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Estimating hyperbolic Green's functions for degenerating surfaces

WP curvature, grad l_α, Hess l_α

WP curvature

D Laplace Beltrami operator for hyperbolic metric

D self adjoint on L^2 spectrum ≤ 0

curvature tensor $R_{\alpha\bar\beta\gamma\bar\delta} = \dfrac{\partial^2 g_{\alpha\bar\beta}}{\partial t_\gamma \partial t_\delta}$

(Tromba, W) for μ_* harmonic Beltrami diffils

$$R_{\alpha\bar\beta\gamma\bar\delta} = -2\int \mu_\alpha \bar\mu_\beta (D-2)^{-1}(\mu_\gamma \bar\mu_\delta)\,dA - 2\int \mu_\alpha \bar\mu_\delta (D-2)^{-1}\mu_\gamma \bar\mu_\beta\,dA$$

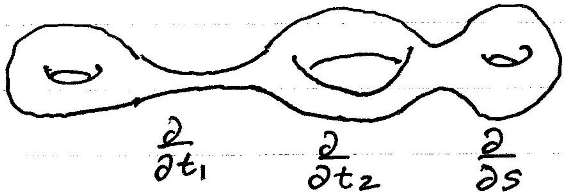
sectional curvatures are negative

negativity follows from Hölder's inequality

examples provide that curvature is neither pinched above/below

Theorem (Zeno Huang) $-c_1/\text{systole}(R) \leq$ WP sec curv $\leq -c_2\,\text{systole}(R)$

cases (may not be finalized)



$\frac{\partial}{\partial t_j}$ tgt opening j^{th} node

$\frac{\partial}{\partial s}$ tgt variation of the THICK region

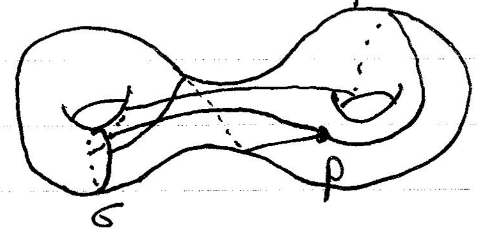
section	curvature
$\frac{\partial}{\partial t_j}, i\frac{\partial}{\partial t_j}$	$-c/\ell_j$
$\frac{\partial}{\partial t_j}, (i)\frac{\partial}{\partial t_k}, j\neq k,$	$-c'\min(\ell_j, \ell_k)$
$(i)\frac{\partial}{\partial t_j}, \frac{\partial}{\partial s}$	$-c''\ell_j$
$\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}$ (variations of a common THICK component)	negative pinched
(variations of distinct THICK components)	$\ell_{\text{separating geodesics}}$

WP gradient of very short geodesic lengths

(W) $\langle \operatorname{grad}\ell_j, \operatorname{grad}\ell_j \rangle = \pi\ell_j + O(\ell_j^3)$

$j\neq k$, $\langle \operatorname{grad}\ell_j, \operatorname{grad}\ell_k \rangle = O(\ell_j^3 + \ell_k^3)$

WP Hessian of geodesic lengths



p has an infinite number of foot points / connecting geodesics on σ

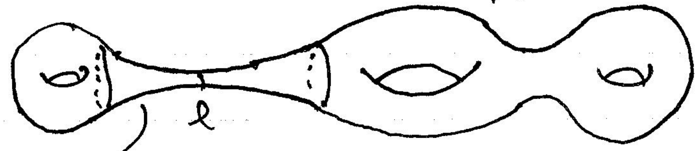
set $\mathbb{P}_\sigma(p) = \sum_{\gamma \text{ geodesic foot paths}} e^{-2\ell(\gamma)}$

(W)

$$\frac{16}{3\pi}\langle\ ,\ \mathbb{P}_\sigma\rangle_{WP} \leq \partial\bar\partial\ell_\sigma \leq \operatorname{Hess}\ell_\sigma \leq 3\partial\bar\partial\ell_\sigma \leq \frac{256}{3\pi}\langle\ ,\ \mathbb{P}_\sigma\rangle_{WP}$$

$\frac{\partial}{\partial t_j}, \frac{\partial}{\partial s}$ cases for $\langle\ ,\ \mathbb{P}_\sigma\rangle_{WP}$ for ℓ_σ very short

Quadratic diffils for very pinched surfaces



collar conformal to $(1+\varepsilon)\, e^{-2\pi^2/\ell} \leq |z| \leq (1-\varepsilon)$

WP dual of $\frac{\partial}{\partial s}$ is a qd quasi constant on THICK and bounded by $c\left(|z| + e^{-2\pi^2/\ell}/|z|\right)$ in collar

small

WP dual of $\frac{\partial}{\partial t}$ is $c\ell^3 e^{2\pi^2/\ell}\left(\frac{dz}{z}\right)^2 + \text{`}O(1)\text{'}$

Estimating Green's functions

Mean value property

consider u satisfying $Du = \lambda u$

in universal cover $u(p) = c(\lambda, r) \int_{B(p;r)} u \, dA$

On surface ?

$$B(p;r) \subset \text{universal cover}$$

$$\text{proj} \downarrow$$

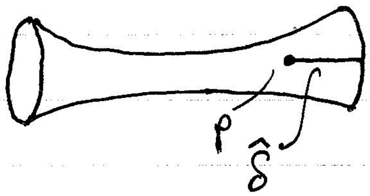
$$\text{surface}$$

proj overlaps images

Margulis lemma (non torsion) overlapping covering transformations are contained in a cyclic group

ρ injectivity radius at image of p then $\#$ overlaps $\leq c/\rho$

enhanced collar/cusp lemma



for a constant c

$$\rho(p)\, e^{\hat{\delta}(p)} \geq c$$

enhanced mean value estimate

$$u \geq 0 \qquad u(p) \leq c\, e^{\hat{\delta}(p)} \int_{B(p;r) \subset \text{universal cover}} u\, dA$$

Green's function

universal cover

$$(D - s(s-1))\, Q_s(d(p,q)) = \text{Dirac delta}(p,q)$$

$$s > 1, \quad 0 < -Q_s \leq c\, e^{-s\, d(p,q)}$$

surface = universal cover$/\Gamma$ $\qquad G_s(p,q) = \sum_{A \in \Gamma} Q_s(d(\overset{A}{\check{p}},q))$

convergence of sum for $r <$ inj rad(q)

$$-G_s(p,q) \leq c(s,r) \underbrace{\sum_{A \in \Gamma} \int_{B(Aq;r)}}_{\text{disjoint balls in universal cover}} \underbrace{-Q_s(d(p,z))}_{e^{-s\, d(p,z)}} \underbrace{dA}_{e^{d(p,z)}\, d(d(p,z))\, d\theta}$$

bounded by

convergent for $s > 1$ (actually sharp)

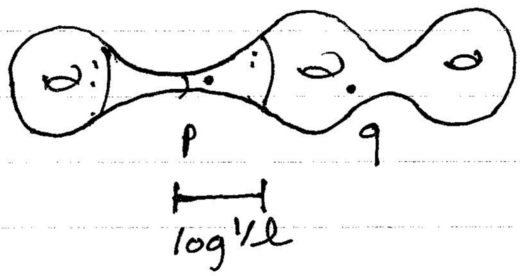
WP curvature for very pinched surfaces

above descriptions of $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$

$-(D-2)^{-1}$ applied to "$\frac{\partial}{\partial t}\overline{\frac{\partial}{\partial t}}$" "$\frac{\partial}{\partial t}\overline{\frac{\partial}{\partial s}}$"

Beltrami diff'l for $\frac{\partial}{\partial t}$ concentrates near the core geodesic

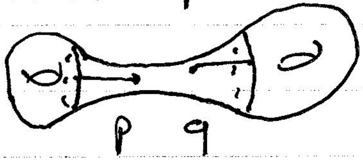
"$\frac{\partial}{\partial t}\overline{\frac{\partial}{\partial s}}$"



$d(p,q) \geq \log 1/\ell$ so in universal cover $\bigcup_A B(Aq;r) \subset$ exterior $B(p;\log 1/\ell - r)$

$-G_s(p,q)$ is bounded by $\displaystyle\int_{\log 1/\ell} e^{(1-s)\delta}\,d\delta \Big|_{s=2} = O(\ell)$

"$\frac{\partial}{\partial t}\overline{\frac{\partial}{\partial t}}$" most important p close to q



$-G_s(p,q) = \sum_{\substack{\text{connecting geodesics}\\ \text{in collar}}} + \sum_{\text{connecting geodesics leave collar}}$

estimate by p and q on a common latitude (I dim'l)

$e^{\hat{\delta}(q)}$ enhanced mean value

bounded in terms of

$\displaystyle\int_{\hat{\delta}(p)+\hat{\delta}(q)} e^{(1-s)\delta}\,d\delta = O(e^{(2-s)\hat{\delta}(q)+(1-s)\hat{\delta}(p)})$

balls contained exterior $B(p;\hat{\delta}(p)+\hat{\delta}(q))$