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The CAT(0) geometry of Teichmüller space

Background

F a reference compact topological oriented surface
a <u>marked</u> <u>Riemann</u> <u>surface</u> (R,f) $\underset{\sim\text{homeo}}{f: F \to R}$

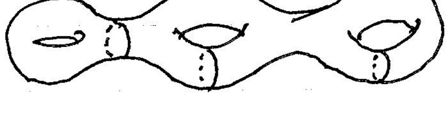
marked surfaces (R,f), (R',f') equivalent provided $f' \circ f^{-1}$ is homotopic to a conformal homeomorphism

the set of equivalence classes is Teichmüller space $\mathcal{T} \approx \mathbb{R}^{6g-6}$
a complex manifold
at $\{(R,f)\}$ holomorphic cotangent space $H^0(\mathcal{O}(dz^{\otimes 2})) \cong$
$Q(R)$: space of holomorphic quadratic differentials

Weil-Petersson cometric: R has hyperbolic metric dh^2
$\varphi, \psi \in Q(R)$ $\qquad \langle \varphi, \psi \rangle = \int_R \varphi \overline{\psi} (dh^2)^{-1}$

WP metric $\longleftrightarrow$ hyperbolic geometry of surfaces
Kähler, <u>not complete</u>, negative sectional curvature
$\sup_{\mathcal{T}}$ sec curv $= 0$ ($\dim_{\mathbb{C}} > 1$) & $\inf_{\mathcal{T}}$ sec curv $= -\infty$
$MCG = Homeo^+(F)/Homeo_0(F)$ acts as biholomorphic isometries
action $[h]: \{(R,f)\} \longrightarrow \{(R, f \circ h^{-1})\}$
$\mathcal{T}/MCG$ is the moduli space of Riemann surfaces

Topological considerations $\quad C(F)$ complex of curves: a k-simplex is a set of $k+1$ free homotopy classes of nontrivial, nonperipheral, mutually disjoint simple closed curves of F

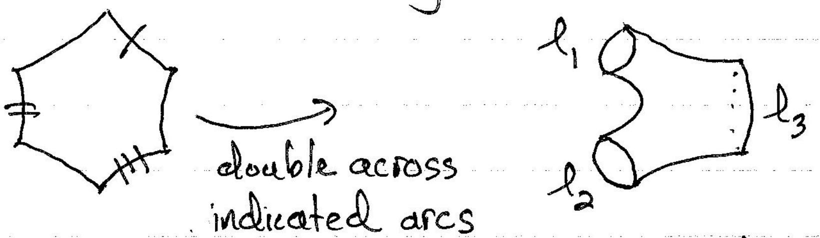


a 2-simplex

Fenchel-Nielsen coordinates

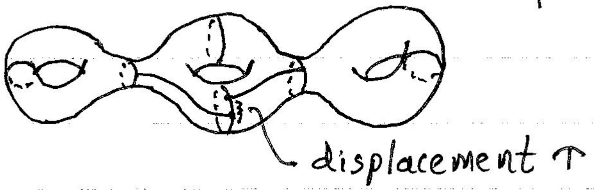
coordinates for $\mathcal{T}$ from gluing-parameters for constructing surfaces from right hyperbolic hexagons

for a pants decomposition $\mathcal{P}$ (a $3g-4$ simplex)
there is a corresponding construction of R



lengths free parameters in $\mathbb{R}^+$

combine boundaries of equal length



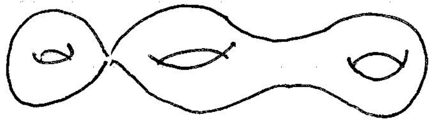
$\mathcal{P}$ realized by geodesics

parameters for construction lengths $\ell_1, \ldots, \ell_{3g-3}$ and twist parameters $\tau_1, \ldots, \tau_{3g-3}$ (measure displacement between foot points and continue the lifting)

$\prod_j (\ell_j, \tau_j) : \mathcal{T} \longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$ real analytic equivalence

WP Kähler form $\omega = \frac{1}{2} \sum_j d\tau_j \wedge d\ell_j$

Bordification after Bailey-Borel <u>augmented</u> <u>Teichmüller</u> <u>space</u> $\overline{\mathcal{T}}$
frontier sets are added to $\mathcal{T}$ by allowing ℓ_j's to assume the value zero with $\theta_j = 2\pi\tau_j/\ell_j$ undefined

$l_j = 0$ parameterizes a pair of cusps 

in particular for a simplex $\sigma \subset \mathcal{P}$ the σ-null stratum
$\mathcal{S}(\sigma) = \{ R \text{ with cusp pairs} \mid l_\alpha(R) = 0 \Leftrightarrow \alpha \in \sigma \}$

introduce $\mathcal{F}_{\mathcal{P}} = \bigcup_{\sigma \subset \mathcal{P}} \mathcal{S}(\sigma)$ frontier spaces subordinate to $\mathcal{P}$

neighborhood basis for $\mathcal{T} \cup \mathcal{F}_{\mathcal{P}}$ prescribed by the requirement

$$((l_\beta, \theta_\beta), l_\alpha) : \mathcal{T} \cup \mathcal{S}(\sigma) \longrightarrow \prod_{\beta \notin \sigma} (\mathbb{R}_+ \times \mathbb{R}) \times \prod_{\alpha \in \sigma} (\mathbb{R}_{\geq 0})$$

is continuous map
for a simplex $\sigma \subset \mathcal{P}, \mathcal{P}'$ the neighborhood systems are equivalent

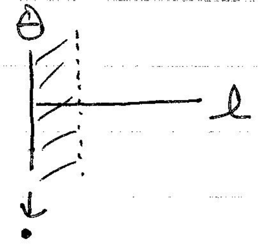
structural property: $\mathcal{S}(\sigma)$ are products of lower dimensional Teichmüller spaces; the limit of the tangential component of the WP metric of $\mathcal{T}$ to the stratum is simply the WP metric of $\mathcal{S}(\sigma)$ (Masur)

Def'n augmented Teichmüller space $\overline{\mathcal{T}} = \mathcal{T} \cup_{\sigma \in C(F)} \mathcal{S}(\sigma)$
a stratified space

convention for $C(F)$: the empty set is a -1-simplex
introduce labeling function $\Lambda : \overline{\mathcal{T}} \to C(F)$
Λ gives the simplex of curves α with $l_\alpha = 0$

Properties of $\overline{\mathcal{T}}$

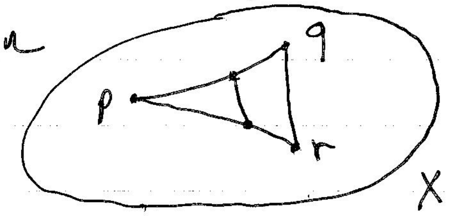
- $\overline{\mathcal{T}}$ is not locally compact

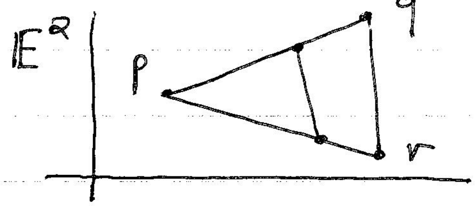


nbhd of $\ell = 0$ is a vertical strip

- (Abikoff, Bers) $\overline{\mathcal{T}}/MCG$ is topologically the Deligne-Mumford stable curve compactification of the moduli space
- (Masur) $\overline{\mathcal{T}}$ is the WP completion of $\mathcal{T}$

Defin CAT(0) metric space: unique geodesic space with triangle comparison

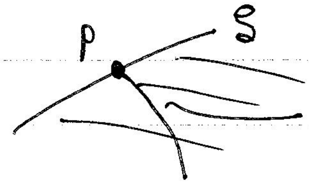




for corresponding edge-lengths chord length bounded by in $\mathbb{E}^2$

- (Farb, Daskalopoulos-Wentworth, W, Yamada*) $\overline{\mathcal{T}}$ is CAT(0)
  - (DW, W, Y*) non refraction   do not have

for $p, q \in \overline{\mathcal{T}}$ and $\widehat{pq}$ unique connecting length-minimizing curve then $\widehat{pq}$ contained in closure of $\Lambda(p) \cap \Lambda(q)$ with $\widehat{pq} \setminus \{p, q\}$ satisfying WP geodesic equation on $\Lambda(p) \cap \Lambda(q)$.
  - use negative sectional curvature to establish triangle comparison
- (W) Strata structure is intrinsic to metric space $(\overline{\mathcal{T}}, d_{WP})$
  Stratum containing a point is the union of all open geodesics containing the point



cannot be prolonged at p

- (W) Flat subspaces are classified ; each a product of geodesics from distinct component Teichmüller spaces ; maximal dimension flats occur for strata a product of 1-dim'l Teichmüller spaces

- (DW, W) The pseudo Anosov elements of MCG have unique axes

Geodesic length functions α a free homotopy class on F

<u>geodesic length</u> $\ell_\alpha(R)$ length of geodesic on R freely homotopic to $f(\alpha)$

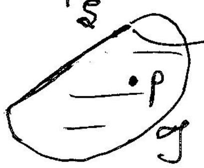
- ℓ_α are strictly convex along WP geodesics & plurisubharmonic

- $|d\ell_\alpha d\ell_\beta| \leq \ell_\alpha \operatorname{Hess} \ell_\beta + \ell_\beta \operatorname{Hess} \ell_\alpha$

- for 'constants' c_1, c_2

$$c_1(\text{systole}(R))\, \ell_\alpha \langle\ ,\ \rangle_{WP} \leq \operatorname{Hess} \ell_\alpha \leq c_2(\text{systole}(R))\, \ell_\alpha \langle\ ,\ \rangle_{WP}$$

- 'geodesic length of σ is a Busemann function for $\mathcal{S}(\sigma)$'



$$\mathcal{S}(\sigma) = \{\ell_\alpha = \ell_\beta = \ell_\gamma = 0\}$$

$$\ell = \ell_\alpha + \ell_\beta + \ell_\gamma$$

$$d(p, \mathcal{S}) \leq (2\pi\ell)^{1/2}$$

$$d(p, \mathcal{S}) = (2\pi\ell)^{1/2} + O(\ell^2) \quad \text{for } \ell \text{ small}$$

$\mathcal{S} = \{\ell = 0\}$ is geodesically convex since ℓ is convex

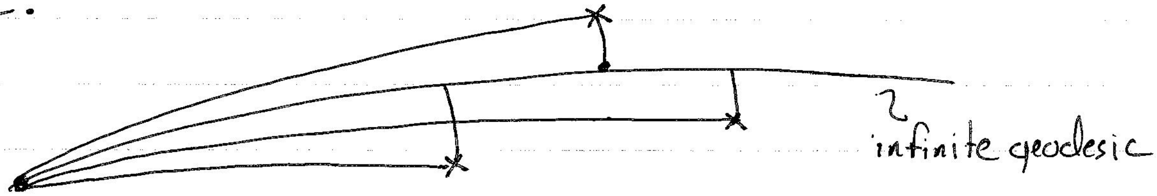
Brock's approximation

Bers: C_g such that each surface has a pants decomposition of total length at most C_g

So each point of $\overline{\mathcal{T}}$ is distance at most $(2\pi C_g)^{1/2}$ to a maximally noded surface

Prop. (B) Tangents to WP geodesics to maximally noded surfaces are dense.

consider



Prop. (W) $\overline{\mathcal{T}}$ is the closed convex hull of the maximally noded surfaces

Theorem (Masur-Wolf, W) The MCG is the full group of WP isometries

- A WP isometry I preserves the intrinsic strata structure and hence partial order on $C(F)$.
- From Ivanov, Korkmaz and Luo a simplicial automorphism of $C(F)$ is induced by a mapping class.
- The maximal simplices of $C(F)$ correspond to unique points in $\overline{\mathcal{T}}$.
- I agrees with a mapping class on the maximally noded surfaces.
- Conclusion follows from the closed convex hull property.