_{-1}\Phi$ guarantees that $\operatorname{Re}K_{-1}\Phi = 0$, the desired conclusion.

2. Second variations and the harmonic projection

2.1. The space $B(\Gamma)$ of Beltrami differentials endowed with the L^∞ norm is a complex Banach space. By definition of the complex structure of the Teichmüller space $T(\Gamma)$ the map taking $\mu\in B(\Gamma)$, $\|\mu\|_\infty < 1$, to the equivalence class of the marked Riemann surface $H/f^\mu\Gamma(f^\mu)^{-1}$ is holomorphic. That the complex structure of $T(\Gamma)$ is independent of Γ depends on analyzing the diagram



where f^μ, f^ρ and f^λ are quasiconformal. In fact the diagram characterizes the manifold structure of $T(\Gamma)$ and is the focus of this chapter. Observe that the Beltrami differential ρ will satisfy

$$\rho(\mu,\lambda) = \frac{(f^\mu\circ(f^\lambda)^{-1})_{\bar{z}}}{(f^\mu\circ(f^\lambda)^{-1})_z} = \left(\frac{\mu-\lambda}{1-\bar{\lambda}\mu}\,\frac{f^\lambda_z}{\overline{f^\lambda_z}}\right)\circ(f^\lambda)^{-1}.$$

Our first result is that for μ, $\nu\in\mathscr{B}$

$$\frac{d}{d\varepsilon_1}\,\frac{d}{d\varepsilon_2}\,\rho(\varepsilon_1\nu+\varepsilon_2\mu,\varepsilon_1\nu)\,|_{\varepsilon_1=\varepsilon_2=0} = \mu K_{-1}\Phi[\nu] - L_{-1}(\mu\overline{\Phi[\nu]}).$$

With this calculation as a foundation we proceed to analyze the variation of the harmonic projection operator $P\colon B\to\mathscr{B}$. The operator P plays a central role in defining the Weil-Petersson metric. The first result for the metric is due to Ahlfors and will be stated in terms of the projection P.

2.2. Of course the starting point of the discussion is the solution of the Beltrami equation.

Definition 2.1. Given $\mu\in B$, $\|\mu\|_\infty < 1$, denote by f^μ the unique homeomorphism f: $\mathbb{C}\to\mathbb{C}$ fixing 0, 1 and ∞ and satisfying

$$\begin{cases} f_{\bar{z}} = \mu f_z & \text{in } H \\ f_{\bar{z}} = \bar{\mu}(\bar{z}) f_z & \text{in } \mathbb{C} - H. \end{cases}$$

The reader can consult [3, 4] for the following standard facts. By uniqueness f^{μ}: $H \to H$ and provided that μ is real analytic, $f^{\varepsilon\mu}$ will be real analytic for ε small and $z \in H$. In particular we shall use for μ real analytic that the ε and z derivatives of $f^{\varepsilon\mu}$ commute and converge in $C_c^{\infty}(H)$, the smooth compact-open topology for H. Furthermore, the solution has an expansion $f^{\varepsilon\nu} = z + \varepsilon F[\nu] + 0(\varepsilon)$, where F is the linear operator of Lemma 1.4 and for $\nu \in \mathscr{B}$, $0(\varepsilon)$ refers to the topology of $C_c^{\infty}(H)$.

An immediate consequence of the hypothesis μ, $\nu \in \mathscr{B}$ is that

$$\rho(\varepsilon_1 \nu + \varepsilon_2 \mu, \varepsilon_1 \nu) = \left(\frac{\varepsilon_2 \mu}{1 - \varepsilon_1 \bar{\nu}(\varepsilon_1 \nu + \varepsilon_2 \mu)} \frac{f_z^{\varepsilon_1 \nu}}{\overline{f_z^{\varepsilon_1 \nu}}} \right) \circ (f^{\varepsilon_1 \nu})^{-1}$$

is real analytic for ε_1, ε_2 small and $z \in H$. Now set

$$R(\mu, \nu) = \frac{d}{d\varepsilon} \rho(\nu + \varepsilon\mu, \nu)|_{\varepsilon=0} = \left(\frac{\mu}{1 - |\nu|^2} \frac{f_z^{\nu}}{\overline{f_z^{\nu}}} \right) \circ (f^{\nu})^{-1} \tag{2.1}$$

the result of one differentiation; in fact the reader can check that the derivative converges in L^{∞}.

Lemma 2.2. *For μ, $\nu \in \mathscr{B}$ then*

$$\frac{d}{d\varepsilon_1} \frac{d}{d\varepsilon_2} \rho(\varepsilon_1 \nu + \varepsilon_2 \mu, \varepsilon_1 \nu)|_{\varepsilon_1 = \varepsilon_2 = 0} = \mu K_{-1} \Phi[\nu] - L_{-1}(\mu \overline{\Phi[\nu]}).$$

Proof. Proceeding with the above calculation we have that

$$\frac{d}{d\varepsilon_1} \frac{d}{d\varepsilon_2} \rho|_{\varepsilon_1 = \varepsilon_2 = 0} = \mu(F[\nu]_z - \overline{F[\nu]_z}) + \frac{d}{d\varepsilon} (\mu_z (f^{\varepsilon\nu})^{-1} + \mu_{\bar{z}} \overline{(f^{\varepsilon\nu})^{-1}})|_{\varepsilon=0}.$$

The inverse map $(f^{\varepsilon\nu})^{-1}$ is characterized by $(f^{\varepsilon\nu})^{-1} \circ f^{\varepsilon\nu} = z$; differentiating in ε yields $\frac{d}{d\varepsilon}((f^{\varepsilon\nu})^{-1} + f^{\varepsilon\nu})|_{\varepsilon=0} = 0$, where we have used the expansion $f^{\varepsilon\nu} = z + \varepsilon F[\nu] + 0(\varepsilon)$. The resulting formula is

$$\frac{d}{d\varepsilon_1} \frac{d}{d\varepsilon_2} \rho|_{\varepsilon_1 = \varepsilon_2 = 0} = \mu(F[\nu]_z - \overline{F[\nu]_z}) - \mu_z F[\nu] - \mu_{\bar{z}} \overline{F[\nu]}.$$

Now the reader can check that $(\mu \overline{F[\nu]})_{\bar{z}} = L_{-1}(\mu \overline{\Phi[\nu]})$ and that for μ harmonic, $\mu_z = \frac{2}{(z - \bar{z})} \mu$, thus

$$\mu F[\nu]_z - \mu_z F[\nu] = \mu \left(F[\nu]_z - \frac{2}{(z - \bar{z})} F[\nu] \right) = \mu K_{-1} \Phi[\nu].$$

The calculation is complete.

2.3. Now we discuss the natural projection operator $P: B \to \mathscr{B}$. We shall see that the first variation of P can be determined from its formal properties. $P: B \to \mathscr{B}$ is the bounded linear operator defined by integration

$$P[\mu] = \frac{-3(z - \bar{z})^2}{\pi} \int_H \frac{\mu(\zeta)}{(\zeta - \bar{z})^4} d\sigma(\zeta)$$

for $\mu \in B$ and $d\sigma$ the Euclidean area element. We shall only use the following properties of the operator [2, 3]

i) for $\mu \in \mathscr{B}$, $P[\mu]=\mu$

ii) $P\gamma^* = \gamma^* P$ for all $\gamma \in \mathrm{SL}(2:\mathbb{R})$

iii) P extends to $\mathscr{H}_{-2}$ and is self adjoint

iv) on $\mathscr{H}_{-2} \cap C^2$, $D_{-2}P = PD_{-2}$.

That P is actually a projection follows from the observation $P: B \to \mathscr{B}$ and property i). An immediate consequence of property ii) is that $P: B(\Gamma) \to \mathscr{B}(\Gamma)$. By an argument of Selberg property iv) follows from property ii) and the regularity of the operator, [18]. Alternatively as an exercise we now derive property iv) from properties i) and iii).

Recall that $D_{-2} = L_{-1}K_{-2} + 2$ and $K_{-2}\mathscr{B} = 0$ thus $D_{-2}P = 2P$. Now using the self adjointness of P and D_{-2} we find $\langle \mu, PD_{-2}\nu\rangle = \langle P[\mu], D_{-2}\nu\rangle = \langle D_{-2}P[\mu], \nu\rangle = \langle 2P[\mu], \nu\rangle = \langle \mu, 2P[\nu]\rangle = \langle \mu, D_{-2}P[\nu]\rangle$, for $\mu \in \mathscr{H}_{-2}$ and $\nu \in \mathscr{H}_{-2} \cap C^2$. In as much as $\mathscr{H}_{-2}$ is a Hilbert space the conclusion follows.

2.4. As background for the first result on the variation of P we introduce the Weil-Petersson metric for Teichmüller space. Accordingly we start with a brief sketch of Teichmüller theory. The reader should check the references [1–4, 6] for a more complete description of the complex structure of $T(\Gamma)$ and the Weil-Petersson metric.

The map Π from the open unit ball in $B(\Gamma)$ to $T(\Gamma)$ given by assigning to μ the equivalence class of the marked Riemann surface H/Γ^μ, $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$, is fundamental to the study of Teichmüller theory. Π is a differentiable map from the Banach space $B(\Gamma)$ to the manifold $T(\Gamma)$. In order to better understand the map Π consider $QD(\Gamma)$, the space of Γ invariant holomorphic quadratic differentials, and the pairing $B(\Gamma) \times QD(\Gamma) \xrightarrow{(\,,\,)} \mathbb{C}$: for $\mu \in B(\Gamma)$, $\phi \in QD(\Gamma)$ define $(\mu, \phi) = \int_{H/\Gamma} \mu\phi$. Let $N(\Gamma) = QD(\Gamma)^\perp \subset B(\Gamma)$ be the null space of the quadratic differentials relative to the pairing. The basic fact is that the kernel at the origin of the differential $d\Pi$ is the subspace $N(\Gamma)$ or equivalently the following theorem.

Theorem 2.3, [1, 3]. *In the above notation, at the point of Teichmüller space representing Γ there are the natural isomorphisms*

$$B(\Gamma)/N(\Gamma) \approx T^{1,0}\,T(\Gamma)$$

$$QD(\Gamma) \approx (T^{1,0})^*\,T(\Gamma)$$

and the pairing

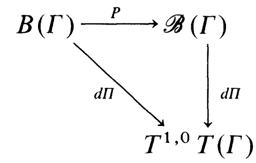
$$B(\Gamma)/N(\Gamma) \times QD(\Gamma) \xrightarrow{(\,,\,)} \mathbb{C}$$

represents the natural pairing $T^{1,0} \times (T^{1,0})^* \to \mathbb{C}$.

Definition 2.4. Given $\mu \in B(\Gamma)$ denote by $\dfrac{\partial}{\partial t(\mu)} \in T^{1,0}\,T(\Gamma)$ the image of μ by the map $d\Pi$.

In order to study tensors on Teichmüller space it will be far simpler if the coset space $B(\Gamma)/N(\Gamma)$ is replaced by a suitable space of tensors on H/Γ. As we shall now explain $\mathscr{B}(\Gamma)$ is naturally isomorphic to the quotient $B(\Gamma)/N(\Gamma)$ and thus it will suffice *for all of our considerations* to derive formulas valid for the harmonic Beltrami differentials.

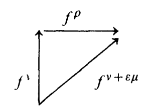
The basic observation is the diagram



(2.2)

for the differential $d\Pi$ and the harmonic projection P. To see this start by noting that the kernel $\operatorname{Ker} P$ of P acting $B(\Gamma)$ coincides with the subspace $N(\Gamma)$. In particular for $\psi \in QD(\Gamma)$ set $v = \frac{(z-\bar{z})^2}{-4}\,\bar{\psi} \in \mathscr{B}(\Gamma)$ and for $\mu \in B(\Gamma)$ the reader will confirm by properties i) and iii) of the projection that $(\mu, \psi) = \langle \mu, v\rangle = \langle \mu, P[v]\rangle = \langle P[\mu], v\rangle$. Consequently $\mu \in N(\Gamma)$ is equivalent to $P[\mu] \in \mathscr{B}(\Gamma)^{\perp} \subset \mathscr{H}_{-2}$ but $P[\mu] \in \mathscr{B}(\Gamma)$ and thus $\mu \in N(\Gamma)$ is equivalent to $P[\mu] = 0$. Finally to establish the commutativity of the diagram (2.2) note that $P[\mu - P[\mu]] = 0$ and thus given $\mu \in B(\Gamma)$, $d\Pi(\mu) = d\Pi(P[\mu] + (\mu - P[\mu])) = d\Pi(P[\mu])$.

Now a change of local coordinates on Teichmüller space may be understood in terms of the diagram



of quasiconformal maps. Informally the *tangent vector* $\frac{d}{d\varepsilon}(v + \varepsilon\mu)|_{\varepsilon=0} \in B(\Gamma)$ considered as a deformation of Γ corresponds to the *tangent vector* $\frac{d}{d\varepsilon}\rho(v + \varepsilon\mu, v)|_{\varepsilon=0} \in B(\Gamma^{v})$ considered as a deformation of Γ^{v}. Since $B(\Gamma^{v})$ is naturally isomorphic to the tangent space of Teichmüller space at Γ^{v} the harmonic projection of $\frac{d}{d\varepsilon}\rho|_{\varepsilon=0} = R(\mu, v)$ (see formula (2.1)) is the canonical representative for the tangent vector. Accordingly, in order to better understand a change of coordinates on Teichmüller space we derive the variational formula for $P[R(\mu, \varepsilon v)]$. We prefer to consider the pullback to H/Γ of the tensor $P[R(\mu, v)]$.

Definition 2.5. Given μ, $v \in B(\Gamma)$ set

$$Q(\mu, v) = P[R(\mu, v)] \circ f^{v}\,\frac{\overline{f_z^{v}}}{f_z^{v}}. \tag{2.3}$$

2.5. Now we are ready to introduce local coordinates and give the formulas for the Weil-Petersson metric. We start by considering Ahlfors' result: the harmonic Beltrami differentials give geodesic coordinates, [1]. Finally we close the chapter by deriving the variational formula for $Q(\mu, \varepsilon v)$.

Given a basis $\kappa_1, \ldots, \kappa_n$ for $\mathscr{B}(\Gamma)$ we wish to consider the associated local coordinates for $T(\Gamma)$. Specifically for $t=(t_1, \ldots, t_n) \in \mathbb{C}^n$ sufficiently small define $\kappa(t)=\sum_j t_j\kappa_j$ and consider the marked surface $S_t = H/\Gamma^{\kappa(t)}$, $\Gamma^{\kappa(t)} = f^{\kappa(t)}\Gamma(f^{\kappa(t)})^{-1}$. The assignment of $t \in \mathbb{C}^n$ to the equivalence class of S_t is a local coordinate chart for $T(\Gamma)$, [1, 2]. Recalling the notation $R(\mu, v) = \frac{d}{d\varepsilon}\rho(v+\varepsilon\mu, v)|_{\varepsilon=0}$ we have for $\mu=\kappa_\alpha$, $\alpha=1, \ldots, n$ that the assignment $t \to P \circ R(\mu, \kappa(t)) \in \mathscr{B}(S_t)$ represents the coordinate vector field $\frac{\partial}{\partial t_\alpha}$ for the above choice of local coordinates.

Definition 2.6. In the above notation, the Weil-Petersson metric $ds^2 = 2\sum g_{\alpha\bar\beta}dt_\alpha \overline{dt_\beta}$ is defined by

$$g_{\alpha\bar\beta}(t) = \langle P \circ R(\kappa_\alpha, \kappa),\ P \circ R(\kappa_\beta, \kappa)\rangle \tag{2.4}$$

where $\kappa=\kappa(t)$ and the Hermitian product is on S_t.

The operator P is self adjoint and thus (2.4) can also be given as

$$g_{\alpha\bar\beta}(t) = \langle P \circ R(\kappa_\alpha, \kappa),\ R(\kappa_\beta, \kappa)\rangle .$$

A change of variables gives the following formulas (the integration is now on $S_0 = H/\Gamma$) for the metric with $\mu=\kappa_\alpha$ and $v=\kappa_\beta$

$$\begin{aligned} g_{\alpha\bar\beta}(t) &= \int_{H/\Gamma} Q(\mu,\kappa)\overline{Q(v,\kappa)}\ (f^\kappa)^* dA \\ g_{\alpha\bar\beta}(t) &= \int_{H/\Gamma} Q(\mu,\kappa)\ \overline{\frac{v}{1-|\kappa|^2}}\ (f^\kappa)^* dA . \end{aligned} \tag{2.5}$$

Before proceeding recall that the derivatives of $Q(\mu,\kappa)$ and $(f^\kappa)^* dA$, $\mu, \kappa \in \mathscr{B}(\Gamma)$, commute and converge in $C_c^\infty(H)$.

Lemma 2.7, [1]. *In the above notation, the derivatives* $\frac{\partial g_{\alpha\bar\beta}}{\partial t_\gamma}(t)$ *and* $\frac{\partial g_{\alpha\bar\beta}}{\partial \bar t_\gamma}(t)$ *vanish at* $t=0$.

Proof. First note that the derivatives of Q and $(f^\kappa)^* dA$ converge uniformly on a fundamental domain for Γ. Now referring to Lemma 1.7 we recall that the first derivatives of $(f^\kappa)^* dA$ are trivial. Using the formulas (2.5) we have that

$$\frac{\partial g_{\alpha\bar\beta}}{\partial t_\gamma} = \left\langle \frac{\partial}{\partial t_\gamma} Q(\mu,\kappa), v\right\rangle + \left\langle \mu, \frac{\partial}{\partial t_\gamma} Q(v,\kappa)\right\rangle$$

and

$$\frac{\partial g_{\alpha\bar\beta}}{\partial t_\gamma} = \left\langle \frac{\partial}{\partial t_\gamma} Q(\mu,\kappa),\ v\right\rangle .$$

Equating the two gives $\left\langle \mu, \frac{\partial}{\partial t_\gamma} Q(v,\kappa)\right\rangle = 0$ and the conclusion follows since μ and v are arbitrary. The analogous argument applies for the $\bar t$ derivatives.

Corollary 2.8. *For μ, ν, $\kappa \in \mathscr{B}(\Gamma)$ then*

$$\left\langle \frac{d}{d\varepsilon} Q(\mu, \varepsilon\kappa)|_{\varepsilon=0}, \nu \right\rangle = 0.$$

An immediate consequence of the lemma is that the Weil-Petersson metric is Kähler and that its Christoffel symbols vanish at the origin for the above local coordinates [1, 2].

Theorem 2.9. *For μ, $\nu \in \mathscr{B}(\Gamma)$*

$$\frac{d}{d\varepsilon} Q(\mu, \varepsilon\nu)|_{\varepsilon=0} = -L_{-1} L_0 (D_0 - 2)^{-1} (\mu\bar{\nu}).$$

Proof. The proof has two steps: First we check that both of the above expressions lie in the orthogonal complement of $\operatorname{Ker}(D_{-2}-2) \subset \mathscr{H}_{-2}$, and then we compute $(D_{-2}-2)$ of both expressions.

First observe that the operators $(D_{-2}-2) = L_{-1}K_{-2}$ and K_{-2} have the same kernel. The inclusion $\operatorname{Ker} K_{-2} \subset \operatorname{Ker}(D_{-2}-2)$ is immediate. For the reverse consider $g \in \operatorname{Ker}(D_{-2}-2)$ then $L_{-1}K_{-2}g = 0$ or equivalently $((z-\bar{z})K_{-2}g)_{\bar{z}} = 0$. Now the tensor $X = (z-\bar{z})K_{-2}g$ is a Γ invariant vector field and $X_{\bar{z}} = 0$ provides that X is a holomorphic vector field. By Riemann Roch X is trivial and $K_{-2}g = 0$ or $g \in \operatorname{Ker} K_{-2}$.

Next we observe that both expressions in the formulas are in the orthogonal complement of $\mathscr{B}(\Gamma) = \operatorname{Ker}(D_{-2}-2)$. That the derivative satisfies this property is the content of Corollary 2.8. For the right hand side consider integration by parts of $g \in \mathscr{H}_{-1} \cap C^1$ arbitrary and $\nu \in \mathscr{B}(\Gamma)$: $\langle \nu, L_{-1}g \rangle = -\langle K_{-2}\nu, g \rangle$ and $\nu \in \mathscr{B}$ implies $K_{-2}\nu = 0$. The claim is established.

At this point it will suffice to establish that

$$\begin{aligned}(D_{-2}-2)\frac{d}{d\varepsilon} Q(\mu, \varepsilon\nu)|_{\varepsilon=0} &= -(D_{-2}-2) L_{-1} L_0 (D_0-2)^{-1}(\mu\bar{\nu}) \\ &= -L_{-1} L_0 (\mu\bar{\nu}).\end{aligned}$$

Start by differentiating the left hand side to obtain (recall the formulae (2.1) and (2.3))

$$\begin{aligned}\frac{d}{d\varepsilon} Q(\mu, \varepsilon\nu)|_{\varepsilon=0} = \frac{d}{d\varepsilon} P[R(\mu, \varepsilon\nu)]|_{\varepsilon=0} &+ \mu_z F[\nu] + \mu_{\bar{z}} \overline{F[\nu]} \\ &+ \mu(\overline{F[\nu]}_z - F[\nu]_z)\end{aligned}$$

where $\mu \in \mathscr{B}$ implies $P[\mu] = \mu$. As with Lemma 2.2 this can be written as

$$\frac{d}{d\varepsilon} Q(\mu, \varepsilon\nu)|_{\varepsilon=0} = \frac{d}{d\varepsilon} P[R(\mu, \varepsilon\nu)]|_{\varepsilon=0} - \mu K_{-1}\Phi[\nu] + L_{-1}(\mu\overline{\Phi[\nu]}).$$

Now we are ready to compute $(D_{-2}-2) = L_{-1}K_{-2}$. The derivatives converge in $C_c^\infty(H)$ and thus

$$L_{-1}K_{-2}\frac{d}{d\varepsilon} P[R] = \frac{d}{d\varepsilon} L_{-1}K_{-2}P[R]$$

where the latter vanishes since $P[R] \in \mathscr{B}$. The final step is to evaluate $L_{-1}K_{-2}(-\mu K_{-1}\Phi[v] + L_{-1}(\mu\overline{\Phi[v]}))$. Substituting $\overline{K_{-1}\Phi[v]} = -\overline{K_{-1}\Phi[v]}$, the result of Lemma 1.7, the desired expression $-L_{-1}L_0(\mu\bar{v})$ is easily obtained.

3. The second variation of area

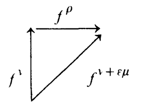
3.1. Our goal is to use the techniques developed in the preceeding chapters to obtain the formula

$$\frac{d}{d\varepsilon_1}\frac{d}{d\varepsilon_2}(f^{\varepsilon_1 v+\varepsilon_2\mu})^* dA|_{\varepsilon_1=\varepsilon_2=0} = -2\mathrm{Re}(\mu\bar{v} + 2(D_0-2)^{-1}(\mu\bar{v}))\,dA$$

valid for $\mu, v \in \mathscr{B}(\Gamma)$. Ahlfors derived an integral formula for the second variation of area, [2]. The integral is over the product $H \times H$ and the kernel is a combination of singular Hilbert kernels for one variable. Recently Royden has also obtained a formula for the second variation of area, [7]. Apparently his formula is similar to the above.

Our considerations begin with showing how the hypothesis $\mu, v \in \mathscr{B}(\Gamma)$ in particular the earlier result $\frac{d}{d\varepsilon}(f^{\varepsilon\mu})^* dA|_{\varepsilon=0} = 0$ greatly simplifies the calculation. Then as with the proof of Lemma 1.7 and Theorem 2.9 we find it easier to evaluate the Laplacian of the desired expression.

3.2. Again consider the diagram



$\rho = \rho(v+\varepsilon\mu, v)$ for a triple of quasiconformal maps. Now by definition of the pullback

$$f^* \frac{d}{d\varepsilon}(f^{v+\varepsilon\mu})^* dA|_{\varepsilon=0} = (f^v)^*\left(\frac{d}{d\varepsilon}(f^\rho)^* dA|_{\varepsilon=0}\right)$$

and by Lemma 1.2. we then obtain

$$(f^v)^*(2\mathrm{Re}(K_{-1}\Phi[R(\mu, v)])\,dA) = 2\mathrm{Re}(K_{-1}\Phi[R(\mu, v)]) \circ f^v (f^v)^* dA.$$

Now we replace v by εv and proceed to evaluate the ε-derivative at $\varepsilon = 0$. By Lemma 1.7 the ε-derivative of $(f^{\varepsilon v})^* dA$ vanishes and it only remains to consider $\frac{d}{d\varepsilon} 2\mathrm{Re}(K_{-1}\Phi[R(\mu, \varepsilon v)]) \circ f^{\varepsilon v}|_{\varepsilon=0} dA$. Recall the $C_c^\infty(H)$ convergence; we proceed and obtain

$$\left(\frac{d}{d\varepsilon} 2\mathrm{Re}K_{-1}\Phi[R(\mu, \varepsilon v)]|_{\varepsilon=0} + 2(\mathrm{Re}K_{-1}\Phi[\mu])_z F[v] + 2(\mathrm{Re}K_{-1}\Phi[\mu])_{\bar{z}}\overline{F[v]}\right) dA.$$

Now we apply Lemma 1.7. one last time in particular $\operatorname{Re} K_{-1}\Phi[\mu]=0$ and we have that the second variation of area is given by

$$\frac{d}{d\varepsilon} 2\operatorname{Re} K_{-1}\Phi[R(\mu,\varepsilon\nu)]|_{\varepsilon=0}\, dA\,.$$

Before proceeding further we discuss convergence. The $C_c^\infty(H)$ convergence guarantees that the ε-derivative and K_{-1} commute. Now for all ε small $R(\mu,\varepsilon\nu)$ is bounded in L^∞ norm; the integral for the potential $\Phi[R]$ converges. On the other hand the convergence of the ε-derivative of $R(\mu,\varepsilon\nu)$ in L^∞ is a delicate question which we wish to avoid. Consequently we shall not use that the ε-derivative commutes with Φ. The following result will serve as a substitute.

Lemma 3.1. *Given* $\mu,\ \nu\in\mathscr{B}(\Gamma)$

$$L_{-1}\frac{d}{d\varepsilon}\Phi[R(\mu,\varepsilon\nu)]|_{\varepsilon=0}=-\frac{d}{d\varepsilon}R(\mu,\varepsilon\nu)|_{\varepsilon=0}=-\mu K_{-1}\Phi[\nu]+L_{-1}(\mu\overline{\Phi[\nu]})\,.$$

Proof. By definition $\Phi[R(\mu,\varepsilon\nu)]=\frac{d}{d\varepsilon_1}f^\rho|_{\varepsilon_1=0},\ \ \rho=\rho(\varepsilon\nu+\varepsilon_1\mu,\varepsilon\nu)$ where all expressions are real analytic in z and ε. Consequently L_{-1} commutes with the ε-derivative: $L_{-1}\frac{d}{d\varepsilon}\Phi=\frac{d}{d\varepsilon}L_{-1}\Phi=-\frac{d}{d\varepsilon}R$ by Lemma 1.4. The second formula is the result of Lemma 2.2.

3.3. The discussion of the previous sections is summarized in the following result.

Lemma 3.2. *Suppose* $G,\ G\in S(-1)$ *satisfies*

i) $L_{-1}G=-\mu K_{-1}\Phi[\nu]+L_{-1}(\mu\overline{\Phi[\nu]})$

and

ii) $\operatorname{Re} K_{-1}G$ *is* Γ *invariant.*

Then $2\operatorname{Re} K_{-1}G\,dA$ *is the second variation of the area element.*

Proof. First observe that by Lemma 3.1 $L_{-1}\left(\frac{d}{d\varepsilon}\Phi[R]|_{\varepsilon=0}-G\right)=0$. Furthermore by (3.1) and hypothesis we have that $g=\operatorname{Re} K_{-1}\left(\frac{d}{d\varepsilon}\ \Phi[R]|_{\varepsilon=0}-G\right)$ is a Γ invariant function. Now as with Lemma 1.7 applying $D_0=K_{-1}L_0$ we find that $D_0 g=2g$ and g is Γ invariant: consequently g vanishes identically. In particular we have that $2\operatorname{Re} K_{-1}G\,dA=2\operatorname{Re} K_{-1}\frac{d}{d\varepsilon}\Phi[R]|_{\varepsilon=0}\,dA$, the desired conclusion.

Theorem 3.3. *Given* $\mu,\ \nu\in\mathscr{B}(\Gamma)$ *then*

$$\frac{d}{d\varepsilon_1}\frac{d}{d\varepsilon_2}(f^{\varepsilon_1\nu+\varepsilon_2\mu})^* dA|_{\varepsilon_1=\varepsilon_2=0}=-2\operatorname{Re}(\mu\bar\nu+2(D_0-2)^{-1}(\mu\bar\nu))\,dA\,.$$

Proof. The plan is to identify that function G satisfying the hypothesis of Lemma 3.2. Certainly we should start by setting $G=\hat G+\mu\overline{\Phi[\nu]}$. Computing the variation of area for the second term we find that $2\mathrm{Re}K_{-1}(\mu\overline{\Phi[v]}) = 2\mathrm{Re}\mu K_1\overline{\Phi[v]} = -2\mathrm{Re}\mu\bar{v}$, an invariant function. In particular we observe that it will suffice to find $\hat{G}$ satisfying i) $L_{-1}\hat{G} = -\mu K_{-1}\Phi[v]$ and ii) $\mathrm{Re}K_{-1}\hat{G}$ is invariant. Following this line of reasoning the quantity $F=(D_{-1}-2)\hat{G}$ will satisfy i) $L_{-1}F = -(D_{-2}-2)\ (\mu K_{-1}\Phi[v])$ and ii) $\mathrm{Re}K_{-1}F = (D_0-2)\mathrm{Re}K_{-1}\hat{G}$ is Γ invariant, since the Laplacians commute with the Maass operators. Before proceeding further recall from Chapter 1 that (D_0-2) is an invertible operator and thus in order to determine the area variation $2\mathrm{Re}K_{-1}\hat{G}$ it will suffice to first calculate $2(D_0-2)\mathrm{Re}K_{-1}\hat{G}$ and then apply the inverse $(D_0-2)^{-1}$.

With this in mind the obvious candidate for F is $-K_{-2}(\mu K_{-1}\Phi[v])$. In particular $(D_{-2}-2) = L_{-1}K_{-2}$ and thus $L_{-1}F = -(D_{-2}-2)\ (\mu K_{-1}\Phi[v])$; our choice of F satisfies i). We now compute $K_{-1}F$; recall that $-K_{-1}\Phi[v] = \overline{K_{-1}\Phi[v]} = L_1\overline{\Phi[v]}$, thus $F = K_{-2}(\mu L_1\overline{\Phi[v]})$ and $K_{-1}F = K_{-1}K_{-2}(\mu L_1\overline{\Phi[v]}) = K_{-1}(\mu K_0 L_1\overline{\Phi[v]})$. Substituting $K_0L_1 = L_2K_1+2$ we find $K_0L_1\overline{\Phi[v]} = (L_2K_1+2)\overline{\Phi[v]} = \overline{(K_{-2}L_{-1}+2)\Phi[v]} = 2\overline{\Phi[v]}$ and therefore in brief $K_{-1}F = K_{-1}2(\mu\overline{\Phi[v]}) = 2\mu\overline{L_{-1}\Phi[v]} = -2\mu\bar{v}$. Indeed $\mathrm{Re}K_{-1}F$ is an invariant function. Therefore our choice of F satisfies the required hypothesis. In summary we have that $2(D_0-2)\mathrm{Re}K_{-1}\hat{G} = 2\mathrm{Re}K_{-1}F = -4\mathrm{Re}\mu\bar{v}$; the final formula follows from this result.

4. The Riemann tensor of the Weil-Petersson metric

4.1. Fixing a basis $\mu_\alpha \in \mathscr{B}(\Gamma)$ for the harmonic Beltrami differentials consider the associated local coordinates (see Section 2.5) $t \in (t_1, \ldots, t_n) \in \mathbb{C}_n$ for $T(\Gamma)$. Ahlfors' result Lemma 2.7 provides that for these coordinates the first derivatives of the metric tensor vanish at the origin. In this case, following Bochner's conventions, the Riemann curvature tensor at the origin is then

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{t}_\delta}, \qquad [8].$$

The main result of this chapter is the formula

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = -2\int_{H/\Gamma}(D_0-2)^{-1}(\mu_\alpha\bar{\mu}_\beta)\ (\mu_\gamma\bar{\mu}_\delta)\,dA$$
$$-2\int_{H/\Gamma}(D_0-2)^{-1}(\mu_\alpha\bar{\mu}_\delta)\ (\mu_\gamma\bar{\mu}_\beta)\,dA$$

for the Riemann tensor of the Weil-Petersson metric. As the reader shall see the elementary fact that $-2(D_0-2)^{-1}$ is a non-negative operator governs the sign of the curvatures. In particular we find that the holomorphic sectional curvature, the Ricci curvature, the scalar curvature and the general sectional curvature are all negative; in fact we obtain upper bounds for the first three. For example the curvature of the holomorphic section spanned by μ_α, $\langle\mu_\alpha, \mu_\alpha\rangle = 1$ is $R_\alpha = 4\int_{H/\Gamma}(D_0-2)^{-1}|\mu_\alpha|^2|\mu_\alpha|^2\,dA$ and we find that $R_\alpha < \dfrac{-1}{2\pi(g-1)}$. Tromba was the first to show that the sectional curvature is indeed negative, [32]. Royden has obtained similar formulas and estimates for the curvatures. In fact Royden was the first to show that the holomorphic curvatures are bounded away from zero and he actually conjectured the above bound for the holomorphic sectional curvature, [16]. Now proceeding in a slightly different vein we consider the first chern form of the cotangent bundle $(T^{1,0})^* T(\Gamma)$. Given a unitary frame $\mu_\alpha \in \mathscr{B}(\Gamma)$, $\alpha = 1, \ldots, n$ and $\nu, \kappa \in \mathscr{B}(\Gamma)$ arbitrary then by definition

$$c_1 = \frac{i}{2\pi} \bar{\partial}\partial \text{ trace } (g_{\alpha\bar{\beta}})$$

thus

$$c_1\left(\frac{\partial}{\partial t(\nu)}, \frac{\partial}{\partial t(\kappa)}\right) = \frac{i}{2\pi} \sum_\alpha \left(-2 \int_{H/\Gamma} (D_0 - 2)^{-1} (\mu_\alpha \bar{\mu}_\alpha)(\nu\bar{\kappa})\, dA \right.$$
$$\left. - 2 \int_{H/\Gamma} (D_0 - 2)^{-1} (\mu_\alpha \bar{\kappa})(\bar{\mu}_\alpha \nu)\, dA \right)$$

from which it follows immediately that c_1 is positive.

4.2. We refer the reader to Bochner's paper for a review of Hermitian geometry, [8]. Introduce local coordinates $t = (t_1, \ldots, t_n)$ for a neighborhood of Γ in $T(\Gamma)$ by choosing a basis $\mu_\alpha \in \mathscr{B}(\Gamma)$, $\alpha = 1, \ldots, n$ (see Section 2.5). By convention Greek indices will run from 1 to n. By Lemma 2.7 the derivatives $\frac{\partial g_{\alpha\bar{\beta}}}{\partial t_\gamma}(0)$ and $\frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{t}_\gamma}(0)$ vanish and hence the curvature tensor is given by $R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{t}_\delta}(0)$, [8].

We start with the following formulas of Section 2.5 for the metric tensor

$$g_{\alpha\bar{\beta}} = \int_{H/\Gamma} Q(\mu, \kappa)\, \overline{Q(\nu, \kappa)}\, (f^\kappa)^*\, dA$$
$$\int_{H/\Gamma} Q(\mu, \kappa)\, \frac{\bar{\nu}}{1 - |\kappa|^2}\, (f^\kappa)^*\, dA$$

where $\kappa = \kappa(t) = \sum_\gamma t_\gamma \kappa_\gamma$ for $t \in \mathbb{C}^n$ small and $Q(\mu, \kappa)$ is defined by (2.3). In order to obtain a more general formula we consider the second real derivative $\frac{d^2}{d\varepsilon^2} g_{\alpha\bar{\beta}}(\varepsilon e_\gamma)|_{\varepsilon=0}$ for $g_{\alpha\bar{\beta}}$ restricted to the line $t = \varepsilon e_\gamma$, e_γ the γth basis vector of $\mathbb{C}^n$, equivalently $\kappa = \varepsilon\kappa_\gamma$. As already discussed the above quantities vary smoothly in $C_c^\infty(H)$ and Γ has a compact fundamental domain: we may differentiate under the integral. At this point the calculation is formal; we denote an ε derivative evaluated at the origin by placing a dot above the corresponding expression.

Given the above formulas (4.1) and the inner product $\langle\ ,\ \rangle$ on $\mathscr{H}_{-2}$ we have that

$$\begin{aligned} \ddot{g}_{\alpha\bar{\beta}} &= \langle \ddot{Q}(\mu, \kappa), \nu \rangle + 2\langle \dot{Q}(\mu, \kappa), \dot{Q}(\nu, \kappa) \rangle + \langle \mu, \ddot{Q}(\nu, \kappa) \rangle + \int \mu\bar{\nu}\, (f^{\ddot{\kappa}})^*\, dA \\ \ddot{g}_{\alpha\bar{\beta}} &= \langle \ddot{Q}(\mu, \kappa), \nu \rangle + 2\langle \mu, \nu|\kappa|^2 \rangle + \int \mu\bar{\nu}\, (f^{\ddot{\kappa}})^*\, dA \end{aligned} \tag{4.2}$$

where we have used Lemma 1.7, $(f^{\dot{\kappa}})^*\, dA = 0$ and that $Q(\rho, \kappa(0)) = \rho$, for $\rho \in \mathscr{B}(\Gamma)$. Equating the two expressions we obtain the following formula

$$\ddot{g}_{\alpha\bar{\beta}} = 4\langle \mu, \nu|\kappa|^2 \rangle - 2\langle \dot{Q}(\mu, \kappa), \dot{Q}(\nu, \kappa) \rangle + \int \mu\bar{\nu}\, (f^{\ddot{\kappa}})^*\, dA\,. \tag{4.3}$$

We are essentially done. The second term is given by Theorem 2.9; the resulting expression will be simplified by integration by parts. The third term is given by Theorem 3.3. In particular $\dot{Q}(\mu,\kappa) = -L_{-1}L_0(D_0-2)^{-1}(\mu\bar{\kappa})$ and therefore intergrating by parts $\langle \dot{Q}(\mu,\kappa), \dot{Q}(\nu,\kappa)\rangle = \langle K_{-1}K_{-2}L_{-1}L_0(D_0-2)^{-1}(\mu\bar{\kappa}), (D_0-2)^{-1}(\nu\bar{\kappa})\rangle$; recalling that $K_{-1}K_{-2}L_{-1}L_0 = (D_0-2)^2+2(D_0-2)$ and that (D_0-2) is self adjoint we find that $\langle \dot{Q}, \dot{Q}\rangle = \langle \mu\bar{\kappa}, \nu\bar{\kappa}\rangle + 2\langle \mu\bar{\kappa}, (D_0-2)^{-1}(\nu\bar{\kappa})\rangle$. Now combining this result with the formula (4.3) and Theorem 3.3 we have that

$$\ddot{g}_{\alpha\bar{\beta}} = -4\langle \mu\bar{\kappa}_\gamma, (D_0-2)^{-1}(\nu\bar{\kappa}_\gamma)\rangle - 4\langle \mu\bar{\nu}, (D_0-2)^{-1}|\kappa_\gamma|^2\rangle. \tag{4.4}$$

We observe that the Hermitian product is now for $\mathscr{H}_0$; in particular given μ, $\nu \in \mathscr{B}(\Gamma)$ then $\mu\bar{\nu} \in \mathscr{H}_0$. To simplify our further considerations we introduce the following notation.

Definition 4.1. $\Delta = -2(D_0-2)^{-1}$ is an operator on $\mathscr{H}_0$.

Recall that Δ is a self adjoint compact integral operator with a positive kernel. Furthermore we note that Δ is the identity on constant functions.

Theorem 4.2. *Given a basis $\mu_\alpha \in \mathscr{B}(\Gamma)$, let $t=(t_1, \ldots, t_n)$ be the associated local coordinates for Teichmüller space and $ds^2 = 2\sum g_{\alpha\bar{\beta}}dt_\alpha d\bar{t}_\beta$ the Weil-Petersson metric, then*

i) $\dfrac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial t_\delta}(0) = 0, \quad \dfrac{\partial^2 g_{\alpha\bar{\beta}}}{\partial \bar{t}_\gamma \partial \bar{t}_\delta}(0) = 0$

and

ii) *the Riemann tensor is given as*

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) = \frac{\partial^2 g_{\alpha\bar{\beta}}}{\partial t_\gamma \partial \bar{t}_\delta}(0) = \langle \Delta(\mu_\alpha\bar{\mu}_\beta), (\bar{\mu}_\gamma\mu_\delta)\rangle + \langle \Delta(\mu_\alpha\bar{\mu}_\delta), (\bar{\mu}_\gamma\mu_\beta)\rangle.$$

Proof. First note that by polarization the real derivative $\dfrac{d^2}{d\varepsilon_1 d\varepsilon_2} g_{\alpha\bar{\beta}}(0)$ is obtained from the above formula (4.4). Property i) is an immediate consequence of the observation that (4.4) is unchanged if κ_γ is replaced by $i\kappa_\gamma$. And finally the formula for the Riemann tensor is an immediate consequence of formula (4.4) and the definition of complex derivatives. The proof is complete.

Before proceeding we wish to remark on the assumption $\mu_\alpha \in \mathscr{B}$ i.e. that the Beltrami differentials are harmonic. We are also interested in evaluating tensors for the general Beltrami differential $\rho \in \mathscr{B}(\Gamma)$. In fact we see that $R(\rho_\alpha, \bar{\rho}_\beta, \rho_\gamma, \bar{\rho}_\delta) = R(P[\rho_\alpha], \overline{P[\rho_\beta]}, P[\rho_\gamma], \overline{P[\rho_\delta]})$ for the projection operator P. This follows from two basic observations: R is a tensor i.e. depends only on a choice of vectors in $T^{1,0}T(\Gamma)$ and $\dfrac{\partial}{\partial t(\rho)} = \dfrac{\partial}{\partial t(\mu)}$ for $\mu = P[\rho]$, $\rho \in \mathscr{B}(\Gamma)$ (see diagram 2.2)).

4.3. Now we shall derive estimates for the sectional, Ricci and scalar curvature of the Weil-Petersson metric. The following inequality is required to show that the sectional curvature is negative.

Lemma 4.3. *Given f, $g \in \mathscr{H}_0$ then*

$$|\Delta(fg)| \leqq |\Delta f^2|^{1/2} |\Delta g^2|^{1/2}.$$

Proof. The operator $\Delta = -2(D_0-2)^{-1}$ is integral with kernel $G(z,w)$ defined on the complement of the diagonal in $H/\Gamma \times H/\Gamma$, [15]. The Green's function G is strictly positive and in a neighborhood of the diagonal $G(z,w)+\frac{1}{2\pi}\log|z-w|$ is continuous, [15]. Given $f,\ g\in\mathscr{H}_0$, $\Delta(fg)$ is defined, the integral converges. G possesses a positive square root and so we may write $G|fg| = G^{1/2}|f|\,G^{1/2}|g|$ and apply the Hölder inequality:

$$|\int GfgdA| \leqq \int G|fg|dA \leqq (\int Gf^2 dA)^{1/2}(\int Gg^2 dA)^{1/2}.$$

This is the desired inequality.

We introduce one last bit of notation in order to simplify the discussion.

Definition 4.4. Given $\mu_* \in \mathscr{B}(\Gamma)$ set

$$(\alpha\bar{\beta},\gamma\bar{\delta}) = \langle \Delta(\mu_\alpha\bar{\mu}_\beta),\ (\bar{\mu}_\gamma\mu_\delta)\rangle .$$

Theorem 4.5. *The Weil-Petersson metric has negative sectional curvature.*

Proof. Given holomorphic tangent vectors $\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2} \in T^{1,0}T(\Gamma)$ associate the real tangent vectors $v_1 = \frac{\partial}{\partial t_1}+\frac{\partial}{\partial \bar{t}_1}$, $v_2 = \frac{\partial}{\partial t_2}+\frac{\partial}{\partial \bar{t}_2} \in T_{\mathbb{R}}T(\Gamma)$. Bochner shows that the curvature of the section spanned by v_1 and v_2 is R/g where [8, formulas 24 and 25]

$$R = R_{1\bar{2}1\bar{2}} - R_{1\bar{2}2\bar{1}} - R_{2\bar{1}1\bar{2}} + R_{2\bar{1}2\bar{1}}$$

and

$$g = 4g_{1\bar{1}}g_{2\bar{2}} - 2|g_{1\bar{2}}|^2 - 2\mathrm{Re}(g_{1\bar{2}})^2$$

for $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ the curvature tensor and $g_{\alpha\bar{\beta}}$ the metric tensor. It is an immediate application of the Cauchy-Schwarz inequality that g is positive provided that v_1 and v_2 are linearly independent. Now we write out the denominator R in terms of the notation of Definition 4.4 $R = 4\mathrm{Re}(1\bar{2},1\bar{2}) - 2(1\bar{2},2\bar{1}) - 2(1\bar{1},2\bar{2})$. Starting with Lemma 4.3 we have that $|\Delta(\mu_1\bar{\mu}_2)| \leqq (\Delta|\mu_1|^2)^{1/2}(\Delta|\mu_2|^2)^{1/2}$ and thus

$$|(1\bar{2},1\bar{2})| \leqq \int|\Delta\mu_1\bar{\mu}_2||\mu_1\bar{\mu}_2|dA \leqq \int(\Delta|\mu_1|^2)^{1/2}\,(\Delta|\mu_2|^2)^{1/2}|\mu_1\bar{\mu}_2|dA$$

and applying the Hölder inequality we obtain $\leqq(\int\Delta|\mu_1|^2|\mu_2|^2dA)^{1/2}$ $(\int\Delta|\mu_2|^2|\mu_1|^2dA)^{1/2}$. The operator Δ is self adjoint and so we finally have that $|(1\bar{2},1\bar{2})| \leqq (1\bar{1},2\bar{2})$. Now for the remaining terms let us write $\mu_1\bar{\mu}_2 = f+ig$, f and g real valued functions. Then certainly $\mathrm{Re}(1\bar{2},1\bar{2}) = \langle\Delta f,f\rangle - \langle\Delta g,g\rangle$ and $(1\bar{2},2\bar{1}) = \langle\Delta f,f\rangle + \langle\Delta g,g\rangle$ in particular $\mathrm{Re}(1\bar{2},1\bar{2}) \leqq (1\bar{2},2\bar{1})$, recall that Δ is positive. Combining the inequalities $|(1\bar{2},1\bar{2})| \leqq (1\bar{1},2\bar{2})$ and $\mathrm{Re}(1\bar{2},1\bar{2}) \leqq (1\bar{2},2\bar{1})$ we have that $R \leqq 0$; the reader can check that equality is not a possibility.

In the conventions of Bochner we have the following additional bounds for the curvatures of the Weil-Petersson metric.

Lemma 4.6.

i) *The holomorphic sectional curvature and Ricci curvature are bounded above by* $\frac{-1}{2\pi(g-1)}$.

ii) *The scalar curvature is bounded above by* $-\dfrac{3(3g-2)}{4\pi}$.

Proof. Bochner shows that the curvature of the holomorphic section spanned by $\mu_\alpha \in \mathscr{B}(\Gamma)$, $\langle \mu_\alpha, \mu_\alpha \rangle = 1$ is $-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = -2\langle \Delta |\mu_\alpha|^2, |\mu_\alpha|^2 \rangle$. To estimate the integral consider the orthogonal expansion $|\mu_\alpha|^2 = \sum_j \psi_j$ of $|\mu_\alpha|^2$ in terms of eigenfunctions of the Laplacian D_0. Indeed we have that $-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} = 4\sum_j \dfrac{\langle \psi_j, \psi_j \rangle}{\lambda_j - 2}$, with the eigenvalues λ_j, $D_0 \psi_j = \lambda_j \psi_j$, non-positive. If ψ_0 is the constant function then certainly the sum is bounded by its first term, $-R_{\alpha\bar{\alpha}\alpha\bar{\alpha}} \leqq -2\langle \psi_0, \psi_0 \rangle$ and equality is ruled out since $|\mu_\alpha|^2$ has zeros. Now ψ_0 is determined by $\int \psi_0 \, dA = \int |\mu_\alpha|^2 dA = \langle \mu_\alpha, \mu_\alpha \rangle = 1$; $\psi_0 = \dfrac{1}{\text{Area}}$ and the estimate follows. The Ricci and scalar curvatures are treated in a similar fashion.

4.4. Recall that associated to a Hermitian metric on a holomorphic vector bundle there is a unique connection compatible with the holomorphic structure, [19]. The Chern forms of the bundle may then be computed from the curvature of this connection. We are interested in QD the bundle of quadratic differentials over $T(\Gamma)$ endowed with the Weil-Petersson metric.

Corollary 4.7. *Let* μ_α *be a unitary basis for* $\mathscr{B}(\Gamma)$ *and* $\mu, \nu \in \mathscr{B}(\Gamma)$ *arbitrary. Then*

$$c_1(\mu, \nu) = \frac{i}{2\pi} \sum_\alpha (\langle \Delta(\mu_\alpha \bar{\mu}_\alpha), \bar{\mu}\nu \rangle + \langle \Delta(\mu_\alpha \bar{\nu}), (\mu_\alpha \bar{\mu}) \rangle)$$

where $c_1(\mu, \nu)$ *is the first Chern form of* QD *evaluated on the holomorphic tangent vectors* $\dfrac{\partial}{\partial t(\mu)}, \dfrac{\partial}{\partial t(\nu)} \in T^{1,0}T(\Gamma)$.

Proof. First recall that for a suitable choice of local holomorphic frame for V, a Hermitian vector bundle, the metric can be written as $h(t) = I + 0(|t|^2)$, t a local coordinate varies in a neighborhood of the origin. Given this normalization the curvature matrix is $\Theta(0) = \bar{\partial}\partial h(0)$ and the first Chern form $c_1(0) = \dfrac{i}{2\pi}$ trace $(\bar{\partial}\partial h(0))$.

For the Weil-Petersson metric Lemma 2.7 provides that a unitary basis $\mu_\alpha \in \mathscr{B}(\Gamma)$ gives the desired local frame of $T^{1,0}T(\Gamma)$. The formula now follows from Theorem 4.2 and the observation that QD is the dual of the tangent bundle and thus $c_1(QD) = -c_1(T^{1,0}T(\Gamma))$ for the dual metric.

5. Characteristic classes of the Teichmüller curve

5.1. Consider the Teichmüller curve $\mathscr{T}_g$, the natural fibre space over the genus g, $g \geqq 2$, Teichmüller space with projection π: $\mathscr{T}_g \to T_g$. The fibre above $p \in T_g$ is a compact Riemann surface, a representative of the conformal equivalence class p. Forming the universal cover of $\mathscr{T}_g$ one obtains the Bers fibre space $\mathscr{B}\mathscr{F}_g$. The mapping class group M_g has an extension by the fundamental group of a genus g surface to a group $\tilde{M}_g$ acting holomorphically and discontinuously on $\mathscr{B}\mathscr{F}_g$. Forming quotients the induced projection $\pi: \mathscr{B}\mathscr{F}_g/\tilde{M}_g \to T_g/M_g$ defines a fibration of V-manifolds, the *universal curve* over the moduli space $\mathscr{M}_g \cong T_g/M_g$.

We shall study the vertical bundle of the fibration $\pi: \mathscr{T}_g \to T_g$. The differential $d\pi: T\mathscr{T}_g \to TT_g (T = T^{1,0})$ has everywhere 1 dimensional kernel, the tangent to the fibre. In particular $\operatorname{Ker} d\pi \subset T\mathscr{T}_g$ defines a line bundle (v) over $\mathscr{T}_g$; informally (v) is the linearization of the fibration. Obviously the restriction of (v) to a fibre is isomorphic to the tangent bundle of the fibre. Consequently the Uniformisation Theorem with parameters provides that the hyperbolic metrics for the fibres piece together to define a smooth Hermitian metric on the line bundle (v), [4]. Now following a general construction for fibre spaces if $c_1(v)$ is the Chern class of (v) then classes are defined on T_g by setting $\tilde{\kappa}_n = \int_{\text{fibre}} c_1(v)^{n+1}$. The classes $\kappa_n = (-1)^{n+1} \tilde{\kappa}_n$ are M_g invariant and have been studied in the work of Mumford, Harris as well as others [5, 9, 10, 13, 14].

Our goal is to calculate the Chern form $c_1(v)$ starting with the hyperbolic metric on (v). We present the result in Theorem 5.5. An immediate consequence is that the line bundle has negative curvature form, a differential geometric analogue of Arakelov's result [5]. Then we compute the forms $\tilde{\kappa}_n$ defined by integration over the fibre. An immediate result is that $\kappa_1 = \frac{1}{2\pi^2}\, \omega_{WP}$, where ω_{WP} is the Weil-Petersson Kähler form. Previously we showed that the cohomology classes of the extensions $\overline{\kappa_1}$, $\overline{\omega_{WP}}$ to the moduli space $\overline{\mathscr{M}}_g$ of stable curves satisfy the relation $[\overline{\kappa_1}] = \frac{1}{2\pi^2}\,[\overline{\omega_{WP}}]$, [26]. By contrast the present result is for the pointwise equality of the characteristic forms.

5.2. The universal cover of $\mathscr{T}_g$ is the Bers fibre space $\mathscr{B}\mathscr{F}_g$, [7]. Our calculations are local and hence it will suffice to consider $\mathscr{B}\mathscr{F}_g$. Bers showed that the Teichmüller space T_g may be embedded as a bounded domain in $\mathbb{C}^n$ and that the fibre space $\mathscr{B}\mathscr{F}_g$ embeds in $\mathbb{C}^n \times \mathbb{C}$ as follows

$$\begin{array}{ccc} \mathscr{B}\mathscr{F}_g & \hookrightarrow & \mathbb{C}^n \times \mathbb{C} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ T_g & \hookrightarrow & \mathbb{C}^n \end{array}$$

for p the projection onto the first factor. In the study of one complex variable variational calculations are generally in the context of maps between domains. We shall follow this approach. Consequently we require the existence of a map between a fixed and the general fibre of $\mathscr{T}_g$. Formally this is a local trivialization of the bundle $\mathscr{T}_g$. One knows at the outset that the fibres of $\pi: \mathscr{T}_g \to T_g$ are not complex isomorphic. Consequently a trivialization of T_g is (at best) given by quasiconformal maps between fibres. We shall now describe a trivialization of this type.

Definition 5.1. Given $\mu \in B$, $\|\mu\|_\infty < 1$, denote by w^μ the unique homeomorphism $w: \mathbb{C} \to \mathbb{C}$ fixing 0,1 and ∞ and satisfying

$$\begin{cases} w_{\bar{z}} = \mu w_z & \text{in } H \\ w_{\bar{z}} = 0 & \text{in } \mathbb{C} - H \end{cases}$$

Denote by $\dot{w}[\mu]$ the derivative $\frac{d}{d\varepsilon}\, w^{\varepsilon\mu}|_{\varepsilon=0}$; $\mu \in B$.

The image $w^{\mu}(H)$ is a quasi-halfplane and the map w^{μ} conjugates Γ into a quasifuchsian group $\Gamma^{\mu}=w^{\mu}\Gamma(w^{\mu})^{-1}$ acting on $w^{\mu}(H)$.

Fix $\mu_{\alpha}\in\mathscr{B}(\Gamma)$ a basis and U a neighborhood of the origin in $\mathbb{C}^n$ such that for $t=(t_1,\ldots,t_n)\in U$ then $\|\mu(t)\|_{\infty}<1$ where $\mu(t)=\sum_j t_j\mu_j$. Consider the map $\Psi: U\times H\to\mathscr{B}\mathscr{F}_g$ defined by $(t,z)\xrightarrow{\Psi}(t,w^{\mu(t)}(z))$.

Theorem 5.2, [7]. *In the above notation, the map $\Psi: U\times H\to\mathscr{B}\mathscr{F}_g$ is a $\Gamma-\Gamma^{\mu}$ equivariant local trivialization of the fibre space $\mathscr{B}\mathscr{F}_g$. $\Psi(t,z)$ is holomorphic in t and quasiconformal in z.*

Throughout the following discussion we shall use z as the coordinate for the 0-fibre of $\mathscr{B}\mathscr{F}_g$ and w as the coordinate for the general fibre. A few remaining preliminary remarks are in order before we proceed with the calculation. We begin with the diagram

$$\begin{array}{ccc} \mathscr{B}\mathscr{F}_g & \hookrightarrow & \mathbb{C}^n\times\mathbb{C} \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle p} \\ T_g & \hookrightarrow & \mathbb{C}^n \end{array}$$

where p is the projection onto the first factor. We observe that $\frac{\partial}{\partial w}$ provides a holomorphic section of the pushforward of the line bundle (v). Let $\left\|\frac{\partial}{\partial w}\right\|$ be the length of $\frac{\partial}{\partial w}$ in the hyperbolic metric, the associated connection 1-form is $\theta=\partial\log\left\|\frac{\partial}{\partial w}\right\|^2$, the associated curvature 2-form is $\Theta=\bar{\partial}\partial\log\left\|\frac{\partial}{\partial w}\right\|^2$ and then the Chern form is $c_1=\frac{1}{2\pi i}\,\partial\bar{\partial}\log\left\|\frac{\partial}{\partial w}\right\|^2$, [19]. In the following paragraphs we shall derive explicit formulas for these quantities.

5.3. The first point is to obtain a suitable expression for the norm $\left\|\frac{\partial}{\partial w}\right\|^2$. The typical fibre of $\mathscr{B}\mathscr{F}_g\subset\mathbb{C}^n\times\mathbb{C}$ is a quasi-halfplane. Perhaps it is easiest to understand the hyperbolic metric for a fibre by considering its uniformisation by the halfplane H. To find this consider f^{μ} the first solution of the Beltrami equation (see Definition 2.1) and w^{μ} the second solution of the Beltrami equation (see Definition 5.1). It is immediate that there is a diagram of maps

$$\begin{array}{ccc} H & \xrightarrow{w^{\mu}} & w^{\mu}(H)\subset\mathbb{C} \\ & {\scriptstyle f^{\mu}}\searrow & \downarrow{\scriptstyle g} \\ & & H \end{array}$$

where g is conformal. Writing $\lambda(w)|dw|$ for the hyperbolic metric in $w^{\mu}(H)$ and $\Lambda(\zeta)|d\zeta|$ for the hyperbolic metric in H then by conformal invariance $\Lambda(g)|g'|=\lambda(w)$ and from the above $f^{\mu}=g\circ w^{\mu}$ thus $f^{\mu}_z=g'(w^{\mu})w^{\mu}_z$. By definition $\left\|\frac{\partial}{\partial w}\right\|=\lambda(w)$ and therefore from the above $\lambda(w^{\mu})=\Lambda(g(w^{\mu}))\,|f^{\mu}_z|/|w^{\mu}_z|$.

Equivalently we have that $\lambda^2(w) = \Lambda^2(g(w^\mu))\,|f_z^\mu|^2(1-|\mu|^2)/|w_z^\mu|^2(1-|\mu|^2)$. Since $f^\mu = g(w^\mu)$ it is immediate that $\Lambda^2(f^\mu)\,|f_z^\mu|^2\,(1-|\mu|^2)\,\frac{i}{2}\,dz \wedge d\bar{z} = (f^\mu)^* dA$, dA the hyperbolic area element. As a matter of notation let us write $[(f^\mu)^* dA]$ for the coefficient of the tensor $(f^\mu)^* dA$. The above considerations are summarized in the following result. This represents the formula that we shall use for $\left\|\frac{\partial}{\partial w}\right\|^2$.

Lemma 5.3. *In the above notation*

$$\left\|\frac{\partial}{\partial w}\right\|^2 = [(f^\mu)^* dA]/|w_z^\mu|^2\,(1-|\mu|^2)\,.$$

Now for $h(t, w)$ a function on the fibre space $\mathscr{B}\mathscr{F}_g \subset \mathbb{C}^n \times \mathbb{C}$ we wish to evaluate the differentials ∂h or $\bar{\partial} h$ in terms of derivatives of the composition $h(t, w(t,z))$ where $w(t,z) = w^{\mu(t)}(z)$ is holomorphic in t and quasiconformal in z. It is immediate that

$$(h(t, w(t,z)))_t = h_t + h_w w_t$$
$$(h(t, w(t,z)))_{\bar{t}} = h_{\bar{t}} + h_{\bar{w}} \bar{w}_{\bar{t}}$$

or equivalently

$$\begin{aligned} h_t &= (h(t, w(t,z)))_t - h_w w_t \\ h_{\bar{t}} &= (h(t, w(t,z)))_{\bar{t}} - h_{\bar{w}} \bar{w}_{\bar{t}}\,. \end{aligned} \tag{5.1}$$

5.4. For the remainder of the discussion we shall use the local coordinates on $\mathscr{B}\mathscr{F}_g$ given by the trivialization Ψ introduced above. Again note that we indicate the coordinate of the 0-fibre by z and the general fibre by w. Before stating the first result recall that for $\mu \in \mathscr{B}(\Gamma)$ we denote by $\frac{\partial}{\partial t(\mu)}$ the associated holomorphic tangent vector of $T_g \subset \mathbb{C}^n$ and also of $\mathscr{B}\mathscr{F}_g \subset \mathbb{C}^{n+1}$ (we are using that the trivialization given by $\Psi(t,z)$ is holomorphic in t).

Lemma 5.4. *In the above notation, the connection* 1*-form* $\theta = \partial \log \left\|\frac{\partial}{\partial w}\right\|^2$ *evaluated at* $(t, w) = (0, z) \in \mathscr{B}\mathscr{F}_g$ *is*

$$\theta\left(\frac{\partial}{\partial z}\right) = \frac{-2}{(z-\bar{z})}$$
$$\theta\left(\frac{\partial}{\partial t(\mu)}\right) = \left(\frac{2}{(z-\bar{z})}\,\dot{w}[\mu] - \dot{w}[\mu]_z\right).$$

Proof. We start with the expression from Lemma 5.3. In particular at $t=0$, $\log\left\|\frac{\partial}{\partial w}\right\|^2 = \log\frac{-4}{(z-\bar{z})^2}$ and immediately $\frac{\partial}{\partial z}\log\frac{-4}{(z-\bar{z})^2} = \frac{-2}{(z-\bar{z})}$, the first result. For the second result again start with $\log\left\|\frac{\partial}{\partial w}\right\|^2 = \log[(f^\mu)^* dA]/|w_z^\mu|^2\,(1-|\mu|^2)$. Lemma 1.7 applied in the present situation provides that $\frac{\partial}{\partial t(\mu)}[(f^\mu)^* dA] = 0$ and thus $\frac{\partial}{\partial t(\mu)}\log\left\|\frac{\partial}{\partial w}\right\|^2 = -\dot{w}[\mu]_z$, at $t=0$ since $w^{\mu(t)}(z)$ is holomorphic in t, $w^0(z) = z$ and $|\mu(t)|^2$ is quadratic in t. Applying the formulas (5.1) the desired result follows.

Before proceeding further with the calculation we wish to consider the second order analogue of the formulas (5.1). The reader will verify that for $h = h(t, w(t,z))$, w holomorphic in t,

$$h_{t\bar{t}}(t,w) = (h(t,w(t,z)))_{t\bar{t}} - h_{w\bar{w}} w_t \bar{w}_t - h_{w\bar{t}} w_t - h_{t\bar{w}} w_t. \tag{5.2}$$

We are now ready to consider the following.

Theorem 5.5. *With the above notation, the curvature 2-form* $\Theta = \bar{\partial}\partial \log \left\| \frac{\partial}{\partial w} \right\|^2$ *evaluated at* $(t,w) = (0,z) \in \mathscr{B}\mathscr{F}_g$ *is*

$$\Theta\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}\right) = \frac{-2}{(z-\bar{z})^2}$$

$$\Theta\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial t(\mu)}\right) = \frac{2}{(z-\bar{z})^2}\, \dot{w}[\mu]$$

$$\Theta\left(\frac{\partial}{\overline{\partial t(\nu)}}, \frac{\partial}{\partial t(\mu)}\right) = \frac{-2}{(z-\bar{z})^2}\, \dot{w}[\mu]\, \overline{\dot{w}[\nu]} + \Delta(\mu\bar{\nu})$$

for $\mu, \nu \in \mathscr{B}(\Gamma)$.

Proof. We start with the first term. Certainly $\Theta\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}\right) = \left(\theta\left(\frac{\partial}{\partial z}\right)\right)_{\bar{z}}$ and the result follows. Similarly $\Theta\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial t(\mu)}\right)$ is $\left(\theta\left(\frac{\partial}{\partial t(\mu)}\right)\right)_{\bar{z}} = \left(\frac{2}{(z-\bar{z})}\, \dot{w}[\mu] - \dot{w}[\mu]_z\right)_{\bar{z}}$ and we recall that $\dot{w}[\mu]_{\bar{z}} = \mu$. Differentiating we obtain $\left(\frac{2}{(z-\bar{z})^2}\, \dot{w}[\mu] + \frac{2}{(z-\bar{z})^2}\, \mu - \mu_z\right)$. Now μ is harmonic and thus $\left(\frac{2}{(z-\bar{z})}\mu - \mu_z\right) = \frac{1}{(z-\bar{z})} K_{-2}\,\mu = 0$; the result follows. We are ready to consider the third term $\Theta\left(\frac{\partial}{\overline{\partial t(\nu)}}, \frac{\partial}{\partial t(\mu)}\right)$; this term is a Hermitian form in $\frac{\partial}{\partial t}$ and thus it will suffice to consider the case $\mu = \nu$. Referring to formula (5.2) we shall consider separately the four terms of the right hand side. Now for the first term $h = \log[(f^\mu)^* dA] / |w_z^\mu|^2 (1 - |\mu|^2)$ and we start by observing $(\log[(f^\mu)^* dA])_{t\bar{t}} = [(f^\mu)^* dA]_{t\bar{t}} / [dA]$ as an immediate application of Lemma 1.7. Now by Theorem 3.3 we have that $[(f^\mu)^* dA]_{t\bar{t}} = (-\mu\bar{\mu} + \Delta(\mu\bar{\mu}))[dA]$. Furthermore $w^{\mu(t)}$ is holomorphic in t and thus $-(\log|w_z^\mu|^2 (1 - |\mu|^2))_{t\bar{t}} = \mu\bar{\mu}$. In summary the first term is simply $\Delta(\mu\bar{\mu})$. The second term for (5.2) has been considered in the preceeding discussion. The result is $-h_{w\bar{w}} w_t \bar{w}_t = \frac{2}{(z-\bar{z})^2}\, \dot{w}[\mu]\, \overline{\dot{w}[\mu]}$. The third and fourth terms are obviously conjugates; it will suffice to consider the fourth. Indeed $h_{t\bar{w}} = \left(\theta\left(\frac{\partial}{\partial t}\right)\right)_{\bar{z}}$ which was also evaluated above; we have that $h_{t\bar{w}} = \frac{2}{(z-\bar{z})^2}\, \dot{w}[\mu]$ and certainly $-h_{t\bar{w}} \bar{w}_t = \frac{-2}{(z-\bar{z})^2}\, \dot{w}[\mu]\, \overline{\dot{w}[\mu]}$. Collecting terms the calculation and proof are now complete.

5.4. Now we wish to consider the formal properties of the Chern form $c_1 = \frac{i}{2\pi}\,\Theta$. In order to better understand the form we introduce a new basis for the tangent space $T\mathscr{B}\mathscr{F}_g$ along a fibre of the projection to T_g. In particular the basis will project to a basis for $T\mathscr{T}_g$ and relative to this basis it will be immediate that c_1 is negative. We start by defining vector fields along the fibres of $\mathscr{B}\mathscr{F}_g$. Our calculations will be pointwise; it will suffice to consider vector fields defined only on the $t=0$ fibre of $\mathscr{B}\mathscr{F}_g$.

Definition 5.6. Given $\mu \in \mathscr{B}(\Gamma)$ define $\tau_\mu = \dot{w}[\mu] \frac{\partial}{\partial z} + \frac{\partial}{\partial t(\mu)}$, a vector field along the $t=0$ fibre of $\mathscr{B}\mathscr{F}_g$.

We wish to show that τ_μ is invariant under the action of the group Γ on $\mathscr{B}\mathscr{F}_g$. Equivalently τ_μ is the lift of a vector field defined along the fibre of $\mathscr{T}_g$. The action of Γ on $\mathscr{B}\mathscr{F}_g$ is holomorphic in particular $\mathscr{B}\mathscr{F}_g/\hat{\Gamma}$ and $\mathscr{T}_g$ are complex isomorphic, [7]. Briefly in order to describe the action, consider $\mu \in \mathscr{B}(\Gamma)$, $\|\mu\|_\infty < 1$ and let w be given by Definition 5.1. Now by the Γ invariance of μ, w^μ and $w^\mu(\gamma)$, $\gamma \in \Gamma$ satisfy the same differential equation (but with different normalizations); in particular $\gamma^\mu \in PSL(2; \mathbb{C})$ exists such that $w^\mu(\gamma) = \gamma^\mu w^\mu$. Now by definition given a point $(t, w) \in \mathscr{B}\mathscr{F}_g \subset \mathbb{C}^{n+1}$ and $\gamma \in \Gamma$ its action is defined by $\hat{\gamma}(t, w^\mu) = (t, \gamma^\mu(w^\mu))$.

Lemma 5.7. *In the above notation, the vector field* τ_μ is $\hat{\Gamma}$ *invariant.*

Proof. We recall the description of the trivialization of $\mathscr{B}\mathscr{F}_g$: $\Psi: U \times H \to \mathscr{B}\mathscr{F}_g$ by the rule $(t, z) \to (t, w^{\mu(t)}(z))$. Now Γ acts on $U \times H$ by: $(t, z) \xrightarrow{\gamma} (t, \gamma(z))$, $\gamma \in \Gamma$ and as $\hat{\Gamma}$ on $\mathscr{B}\mathscr{F}_g$ by: $(t, w) \xrightarrow{\hat{\gamma}} (t, \gamma^{\mu(t)}(w))$, $\gamma^\mu \in \Gamma^\mu$. As already mentioned the trivialization is $\Gamma - \hat{\Gamma}$ equivariant: $\Psi \circ \gamma = \hat{\gamma} \circ \Psi$ for all $\gamma \in \Gamma$. In particular we have that $\Psi_* \circ \gamma_* = \hat{\gamma}_* \circ \Psi_*$ for the action on tangent vectors. The proof is now the consequence of two observations: $\tau_\mu = \Psi_*\left(\frac{\partial}{\partial t(\mu)}\right)$, and for the action of Γ on $U \times H$ $\gamma_*\left(\frac{\partial}{\partial t(\mu)}\right) = \frac{\partial}{\partial t(\mu)}$, $\gamma \in \Gamma$. In brief we have that $\tau_\mu = \Psi_* \circ \gamma_*\left(\frac{\partial}{\partial t(\mu)}\right) = \hat{\gamma}_* \circ \Psi_*\left(\frac{\partial}{\partial t(\mu)}\right) = \hat{\gamma}_*(\tau_\mu)$, the desired conclusion.

We note in passing that the assignment $\mu \to \tau_\mu$, $\mu \in \mathscr{B}(\Gamma)$ defines a canonical lifting of TT_g to $T\mathscr{T}_g$. The results of Theorem 5.5 can now be reformulated as

$$\Theta\left(\frac{\partial}{\partial \bar{z}}, \frac{\partial}{\partial z}\right) = \frac{-2}{(z-\bar{z})^2}$$
$$\Theta\left(\frac{\partial}{\partial \bar{z}}, \tau_\mu\right) = 0 \tag{5.3}$$
$$\Theta(\overline{\tau_\nu}, \tau_\mu) = \Delta(\mu\bar{\nu})$$

where $\Delta = -2(D_0 - 2)^{-1}$. Recall that Δ is an integral operator with positive kernel, $\Delta(\mu\bar{\mu})$ is everywhere positive. The Chern form of (v) is given as $c_1(v) = \frac{i}{2\pi}\Theta$ and the following is an immediate consequence.

Lemma 5.8. *The vertical bundle of the fibration* $\pi: \mathscr{T}_g \to T_g$ *is a negative line bundle.*

It was known previously that the dual line bundle, the *relative dualizing sheaf*, is numerically effective, on the compactification of the universal curve, [14].

5.5. Finally we wish to consider the characteristic classes $\tilde{\kappa}_n = \int_{\text{fibre}} c_1^{n+1}$. The characteristic forms $\tilde{\kappa}_n$ are of type (n, n) and invariant under the action of the mapping class group M_g. We shall derive the formula for $\tilde{\kappa}_n$ starting with the above expression for $c_1(v)$. As an immediate application we have that $\tilde{\kappa}_1 = \frac{1}{2\pi^2}\,\omega_{WP}$ where ω_{WP} is the Kähler form of the Weil-Petersson metric. The exterior power c_1^{n+1} is a $(n+1, n+1)$ form. We wish to emphasize the type decomposition; accordingly we shall evaluate forms on $T^{1,0}\mathscr{T}_g$.

Lemma 5.9. *In the above notation, given $\mu_j \in \mathscr{B}(\Gamma)$, $j = 1, \ldots, n$ set $\tau_{\mu_j} = \tau_j$ and then*

$$\Theta^{n+1}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}, \tau_1, \bar{\tau}_1, \ldots, \tau_n, \bar{\tau}_n\right)$$
$$= (n+1)!\; \Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \sum_{\sigma \in \mathscr{S}_n} \varepsilon(\sigma)\, \Theta(\tau_1, \overline{\tau_{\sigma(1)}}) \ldots \Theta(\tau_n, \overline{\tau_{\sigma(n)}})$$

where $\mathscr{S}_n$ is the permutation group on n letters and ε is the sign character.

Proof. The exterior power Θ^{n+1} is evaluated by summing over all permutations. Now the vanishing $\Theta\left(\frac{\partial}{\partial \bar{z}}, \tau_j\right) = 0$ (see (5.3)) immediately reduces the expression to $\Theta^{n+1} = (n+1)\,\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \otimes \Theta^n$ where $\otimes$ is the symmetric tensor. Since Θ is of type (1,1) the quantities $\Theta(\tau_j, \tau_k)$, $\overline{\Theta(\tau_j, \tau_k)}$ will also vanish. We are left to consider

$$\Theta^{n+1} = (n+1)\;\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \sum_{(\sigma, \rho) \in \mathscr{S}_n \times \mathscr{S}_n} \varepsilon(\sigma\rho)\, \Theta(\tau_{\sigma(1)}, \overline{\tau_{\rho(1)}}) \ldots \Theta(\tau_{\sigma(n)}, \overline{\tau_{\rho(n)}}).$$

Transferring the action of permutations to the second index we find the desired formula

$$\Theta^{n+1} = (n+1)!\; \Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) \sum_{\sigma \in \mathscr{S}_n} \varepsilon(\sigma)\, \Theta(\tau_1, \overline{\tau_{\sigma(1)}}) \ldots \Theta(\tau_n, \overline{\tau_{\sigma(n)}}).$$

In order to relate our results to those of other authors we now shift our attention to the dual of the vertical line bundle, often referred to as the *relative dualizing sheaf.* Specifically we consider the classes

$$\kappa_n = \left(\frac{1}{2\pi i}\right)^{n+1} \int_{\text{fibre}} \Theta^{n+1} \tag{5.4}$$

By polarization it will suffice to obtain a formula for $\kappa_n(\mu_1, \overline{\mu_1}, \ldots, \mu_n, \overline{\mu_n})$ where we have abbreviated $\frac{\partial}{\partial t(\mu)}$, $\mu \in \mathscr{B}$, by simply writing μ.

Lemma 5.10. *In the above notation*

$$\kappa_n(\mu_1, \overline{\mu_1}, \ldots, \mu_n, \overline{\mu_n})$$
$$= (n+1)!\left(\frac{i}{2\pi}\right)^{n+1} (-i) \int_{H/\Gamma} \sum_{\sigma \in \mathscr{S}_n} \varepsilon(\sigma)\, \Delta(\mu_1 \overline{\mu_{\sigma(1)}}) \ldots \Delta(\mu_n \overline{\mu_{\sigma(n)}})\, dA.$$

Proof. To start the considerations recall the formulas (5.3) and Lemma 5.9. And note that if dA is the hyperbolic area element on H, an exterior 2-form, then $dA=\frac{i}{2}\frac{dz\wedge d\bar{z}}{(\operatorname{Im} z)^2}$ and thus

$$\Theta\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)=\frac{2}{(z-\bar{z})^2}=\frac{-1}{2(\operatorname{Im} z)^2}=\mathrm{i}dA\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right).$$

Combining these remarks with the definition (5.4) the formula follows.

Corollary 5.11. *In the above notation,* $\kappa_1=\omega_{WP}$, *where* ω_{WP} *is the Weil-Petersson Kähler form.*

Proof. Starting from the above result we find that $\kappa_1(\mu,\bar{\nu})=\frac{i}{2\pi^2}\int \Delta(\mu\bar{\nu})\,dA$, μ, $\nu\in\mathscr{B}(\Gamma)$. Now for an arbitrary function $f\in\mathscr{H}_0$ we have that $\int\limits_{H/\Gamma}\Delta f dA=\int\limits_{H/\Gamma} f dA$ and the result follows since $\omega=i\int\limits_{H/\Gamma}\mu\bar{\nu}dA$ is the Kähler form given on $T^{1,0}T_g$.

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