Collective Dynamics: consensus, emergence of patterns and social hydrodynamics

Leçons Jacques-Louis Lions 2016

Lecture #1. Mathematical models for collective dynamics

Eitan Tadmor

Center for Scientific Computation and Mathematical Modeling (CSCAMM)
Department of Mathematics, Institute for Physical Science & Technology
University of Maryland

Laboratoire Jacques-Louis Lions

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Mathematical aspects of collective dynamics...

- A basic paradigm for collective dynamics — environmental averaging

1. Examples of mathematical models for collective dynamics
   - Krause-Hegselmann model for opinion dynamics
   - Vicsek model for flocking; phase transition
   - Cucker-Smale model for flocking — near and far from equilibrium

Outline

- A basic paradigm for collective dynamics — environmental averaging

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A basic paradigm in collective dynamics

- A general class of $N \gg 1$ agents identified w/ "traits" $\{\mathbf{p}_i(t)\}_{i=1}^N$:

- Environmental Averaging: $$\mathbf{p}_i(t+\Delta t) = \sum_j a_{ij}\mathbf{p}_j(t) \qquad \sum_j a_{ij} = 1$$

Alignment w/frequency $\alpha \sim \frac{1}{\Delta t}$ $$\frac{\mathbf{p}_i(t+\Delta t) - \mathbf{p}_i(t)}{\Delta t} = \alpha\Big(\sum_{j\neq i} a_{ij}(\mathbf{p}_j - \mathbf{p}_i)\Big)$$

- Nonlinear influence function ϕ: $$a_{ij} = \frac{1}{\deg_i}\phi(\mathbf{p}_i, \mathbf{p}_j) \geqslant 0$$

- $$\deg_i := \sum_j \phi(\mathbf{p}_i, \mathbf{p}_j)$$ — the degree of influence on agent$_i$

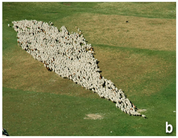
- Observations: considerably different models in different contexts
- Despite the variety – similar fundamental features in collective dynamics; notably ...
- A large number of agents, $N \gg 1$ $\rightsquigarrow$ emergence of large scale coherent structures: swarms, colonies, parties, clusters, consensus, flocks, ...

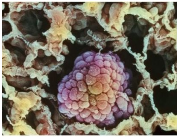
Examples of collective dynamics

- Examples of "living agents"— averaging of orientations, velocities,...

⊙ flocks of birds; schools of fish, colonies of ants, locust, bacteria, ...



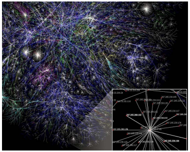




- Examples of "thinking agents" (social dynamics) — averaging of opinions and other traits, ...

⊙ Human crowd; traffic jam; "opinion dynamics", neural networks



- Examples of "non-living agents" — averaging of "positions"

⊙ Robots – the rendezvous problem, UAVs, peridynamics, nematic fluids,...

Outline

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Example#1: Krause model[1] for opinion dynamics

- State space — vectors of "opinions" $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\mathbf{x}_i(t)\}_{i=1}^N$
- Krause-Hegselmann model (1997) — interaction through **local** averaging:

$$\mathbf{x}_i(t+\Delta t) = \frac{1}{N_i}\sum_{|\mathbf{x}_i-\mathbf{x}_j|\leqslant R} \mathbf{x}_j(t), \qquad N_i := \#\{\mathbf{x}_j : |\mathbf{x}_j - \mathbf{x}_i| < R\}$$

- "Environmental averaging": $\boxed{\mathbf{x}_i(t+\Delta t) = \sum_j a_{ij}\mathbf{x}_j(t)}$ $\quad \sum_j a_{ij} = 1$
- Act on difference of opinions: $\boxed{a_{ij} = \dfrac{\phi(|\mathbf{x}_i - \mathbf{x}_j|)}{\deg_i}}$ $\quad \phi(r) = \mathbb{1}_{[0,R)}(r)$
- $\boxed{\deg_i = \sum_k \phi(|\mathbf{x}_i - \mathbf{x}_k|) \rightsquigarrow N_i}$ — the degree of influence on agent$_i$
- A local model: agent$_i$ influenced by N_i "nearest" (?) neighbors

[1] U. Krause (1997,2000), R. Hegselmann & U.Krause: Opinion dynamics and bounded confidence models, analysis and simulation (2002,2004)

Environmental averaging: $\mathbf{x}_i(t+\Delta t) = \sum_j a_{ij}\mathbf{x}_j(t)$

- Alignment — models environmental averaging ($\sum_j a_{ij} = 1$):

$$\frac{\mathbf{x}_i(t+\Delta t) - \mathbf{x}_i(t)}{\Delta t} \quad \frac{d}{dt}\mathbf{x}_i(t) = \alpha\Big(\sum_j a_{ij}\mathbf{x}_j(t) - \mathbf{x}_i(t)\Big)$$

Frequency $\alpha \sim \dfrac{1}{\Delta t}$

$$a_{ij}(\mathbf{x}(t)) = \frac{1}{\deg_i}\phi(|\mathbf{x}_i - \mathbf{x}_j|), \quad \deg_i = \text{degree of influence on agent}_i$$

- Local models involve nearby neighbors: $\deg_i := \sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|) \rightsquigarrow N_i$
  involve 'neighboring' agents in finite $\mathrm{Supp}\{\phi\}$[2]
- Global models involve all agents: $\deg_i = \sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|) \rightsquigarrow N$

  All agents interact within a possibly infinite $\mathrm{Supp}\{\phi\}$

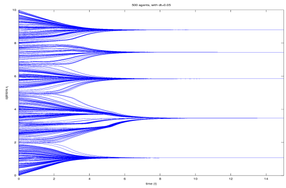
$\star$ wlog $\phi \leqslant 1$ — set $\phi_{ii} := N - \sum_{j\neq i}\phi_{ij} \in [0,N]$ s.t. $\sum_j a_{ij} = \frac{1}{N}\sum_j \phi_{ij} = 1$

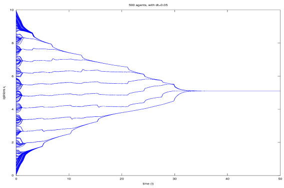
$\rightsquigarrow$ $a_{ij} \in [0,1]$ but may be "far from equilibrium": $\{a_{\cdot j}\} \notin \mathcal{U}[0,1]$

[2] Compared with usual Laplacian — $N_i \equiv 4$

Large-time opinion dynamics: parties and consensus

- d $= 1$ — 100 uniformly distributed opinions on $[0, 10]$





- d $= 2$ — 100 opinion lead to the formation of $K = 17$ parties ...

- How to measure difference of opinions — $|\mathbf{x}_i - \mathbf{x}_j|$ in d $\geqslant 2$?

Example #2: Sensor-based networks

- State vector — vectors of "positions'" $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\mathbf{x}_i(t)\}_{i=1}^N$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \frac{\alpha}{\deg_i}\sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|)\,(\mathbf{x}_j(t) - \mathbf{x}_i(t)) \equiv -\frac{\alpha}{\deg_i}\nabla_{\mathbf{x}}\mathcal{K}(\mathbf{x}_i(t))$$

- Gradient descent $\quad \mathcal{K}(\mathbf{x}) = \frac{1}{2}\sum_{\alpha,\beta} k(|\mathbf{x}_\alpha - \mathbf{x}_\beta|), \quad k(r) := \int_0^r s\phi(s)ds$

$$\nabla_{\mathbf{x}}\mathcal{K}(\mathbf{x})_{|\mathbf{x}=\mathbf{x}_i} = \sum_j k'(|\mathbf{x}_i - \mathbf{x}_j|)\frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|} = -\phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{x}_j - \mathbf{x}_i)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{K}(\mathbf{x}(t)) = \sum_i \langle \dot{\mathbf{x}}_i, \nabla_{\mathbf{x}}\mathcal{K}(\mathbf{x}_i(t))\rangle = -\frac{1}{\alpha}\sum_i \langle \dot{\mathbf{x}}_i, \deg_i \dot{\mathbf{x}}_i\rangle \leqslant 0$$

- Robotic agents — the "rendezvous problem": $\mathbf{x}_i(t) - \mathbf{x}_j(t) \xrightarrow{t\to\infty} 0?$

Example #3: Vicsek model[3] — alignment of orientations

- Fix a speed s. Averaging of orientations $\{\mathbf{p}_i(t)\}_{i=1}^N \rightsquigarrow \{\boldsymbol{\omega}_i(t)\}_{i=1}^N \in \mathbb{S}^{d-1}$.

$$\boldsymbol{\omega}_i(t+\Delta t) = \Big(s\sum_j a_{ij}\,\boldsymbol{\omega}_j(t) + \text{noise}\Big) \times \frac{1}{\big|s\sum_j a_{ij}\,\boldsymbol{\omega}_j(t) + \text{noise}\big|}$$

- 2D additive Noise$=$ uniform in angle in $[-\tau, \tau]$
- $a_{ij} = \dfrac{\phi(|\mathbf{x}_i - \mathbf{x}_j|)}{\deg_i} \rightsquigarrow \mathbf{v}_i(t) := \dfrac{s}{\deg_i} \displaystyle\sum_{j:|\mathbf{x}_i - \mathbf{x}_j|<R} \phi(|\mathbf{x}_i - \mathbf{x}_j|)\boldsymbol{\omega}_j(t)^{2b}$
- A second-order model $a_{ij}(\mathbf{x}(t))$: $\quad \mathbf{x}_i(t+\Delta t) = \mathbf{x}_i(t) + \Delta t \cdot \mathbf{v}_i(t)$

Vicsek model and alignment—Degond & Motsch (2008)

$$\boldsymbol{\omega}_i(t+\Delta t) - \boldsymbol{\omega}_i(t) d\boldsymbol{\omega}_i = \Delta t\,\mathbb{P}_i\left(\frac{\mathbf{v}_i(t)}{|\mathbf{v}_i(t)|} - \boldsymbol{\omega}_i(t)\right)\alpha\mathbb{P}_i\left(\frac{\mathbf{v}_i}{|\mathbf{v}_i|}\right) dt + \Delta t\,\mathbb{P}_i^n \text{ r}$$

- Projection

$\underline{\mathbb{P}_i = \mathbb{P}^{n+1n+\frac{1}{2}n}(\boldsymbol{\omega}_i) := \mathbb{I} - \boldsymbol{\omega}_i\boldsymbol{\omega}_i^{\perp}}(\boldsymbol{\omega}_i) = 0 \rightsquigarrow \boldsymbol{\omega}_i(t) + \Delta t) \in \mathbb{S}^{d-1}$

[3] T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet (PRL 1995); [3b] $\phi = \mathbb{1}_{[0,R)}$

Vicsek model: $\mathbf{v}_i(t) = \frac{s}{\deg_i}\sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|)\boldsymbol{\omega}_j(t) + \text{noise}_{[-\tau,\tau]}$

- Phase transition[4,4b] in order parameter $\varphi := \frac{1}{sN}\left|\sum_i \mathbf{v}_i\right|$:
  From global alignment to disorder: $\tau \approx 0 \rightsquigarrow \varphi \approx |\mathbf{v}^\infty|$ $\tau \gg 1 \rightsquigarrow \varphi \approx 0$

- Transitions observed[4c] in meso-scopic scales of 10s–100s
- Act local think global [4d,4e,4f] (near critical point):

⋆ Exploration of resources; defense mechanism; improved decision process

[4*] Vicsek & Zaferis, Collective motion (2012); [4b]Chaté et. al. (2004,2008)
[4c]Cavagna et. al., The starflag project (2008) [4d]Bialek et. al. (2013);
[4e]E. O Wilson (ants) Sociobiology (1975) [4f]I. Couzin (locust, fish)

Example#4:Cucker-Smale model[5]—alignment of velocities

- State space of velocities $\{\mathbf{v}_i(t)\}_{i=1}^N \in \mathbb{R}^\mathrm{d}$

$$\frac{\mathbf{v}_i(t+\Delta t)-\mathbf{v}_i(t)}{\Delta t}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \alpha\Big(\sum_j a_{ij}\mathbf{v}_j(t) - \mathbf{v}_i(t)\Big) = \alpha\sum_j a_{ij}\Big(\mathbf{v}_j(t) -$$

- A second-order model:

$$a_{ij}(\mathbf{x}(t)) = \frac{1}{\mathrm{deg}_i}\phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|), \qquad \frac{\mathbf{x}_i(t+\Delta t)-\mathbf{x}_i(t)}{\Delta t}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}_i(t) = \mathbf{v}_i(t)$$

- Global vs. local models, depending on deg_i:
  - ⋆ Global models: interaction involve all agents $\mathrm{deg}_i \rightsquigarrow N$

    Example of Cucker-Smale: $\quad \phi(r) = \dfrac{1}{1+r^{2\beta}}, \quad \beta > 0$
  - ⋆ Local models: involve nearby neighbors $\mathrm{deg}_i \rightsquigarrow N_i$

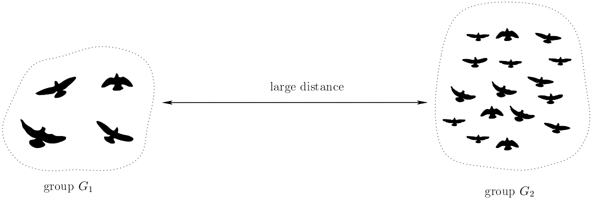
    "Environmental averaging": $\quad \big[\mathrm{Supp}\{\phi\}\big] \ll \big[\mathbf{x}(t)\big] := \max_{ij} |\mathbf{x}_i - \mathbf{x}_j|$
- More on C-S dynamics: Carrillo, Fornasier, S.-Y. Ha, Illner, Karper, J.-G. Liu,...

Limitations C-S model: state space is not homogeneous

- Global C-S: $\deg_i = N$ — a total of N agents in group cluster G

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{\alpha}{N}\sum_{j\in G}\phi_{ij}\left(\mathbf{v}_j(t) - \mathbf{v}_i(t)\right) \qquad \boxed{\phi_{ij} := \phi(|\mathbf{x}_i - \mathbf{x}_j|)} \qquad \phi(\cdot)\downarrow$$

- What can go wrong? say $G = G_1 \cup G_2$:



- C-S alignment: $N_1 = \#G_1,\ N_2 = \#G_2,\ G_2 = \{j :\ |x_j - x_i| \gg R\}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}v_i(t) = \frac{\alpha}{N_1+N_2}\left[\sum_{1\leqslant j\leqslant N_1}\phi_{ij}\left(\mathbf{v}_j(t)-\mathbf{v}_i(t)\right) + \sum_{1\leqslant j\leqslant N_2}\phi_{ij}\left(\mathbf{v}_j(t)-\mathbf{v}_i(t)\right)\right]$$

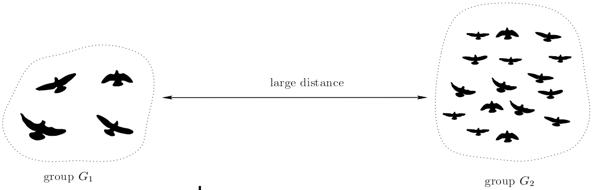
- If $N_2 \gg N_1$, then the far-away flock G_2 causes G_1 to stop

Example #5: Far from equilibrium

- (with S. Motsch)[6]:

$$a_{ij}(\mathbf{x}(t)) = \frac{1}{\deg_i}\phi_{ij}, \quad \phi_{ij} := \phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|)$$

$$\frac{\mathbf{v}_i(t+\Delta t) - \mathbf{v}_i(t)}{\Delta t}\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{\alpha}{\deg_i}\sum_j \phi_{ij}\Big(\mathbf{v}_j(t) - \mathbf{v}_i(t)\Big), \quad \deg_i = \sum_k \phi_{ik}$$



$\star$ $\deg_i(i \in G_1) \approx N_1\phi_0 \quad \rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}_i(t) = \frac{\alpha}{N_1\phi_0}\sum_{G_1}\phi_{ij}\left(\mathbf{v}_j(t) - \mathbf{v}_i(t)\right)$

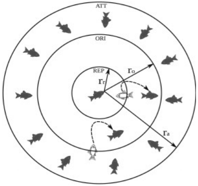
- a_{ij} is <u>not</u> symmetric

[6] Motsch & ET, A new model for self-organized dynamics and flocking behavior(2011)

Rules of engagement for self-propelled collective dynamics

- Craig Reynolds (1987)[7]

- Three zone (AAAs) models:
  - ⋆ Avoidance (Repulsion)
  - ⋆ Attraction (Cohesion)
  - ⋆ Alignment: $a_{ij} \geqslant 0$



... 1998 Academy Scientific and Technical Award in recognition of "pioneering contributions ... development of 3D computer animation for motion picture production."

[7] Flocks, herds and schools: A distributed behavioral model (1987);

Example #6: Collective synchronization

- Kuramoto model[8] $\{\mathbf{p}_i\} \rightsquigarrow$ phases $\{\theta_i\}$ or frequencies $\{\omega_i = \dot{\theta}_i\}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \Omega_i + \frac{\alpha}{N}\sum_j \sin(\theta_j - \theta_i) \rightsquigarrow \Omega_i + \frac{\alpha}{\deg_i}\sum_j a_{ij}(\theta_j - \theta_i), \; a_{ij} = \frac{\sin(\theta_j - \theta_i)}{\theta_j - \theta_i}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\theta_i(t) = \Omega_i + \alpha r \sin(\psi - \theta_i) \rightsquigarrow \Omega_i - \alpha r \sin(\theta_i), \quad re^{i\psi} := \frac{1}{N}\sum_j e^{i\theta_j}, \; \langle\Omega\rangle = 0$$

- $|\Omega_i| < \alpha r$: steady states; more oscillators recruited into synchronized clusters as $\alpha > \alpha_c$ increases
- $|\Omega_i| < \alpha r$: no synchronization is possible for $\alpha < \alpha_c$

- As a second-order model:

Averaging of frequencies[8b] $\displaystyle \frac{\mathrm{d}}{\mathrm{d}t}\omega_i(t) = \frac{\alpha}{N}\sum_j a_{ij}(\omega_j - \omega_i), \; a_{ij} := \cos(\theta_j - \theta_i)$

[8] Kuramoto, Lecture Notes Phys. (1975, 1984)...Acebron et. al. RevModPhys (2005)
[8b] Ha et. al (2010 –)

Collective Dynamics:
consensus, emergence of patterns and social hydrodynamics

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Lecture #2. $t \to \infty$: alignment and self-organization; consensus, flocking, ...

Eitan Tadmor

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Focus on two important limits

Lecture #2. Large time behavior – what happens when $t \to \infty$?

- Alignment self-organizes into K clusters,
  and in particular, into a flock, consensus, ... ($K = 1$)

- A distinction between global and local models:

$\odot$ Global models with unconditional consensus;
$\odot$ Local models — clusters, connectivity and heterophilious dynamics[1]

Lecture #3. Large number of "agents" – what happens when $N \gg 1$?

- Social hydrodynamics (opinions, flocking, ...)

- Critical thresholds in social hydrodynamics

[1] Motsch & ET., SIAM Review 2014

Outline

- $t \to \infty$: alignment and self-organization — consensus, flocking, ...
  - Global models — unconditional consensus/flocking
  - Local models — clusters, connectivity and heterophilious dynamics

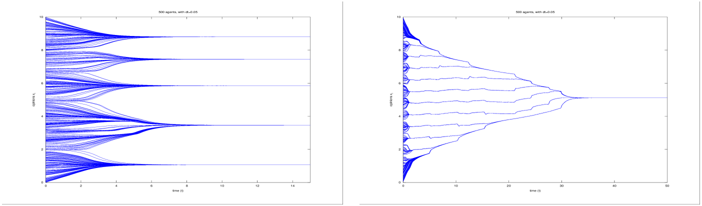
From local interactions to emerging consensus?

- How group pressure in small scales
  may lead to a consensus on larger scales

Large-time opinion dynamics: parties and consensus

$$\frac{\mathbf{x}_i(t+\Delta t)-\mathbf{x}_i(t)}{\Delta t} = \alpha\Big(\sum_j a_{ij}\mathbf{x}_j(t) - \mathbf{x}_i(t)\Big) \qquad a_{ij} = \frac{\phi(|\mathbf{x}_i(t)-\mathbf{x}_j(t)|)}{\deg_i(\mathbf{x}(t))}$$

- $\alpha > 0$ is a scaling factor s.t. $\phi(\cdot) \lesssim 1$
- Clustering – formation of K "parties":
- d $= 1$ — 100 uniformly distributed opinions on $[0, 10]$



- $K = 1$ — When does "consensus" emerge: $\mathbf{x}_i(t) \longrightarrow \mathbf{x}^\infty$ as $t \to \infty$?

Large-time social dynamics: flocking

$$\frac{\mathbf{v}_i(t+\Delta t)-\mathbf{v}_i(t)}{\Delta t} = \alpha\Big(\sum_j a_{ij}\mathbf{v}_j(t) - \mathbf{v}_i(t)\Big)$$

$$a_{ij}(\mathbf{x}(t)) = \frac{1}{\deg_i}\phi(|\mathbf{x}_i(t)-\mathbf{x}_j(t)|), \quad \frac{\mathbf{x}_i(t+\Delta t)-\mathbf{x}_i(t)}{\Delta t} = \mathbf{v}_i(t)$$

- Again — $\alpha > 0$ is a scaling factor s.t. $\phi(\cdot) \lesssim 1$
- Clustering – formation of K "flocks":
- $K = 1$ — When does a "flock" emerge:

$$\left\{ \begin{array}{l} \mathbf{v}_i(t) \longrightarrow \mathbf{v}^\infty \\ |\mathbf{x}_i(t)-\mathbf{x}_j(t)| \leqslant D_\infty \end{array} \right\} \text{as } t \to \infty \ ?$$

Emergence as $t \to \infty$

$$\frac{\mathbf{p}_i(t+\Delta t)-\mathbf{p}_i(t)}{\Delta t} = \alpha\Big(\sum_{j\neq i} a_{ij}\mathbf{p}_j - \mathbf{p}_i\Big), \quad \alpha \mapsto 1 \text{ (rescale } \Delta t) \text{ A is}$$

stochastic: $\displaystyle\sum_j a_{ij} \equiv 1$

Nonlinear Alignment

$$\mathbf{p}(t+\Delta t) = (1-\Delta t)\mathbf{p}(t) + \Delta t\, A\,\mathbf{p}(t)$$

$$A = A(\mathbf{p}(t))$$

- Seek <u>contractive</u> diameter $[...]$ such that $[A\mathbf{p}] \leqslant (1-\eta)[\mathbf{p}]$

  if $[A\mathbf{p}] \leqslant (1-\eta)[\mathbf{p}], \quad \eta = \eta(t) > 0$

  then $[\mathbf{p}(t+\Delta t)] \leqslant (1-\Delta t)[\mathbf{p}(t)] + \Delta t[A\,\mathbf{p}(t)] \leqslant (1-\eta\Delta t)[\mathbf{p}(t)]$:

$$\int^{\infty}\eta(s)ds = \infty \quad \rightsquigarrow$$

Contraction

$$\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{p}(t)] \leqslant -\eta(t)[\mathbf{p}(t)] < 0 \ \ldots \ \mathbf{p}_i(t) \xrightarrow{t\to\infty} \mathbf{p}^\infty(t)$$

- but[2] $\sum a_{ij} = 1$ or $A\mathbf{1} = \mathbf{1} \rightsquigarrow$ contract states "separated" from $\mathbf{p}^\infty \equiv \mathbf{1}$

Contractivity for stochastic A's (Dobrushin '56,...MT 2014)

Fix any i and j; set $\eta_k := \min\{a_{ik}, a_{jk}\}$ so that $a_{ik} - \eta_k \geqslant 0$, $a_{jk} - \eta_k \geqslant 0$.
Then, for arbitrary $\mathbf{w} \in \mathbb{R}^d$ and $\eta = \min_{ij} \sum_k \min\{a_{ik}, a_{jk}\}$

$$
\begin{aligned}
\langle (A\mathbf{p})_i - (A\mathbf{p})_j, \mathbf{w}\rangle &= \sum_k a_{ik}\langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k a_{jk}\langle \mathbf{p}_k, \mathbf{w}\rangle \\
&= \sum_k (a_{ik} - \eta_k)\langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k (a_{jk} - \eta_k)\langle \mathbf{p}_k, \mathbf{w}\rangle \\
A \text{ is raw stochastic } &\leqslant \sum_k (a_{ik} - \eta_k)\max_k \langle \mathbf{p}_k, \mathbf{w}\rangle - \sum_k (a_{jk} - \eta_k)\min_k \langle \mathbf{p}_k, \mathbf{w}\rangle \\
\eta_k = \eta_k(i,j)\ldots &= (1 - \sum_k \eta_k)\left(\max_k \langle \mathbf{p}_k, \mathbf{w}\rangle - \min_k \langle \mathbf{p}_k, \mathbf{w}\rangle\right) \\
\eta := \min_{ij} \sum \eta_k(i,j)\ldots &\leqslant (1 - \eta)\max_{k\ell} \langle \mathbf{p}_k - \mathbf{p}_\ell, \mathbf{w}\rangle \leqslant (1 - \eta)\max_{k,\ell} |\mathbf{p}_k - \mathbf{p}_\ell||\mathbf{w}|_*
\end{aligned}
$$

Choose $[A\mathbf{p}] := \max_{k\ell} |(A\mathbf{p})_k - (A\mathbf{p})_\ell| = |(A\mathbf{p})_i - (A\mathbf{p})_j|$; then

$$
\big[A\mathbf{p}\big] = |(A\mathbf{p})_i - (A\mathbf{p})_j| = \sup_{\mathbf{w}\neq 0} \frac{\langle (A\mathbf{p})_i - (A\mathbf{p})_j, \mathbf{w}\rangle}{|\mathbf{w}|_*} \leqslant (1 - \eta)\max_{k,\ell} \big[\mathbf{p}\big]
$$

Global models: $a_{ij}(\mathbf{x}) = \dfrac{1}{\deg_i}\phi(|\mathbf{x}_i - \mathbf{x}_j|) \geqslant \eta > 0$

- ℓ_∞-diameter — $\big[\mathbf{p}\big] := \max\limits_{i,j} |\mathbf{p}_i - \mathbf{p}_j| \ \rightsquigarrow\ \big[A\mathbf{p}\big] \leqslant (1-\eta)\big[\mathbf{p}\big], \quad \mathbf{p} \perp \mathbf{1}$

  $1-\eta$ is the coefficient of ergodicity[3] $\quad \eta = \eta_E := \min\limits_{ij} \sum \min(a_{ik}, a_{jk})$

- Opinion dynamics: $\qquad \eta_E = \min \sum \min(a_{ik}, a_{jk}) \geqslant \dfrac{N^k}{\deg_i} \min\limits_{r \leqslant [\mathbf{x}_0]} \phi(r)$

- If $\{\mathbf{x}_0\} \subset \mathrm{Supp}\{\phi\}$ then:

  $$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{x}(t)\big] \leqslant -\eta\big[\mathbf{x}(t)\big] \ \rightsquigarrow\ \big[\mathbf{x}(t)\big] \leqslant e^{-\eta t}\big[\mathbf{x}_0\big], \quad \eta = \min_{r \leqslant [\mathbf{x}_0]} \phi(r) > 0$$

- Global connectivity $\rightsquigarrow$ unconditional consensus: $\mathbf{x}(t) \rightarrow \mathbf{x}^\infty \in \mathrm{Conv}\{\mathbf{x}_0\}$

  Symmetric case ($\deg_i = N$): $\mathbf{x}^\infty = \dfrac{1}{N}\sum\limits_i \mathbf{x}_i(t=0)$

  Non-symmetric case ($\deg_i = \sum_k \phi_{ik}$) Q. #1: characterize $\mathbf{x}^\infty$

- Q. #2: Conditional consensus — dependence on initial configuration

[3] Ipsen & Selee (2011): $1-\eta_E = \frac{1}{2}\max_{ij}\sum_k |a_{ik} - a_{jk}|$; Krause (2014): scrambling

Finite time collision — the rendezvous problem

- Assume the influence function $\phi(\cdot)$ is decreasing ...
- If $\{\mathbf{x}(0)\} \subset \text{Supp}\{\phi\}$ then:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{x}(t)\big] \leqslant -\phi(\big[\mathbf{x}(t)\big])\big[\mathbf{x}(t)\big] = k'(\big[\mathbf{x}(t)\big]), \quad k(r) = \int_0^r s\phi(s)ds$$

- Osgood condition:[4]

$$\int_0^1 \frac{dr}{r\phi(r)} < \infty \quad \rightsquigarrow \text{ Finite-time collision — } \big[\mathbf{x}(t_c)\big] = 0;$$

...the role for repulsion forces

$$\int_0^1 \frac{dr}{r\phi(r)} = \infty^{4b} \rightsquigarrow \text{ Global regularity } \quad \text{— } \big[\mathbf{x}(t)\big] > 0$$

Example. $\phi(r) \sim r^{-\beta}$, $\beta \leqslant 1 \rightsquigarrow$ unconditional rendezvous[4c]

[4] Bertozzi,Carrillo,Laurent (2009): symmetric $\dot{R}(t) \lesssim k'(R(t)), R := \max_i |\mathbf{x}_i(t) - \mathbf{x}^\infty|$

[4b] $\phi'(r)r \gtrsim -\phi(r)$; [4c] Unconditional rendezvous with bearing-only $\phi(r) = 1/r$

Global models: $a_{ij}(\mathbf{x}) = \frac{1}{\deg_i}\phi(|\mathbf{x}_i - \mathbf{x}_j|) \geqslant \eta > 0$

$\boxed{\text{Flocking dynamics}}$ with a decreasing influence function ϕ:

Diameter $[\mathbf{x}(t)]$ may grow $\frac{d}{dt}\big[\mathbf{x}(t)\big] \leqslant \big[\mathbf{v}(t)\big]$ — but remains bounded:

$$\frac{d}{dt}\big[\mathbf{v}(t)\big] \leqslant -\phi(\big[\mathbf{x}(t)\big])\big[\mathbf{v}(t)\big] \rightsquigarrow \mathcal{E}(t) := \big[\mathbf{v}(t)\big] + \int_{[\mathbf{x}_0]}^{[\mathbf{x}(t)]} \phi(s)ds \text{ decreases}^5$$

$$\int_{[\mathbf{x}_0]}^{[\mathbf{x}(t)]} \phi(s)ds \leqslant \big[\mathbf{v}_0\big] \text{ If } \big[\mathbf{v}_0\big] < \int_{[\mathbf{x}_0]}^{\infty} \phi(s)ds \text{ then } \exists D_\infty : \ [\mathbf{v}_0] = \int_{[\mathbf{x}_0]}^{D_\infty} \phi(s)ds$$

- Unconditional flocking: $\big[\mathbf{x}(t)\big] \leqslant D_\infty \rightsquigarrow \big[\mathbf{v}(t)\big] \leqslant e^{-\alpha\phi(D_\infty)t}\big[\mathbf{v}_0\big]$

$\int^\infty \phi(s)ds = \infty$ — global communication implies unconditional flocking

Example. $\phi(r) = \frac{1}{1+r^{2\beta}}, \quad \beta < \frac{1}{2} \rightsquigarrow$ flocking velocity $\mathbf{v}(t) \to \mathbf{v}^\infty$

- Q. #1. What is $\mathbf{v}^\infty$ in the non-symmetric case?

[5] S. Y. Ha & J.-G. Liu (2009); non-symmetric case in Motsch & ET (2014)

From ℓ_∞ to ℓ_2 contractivity ... $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_i = \sum_j a_{ij}\mathbf{p}_j - \mathbf{p}_i$

- $\displaystyle\max_{\mathbf{p}} \frac{[A\mathbf{p}]}{[\mathbf{p}]} = 1-\eta$ — set the ℓ_2 diameter $[\mathbf{p}]^2 := \sum |\mathbf{p}_i - \mathbf{p}_j|^2$
- $A\mathbf{1} = \mathbf{1}$; If $\mathbf{p} \perp \mathbf{1}$ then (i) $[\mathbf{p}]^2 = 2N|\mathbf{p}|^2$ and (ii) $A\mathbf{p} \perp \mathbf{1}$ (A is symmetric)

$$\rightsquigarrow \max_{\mathbf{p}} \frac{[A\mathbf{p}]}{[\mathbf{p}]} = \max_{\mathbf{p}\perp\mathbf{1}} \frac{[A\mathbf{p}]}{[\mathbf{p}]} = \max_{\mathbf{p}\perp\mathbf{1}} \frac{|A\mathbf{p}|}{|\mathbf{p}|} = \max_{\mathbf{p}\perp\mathbf{1}} \frac{|\langle A\mathbf{p},\mathbf{p}\rangle|}{|\mathbf{p}|^2}$$

- Set the graph Laplacian[6] $L_A := I - A$ (note the positivity)

$a_{ij} = \dfrac{\phi_{ij}}{\deg_i}$ is symmetriazble with eigenvalues $0 \leqslant \mu_N \leqslant \ldots \mu_2 \leqslant \mu_1 = 1$

$L_A = I - A$ w/corresponding real eigenvalues $1 \geqslant \lambda_n \geqslant \ldots \lambda_2 \geqslant \lambda_1 = 0$

$$\max_{\mathbf{p}\perp\mathbf{1}} \frac{\langle A\mathbf{p},\mathbf{p}\rangle}{|\mathbf{p}|^2} = 1 - \min_{\mathbf{p}\perp\mathbf{1}}\left(1 - \frac{\langle A\mathbf{p},\mathbf{p}\rangle}{|\mathbf{p}|^2}\right) = 1 - \min_{\mathbf{p}\perp\mathbf{1}} \frac{\langle L_A\mathbf{p},\mathbf{p}\rangle}{|\mathbf{p}|^2} = 1 - \lambda_2(L_A)$$

- $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = (A-I)\mathbf{p} = -L_A\mathbf{p}$ contractivity: $\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{p}] \leqslant -\eta[\mathbf{p}]$, $\eta = \lambda_2(L_A)$
- ℓ_2-contarctivity ("shortcut"): $\frac{\mathrm{d}}{\mathrm{d}t}[\mathbf{p}]^2 = -2\langle L_A\mathbf{p},\mathbf{p}\rangle \leqslant -2\lambda_2(L_A)[\mathbf{p}]^2$

[6] Laplacian $L_\Phi = D - \Phi$ and its normalized $L_A = I - D^{-1}\Phi$

Geometric aspects of spectral graph theory

- A graph $G = (V, E)$;
  vertex set $V = \{\mathbf{p}_i\} \subset \mathbb{R}^N$ and edge set $E = \{\mathbf{e}_{ij}\} \subset \mathbb{R}^N \times \mathbb{R}^N$

  The gradient $\nabla_\phi : V \mapsto E$ $\qquad \nabla_\phi(\mathbf{p})_{ij} := \sqrt{\phi_{ij}}(\mathbf{p}_i - \mathbf{p}_j)$

  The divergence $\text{div}_\phi : E \mapsto V$ $\qquad (\text{div}_\phi(\mathbf{e}))_i := \sum_j \sqrt{\phi_{ij}}(\mathbf{e}_{ij} - \mathbf{e}_{ji})$

  with the usual duality $\langle \nabla \mathbf{p}, \mathbf{u} \rangle = \langle \mathbf{p}, \text{div}\mathbf{u} \rangle$ ... and the Laplacian:

  $$\Delta_\phi := -\frac{1}{2}\text{div}_\phi \circ \nabla_\phi : V \mapsto V \qquad \Delta_\phi(\mathbf{p})_i = \sum_j \phi_{ij}(\mathbf{p}_i - \mathbf{p}_j)$$

- Alignment process: $\quad \dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) = \dfrac{-1}{\deg(\mathbf{p}(t))}\Delta_\phi(\mathbf{p}(t)) = -L_A\mathbf{p}(t)$

- Laplacians[7]: $\quad L_A = I - D^{-1}\Phi, \ L_\Phi := D - \Phi, \ L_{\text{sym}} = I - D^{-\frac{1}{2}}\Phi D^{-\frac{1}{2}}$

$\dfrac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -L_A\mathbf{p} \ \rightsquigarrow \ \dfrac{1}{2}\dfrac{\mathrm{d}}{\mathrm{d}t}|\mathbf{p}|^2 = -\langle L_A\mathbf{p}, \mathbf{p}\rangle$ but lacks ℓ_2-coercivity $\langle L_A\mathbf{p}, \mathbf{p}\rangle \not\gtrsim |\mathbf{p}|^2$

- L^p Sobolev inequalities[7c], Poincare inequality[7d]

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[7] Merris (1994) Laplacian matrices on graphs; Chung (1997) Spectral graph theory,...
[7b] Demmel (2009) Lecture notes, Luxborg (2007)... ,[7c] Badr & Russ (2009, 2012)
[7d] Coulhon & Koskela (2004)

revisiting the coercivity of graph Laplacians

- Symmetric Self-alignment ($\deg_i = N$):

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p} = -\frac{1}{2N}\mathrm{div}\nabla\mathbf{p} \rightsquigarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{p}|^2 = -\frac{1}{2N}|\nabla\mathbf{p}|^2 = -\frac{1}{2N}\sum_{ij}\phi_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$$

- And the corresponding non-symmetric case $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{p}|^2 = \langle L_A\mathbf{p}, \mathbf{p}\rangle$:

$$\langle L_\Phi\mathbf{p}, \mathbf{p}\rangle = \sum_i \deg_i|\mathbf{p}_i|^2 - \sum_{ij}\phi_{ij}\langle\mathbf{p}_i, \mathbf{p}_j\rangle = \frac{1}{2}\sum_{ij}\phi_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \;\; \gtrsim [\mathbf{p}]^2 \; ?$$

- Symmetry[8] — $\sum \mathbf{p}_i(t) = \sum \mathbf{p}_i(0)$

$$\frac{1}{2N}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 = -\frac{1}{N}\sum_{ij}\phi_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2 \leqslant -\eta\sum_{ij}|\mathbf{p}_i - \mathbf{p}_j|^2$$

- Coercivity with $\eta = \dfrac{1}{N}\min_{ij}\phi_{ij}$; ... $\eta = \lambda_2(L_\Phi)$

[8] Raw stochastic and column stochastic matrices

Fiedler number and spectral graph theory

- $\lambda_2(L_\Phi) > 0$ – Fiedler number[9] dictates algebraic connectivity of $\{\phi_{ij}\}\geqslant 0$
- A (weighted) graph $G = (V, E)$: vertex set of positions: $V = \{\mathbf{p}_i\}$ and edge set of connectors: $E = \{\phi_{ij}\}$, $\phi_{ij} = \phi(|\mathbf{p}_i - \mathbf{p}_j|)$
- Connectivity of graph $G_\Phi(\mathbf{p}, \Phi(\mathbf{p}))$: $\forall \mathbf{p}_i, \mathbf{p}_j$ $\exists$a (shortest) path Γ_{ij} s.t.

$$\min_{k_\ell \in \Gamma_{ij}} \phi_{k_\ell, k_{\ell+1}} \geqslant \mu > 0, \quad \Gamma_{ij} = \{i = k_0 < k_2 < \ldots < k_r = j\}, \ \text{length}(\Gamma_{ij}) =: r_{ij}$$

$$\sum_{ij} \mu |\mathbf{p}_i - \mathbf{p}_j|^2 \leqslant r_{ij} \sum_{k_\ell \in \Gamma_{ij}} \mu |\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2 \qquad \mathbf{p}_i - \mathbf{p}_j = \sum_{\ell=0}^{r_{ij}} (\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell})$$

$$\leqslant r_{ij} \sum_{k_\ell \in \Gamma_{ij}} \phi_{k_{\ell+1}, k_\ell} |\mathbf{p}_{k_{\ell+1}} - \mathbf{p}_{k_\ell}|^2 \leqslant \max_{ij}(r_{ij})\, N^2 \sum_{\alpha\beta} \phi_{\alpha\beta} |\mathbf{p}_\alpha - \mathbf{p}_\beta|^2$$

$$\rightsquigarrow \ \lambda_2(L_\Phi) = \min_{\mathbf{p} \perp \mathbf{1}} \frac{\langle L_\Phi \mathbf{p}, \mathbf{p} \rangle}{|\mathbf{p}|^2} = \min_{\mathbf{p} \perp \mathbf{1}} \frac{\frac{1}{2} \sum_{\alpha\beta} \phi_{\alpha\beta} |\mathbf{p}_\alpha - \mathbf{p}_\beta|^2}{\frac{1}{2N} [\mathbf{p}]^2} \geqslant \frac{\mu}{\max_{ij}(r_{ij})\, N}$$

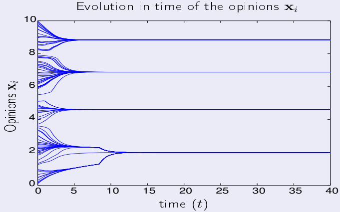
- Algebraic connectivity[9b]: $\lambda_2(L_\Phi) \geqslant \dfrac{\mu}{\mathrm{d}(G) N}, \geqslant \dfrac{\mu}{N^2}, \qquad \mathrm{d}(G) := \max_{ij}(r_{ij}) \leqslant N$

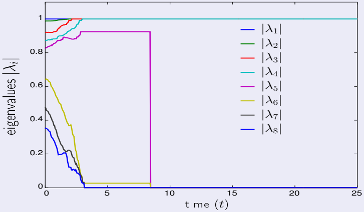
[9] M. Fiedler, Algebraic connectivity of graphs (1973); [9b] Mohar (1991) $\lambda_2(L_\Phi) \geqslant \frac{4\mu}{N^2}$

On the geometric aspects of Fiedler eigenpair

- The multiplicity of $\lambda_1(L_\Phi) = 0 \rightsquigarrow$ the number of connected components

$$\dot{\mathbf{x}} = \frac{-1}{\deg(\mathbf{x})}\Delta_\phi(\mathbf{x}), \quad \phi(r) = \mathbf{1}_{[0,1]} \quad \text{vs.} \quad \mu_i(A) = 1 - \lambda_i(L_A)$$





- $L_A = I - D^{-1}\Phi$ similar to $L_{\text{sym}} = I - D^{-\frac{1}{2}}\Phi D^{-\frac{1}{2}}$ congruent to $L_\Phi = D - \Phi$

- Fiedler (1973): $\lambda_2(L_A)$ increases as A becomes "more" connected:

  If (V, E) is a sub-graph of $(V, F \supset E)$ then $\lambda_2(L_E) \leqslant \lambda_2(L_F)$

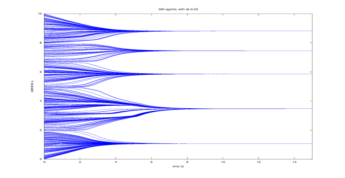
- Spectral bisection[10] (1975): The Fiedler vector $\mathbf{v}_2(L_{\text{sym}})$ induces connected decomposition $V = V_- \cup V_+, \quad V_\pm = \{i \ : \ \pm\mathbf{v}_2(i) > 0\}$

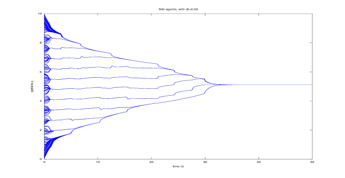
[10] Pothen, Simon, Liou (1990); Newmann (2003): The structure ... of complex networks

Propagation of connectivity

- Consensus/flocking: time-dependence on the underlying graph $G_{A(\mathbf{p}(t))}$
- Loss of connectivity (in local models):

emergence clusters (K=5) vs. consensus (K=1)





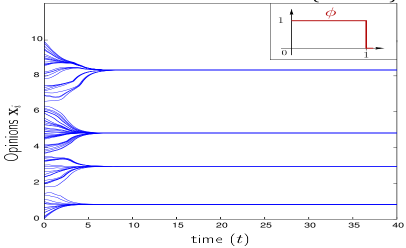
1. Determine the number of "parties" K[11]
2. In particular — when $K = 1$ (consensus/flocking) depending on ...
   - Propagation of connectivity: $\lambda_2(L_{A(\mathbf{p}(0))}) > 0 \rightsquigarrow \lambda_2(L_{A(\mathbf{p}(t))}) > 0$?
   - "Intensity" of connectivity – $\{A^N(\mathbf{p}(t))\}_{ij} \geqslant \mu^N(t)$: $\displaystyle\int^\infty \mu(t)dt = \infty$

[11] Alignment yields finite # of parties, Jabin & Motsch (2014), Motsch & ET (2014)

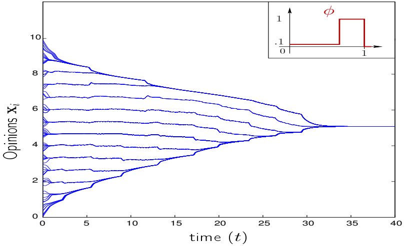
How “rules of engagement” influence the emergence of consensus?

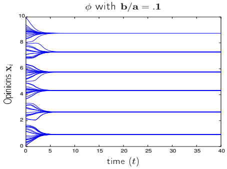
- 100 uniformly distributed opinions: $\phi(r) = a\mathbf{1}_{\{r\leqslant \frac{1}{\sqrt{2}}\}} + b\mathbf{1}_{\{\frac{1}{\sqrt{2}}\leqslant r<1\}}$

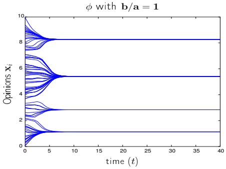
$a = b = 1: \ \phi(r) = \mathbf{1}_{\{0<r<1\}}$

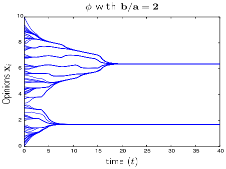


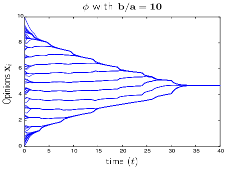
$(a, b) = (0.1, 1)$











- Homophilious dynamics: align with those that think alike ($a \gg b$) vs.
- Heterophilious dynamics: "bonding with the different" ($a \ll b$)

Collective Dynamics:
consensus, emergence of patterns and social hydrodynamics

Leçons Jacques-Louis Lions 2016

Lecture #3. $N \to \infty$: from agent-based models to social hydrodynamics

Eitan Tadmor

Center for Scientific Computation and Mathematical Modeling (CSCAMM)
Department of Mathematics, Institute for Physical Science & Technology
University of Maryland

Laboratoire Jacques-Louis Lions

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Mathematical aspects of collective dynamics...

- A basic paradigm for collective dynamics — environmental averaging

1. Examples of mathematical models for collective dynamics
   - Krause-Hegselmann model for opinion dynamics
   - Vicsek model for flocking; phase transition
   - Cucker-Smale model for flocking — near and far from equilibrium
   - Questions that arise in different fields
2. $t \to \infty$: alignment and self-organization — consensus, flocking, ...
   - Global models — unconditional consensus/flocking
   - Local models — clusters, connectivity and heterophilious dynamics
   - A new paradigm — tendency to move ahead; emergence of leaders
3. $N \to \infty$: from agent-based models to social hydrodynamics
   - Liouville equation
   - Kinetic description
   - From kinetic to hydrodynamic description of flocking
   - Hydrodynamic alignment — smooth solutions must flock
   - Critical thresholds in flocking hydrodynamics

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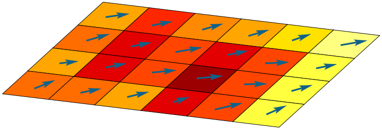
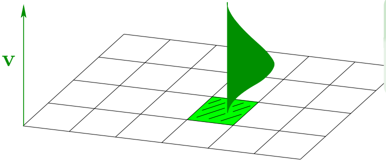
Multi-scale phenomena - different models at different scales



Macroscopic models: $\rho, \mathbf{u}$

$$\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$$
$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} + \nabla_{\mathbf{x}} P = 0$$

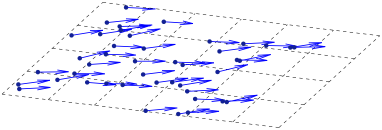
$\Uparrow$ Human scale: $\varepsilon \to 0, \quad (t' = \varepsilon t, \mathbf{x}' = \varepsilon \mathbf{x})$



Kinetic models: $f(t, \mathbf{x}, \mathbf{v})$

$$f_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f, f)$$

$\Uparrow$ $N \to \infty$ (including meso-scale phenomena)



Agent-based models: $\{\mathbf{x}_i, \mathbf{v}_i\}_{1 \leqslant i \leqslant N}$

$$\begin{cases} \dot{\mathbf{x}}_i = \mathbf{v}_i \\ \dot{\mathbf{v}}_i = \mathbf{F}_i, \qquad \mathbf{F}_i \mapsto \mathbf{F}(\mathbf{x}_i, \mathbf{v}_i)(\mathbf{x}, \mathbf{v}) \end{cases}$$

Empirical distribution

- Phase space: $\{(\mathbf{x}_i, \mathbf{v}_i)\} \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \qquad i = 1, 2, \ldots, N, \quad N \gg 1$

Agent based models — in terms of Empirical distribution

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i, \quad \frac{d\mathbf{v}_i(t)}{dt} = F[f^N](\mathbf{x}_i, \mathbf{v}_i), \quad f^N(t, \mathbf{x}, \mathbf{v}) := \frac{1}{N}\sum_{j=1}^{N} \delta_{\mathbf{x}_j(t)} \otimes \delta_{\mathbf{v}_j(t)}$$

- $F(f^N)$ dictated by interaction kernel — $a(\mathbf{x}, \mathbf{y}) = \dfrac{\phi(|\mathbf{x}-\mathbf{y}|)}{\deg(\mathbf{x})}$

$$F(f)(\mathbf{x}, \mathbf{v}) := \iint_{\mathbf{x}', \mathbf{v}'} a(\mathbf{x}, \mathbf{y})(\mathbf{v}' - \mathbf{v}) f(t, \mathbf{x}', \mathbf{v}') d\mathbf{x}' d\mathbf{v}'_{|f=f^N}$$

$$\rightsquigarrow \quad F(f^N)(\mathbf{x}, \mathbf{v}) = \frac{1}{N \cdot \deg(\mathbf{x})} \sum_j \phi(|\mathbf{x}_j - \mathbf{x}|)(\mathbf{v}_j - \mathbf{v})$$

The passage to limit: $f^N = \frac{1}{N}\sum_j \delta_{\mathbf{x}_j(t)} \otimes \delta_{\mathbf{v}_j(t)} \ \to f$

- Agents are indistinguishable: $f^N(\mathbf{x}_{\sigma(1)}, \mathbf{v}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(N)}, \mathbf{v}_{\sigma(N)})$
- Liouville equation for empirical function f^N: $\dot{\mathbf{x}}_i = \mathbf{v}_i,\ \dot{\mathbf{v}}_i = F(\mathbf{x}_i, \mathbf{v}_i) \ \rightsquigarrow$

$$\partial_t f^N + \overbrace{\sum_{i=1}^N \mathbf{v}_i \cdot \nabla_{\mathbf{x}_i} f^N}^{T(f(\mathbf{x},\mathbf{v}))} + \overbrace{\frac{\alpha}{N}\sum_{i=1}^N \frac{1}{\deg_i}\nabla_{\mathbf{v}_i} \cdot \left(\sum_{j=1}^N \phi(|\mathbf{x}_i - \mathbf{x}_j|)(\mathbf{v}_j - \mathbf{v}_i) f^N\right)}^{I(f^N(\mathbf{x},\mathbf{v}))} = 0$$

- Marginal distribution: we "probe" it by <u>any</u> of its pairs — take $(\mathbf{x}_1, \mathbf{v}_1)$:

$$f^N(\mathbf{x}_1,\mathbf{v}_1) := \int f^N(\mathbf{x}_1,\mathbf{v}_1,\mathbf{x}_-,\mathbf{v}_-)d\mathbf{x}_- d\mathbf{v}_-, \quad (\mathbf{x}_-,\mathbf{v}_-) := (\mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N)$$

$$\int_{\mathbb{R}^{d(N-1)}} T(f^N(\mathbf{x},\mathbf{v}))d\mathbf{x}_- d\mathbf{v}_- = \mathbf{v}_1 \cdot \nabla_{\mathbf{x}_1} f^N(\mathbf{x}_1, \mathbf{v}_1)$$

$$\int I(f^N) d\mathbf{x}_- d\mathbf{v}_- = \frac{1}{\deg_1}\int \sum_{j=2}^N \nabla_{\mathbf{v}_1} \cdot \left(\phi(|\mathbf{x}_j - \mathbf{x}_1|)(\mathbf{v}_j - \mathbf{v}_1) f^N\right) d\mathbf{x}_- d\mathbf{v}_-$$

The passage to a limit - formalities cont'd

- But agents are indistinguishable $(\mathbf{x}_j,\mathbf{v}_j)\leftrightarrow(\mathbf{x}_2,\mathbf{v}_2)$:

$$\int I(f^N)\mathrm{d}\mathbf{x}_-\,\mathrm{d}\mathbf{v}_- = \frac{\alpha}{N}\frac{1}{\deg_1}\int\sum_{j=2}^{N}\nabla_{\mathbf{v}_1}\cdot\Big(\phi(|\mathbf{x}_j-\mathbf{x}_1|)(\mathbf{v}_j-\mathbf{v}_1)f^N\Big)\mathrm{d}\mathbf{x}_-\mathrm{d}\mathbf{v}_-$$

$$\rightsquigarrow\quad \alpha\frac{N-1}{N}\frac{1}{\deg_1}\int\nabla_{\mathbf{v}_1}\cdot\Big(\phi(|\mathbf{x}_2-\mathbf{x}_1|)(\mathbf{v}_2-\mathbf{v}_1)g^N\Big)\mathrm{d}\mathbf{x}_2\mathrm{d}\mathbf{v}_2$$

where $g^N(\mathbf{x}_1,\mathbf{v}_2,\mathbf{x}_2,\mathbf{v}_2) := \int_{\mathbb{R}^{2\mathrm{d}(N-2)}} f^N(\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{x}_3 d\mathbf{v}_3\ldots\mathrm{d}\mathbf{x}_N\mathrm{d}\mathbf{v}_N$

- Propagation of chaos[1,1b]: $f^N\to f(\mathbf{x}_1,\mathbf{v}_1),\quad g^N\to f(\mathbf{x}_1,\mathbf{v}_1)f(\mathbf{x}_2,\mathbf{v}_2)$

... $\exists$ f satisfies the Vlasov equation: $\boxed{f_t+\mathbf{v}\cdot\nabla_{\mathbf{x}}f+\nabla_{\mathbf{v}}\cdot(F(f)f)=0}$

- From deterministic to a statistical description[1c]:
  $f(t,\mathbf{x},\mathbf{v})$ — probability of finding agents in $\mathrm{d}\mathbf{x}\mathrm{d}\mathbf{v}$ at time t

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[1] Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy
[1b] Convergence of expectation (Sznitman)
[1c] F. Golse Lecture notes; Gallagher, Saint-Raymond & Texier

The passage to the limit[2] $N \to \infty$

- $$f^N = \frac{1}{N}\sum_j \delta_{\mathbf{x}-\mathbf{x}_j(t)} \otimes \delta_{\mathbf{v}-\mathbf{v}_j(t)} \ \to f(t,\mathbf{x},\mathbf{v})$$

Liouville $\mapsto$ Vlasov eq. for probability distribution $f(t,\mathbf{x},\mathbf{v})$

$$f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \alpha(\mathbf{x})\nabla_{\mathbf{v}}\cdot Q(f,f) = 0\tau\Delta_{\boldsymbol{\omega}} f$$

- $Q(f,f)$ assembles binary interactions: alignment, repulsion, noise, ...

$$Q(f,f) = \int_{\mathbb{R}^{2d}} \frac{\phi(|\mathbf{x}-\mathbf{y}|)}{\deg(t,\mathbf{x})}(\mathbf{w}-\mathbf{v})f(t,\mathbf{x},\mathbf{v})f(t,\mathbf{y},\mathbf{w})d\mathbf{y}d\mathbf{w} = (\bar{\mathbf{v}}_{\text{mean}} - \mathbf{v})f$$

- Align w/mean $\quad \bar{\mathbf{v}}_{\text{mean}} := \dfrac{\overline{v\phi * f}}{\overline{\phi * f}}, \quad \bar{g} := \displaystyle\int_{\mathbf{w}} g(\mathbf{w})d\mathbf{w} \quad \alpha(\mathbf{x}) = \dfrac{\overline{\phi * f}}{\deg(t,\mathbf{x})}$

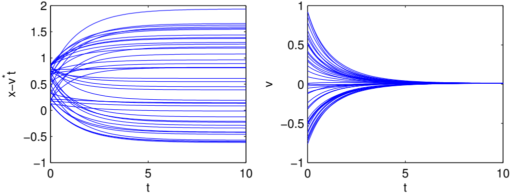
$$\deg(t,\mathbf{x}) \begin{cases} \equiv 1 & \text{Cucker-Smale} \quad \rightsquigarrow \quad \alpha(\mathbf{x}) = \overline{\phi * f} \\ \\ = \overline{\phi * f} & \text{Motsch-ET} \quad \rightsquigarrow \quad \alpha(\mathbf{x}) \equiv 1 \end{cases}$$

[2] Ha & ET (2008); Canizo, Carrillo, Rosado (2009)

Emergence of Dirac masses in velocity space

- Kinetic description: $f_t + \mathbf{v}\cdot\nabla_{\mathbf{x}} f + \alpha\nabla_{\mathbf{v}}\cdot Q(f,f) = 0$

- Flocking[3] $(K=1)$ $\quad f \ \rightsquigarrow\ \rho(t,\mathbf{x})\delta(\mathbf{v}-\mathbf{u}(t,\mathbf{x}))$



Shifted position $\mathbf{x}-\mathbf{u}t$ Velocity $\mathbf{v}-\mathbf{u}$

- Recover $\begin{bmatrix} \text{density} \ldots\ldots\ldots & \rho \\ \text{momentum} \ldots & \rho\mathbf{u} \end{bmatrix} = \int \begin{bmatrix} 1 \\ \mathbf{v} \end{bmatrix} f(t,\mathbf{x},\mathbf{v})d\mathbf{v}$

  Requires closure in terms of $P_{ij} := \int (\mathbf{v}_i - \mathbf{u}_i)(\mathbf{v}_j - \mathbf{u}_j) f(t,\mathbf{x},\mathbf{v})d\mathbf{v}$

[3] Ha & ET (2008); Carrillo-Fornasier-Rosado-Toscani (2009), Motsch-ET (2014):
Unconditional flocking: $\sup_{\text{supp} f(t)} |\mathbf{x}-\mathbf{y}| < \infty$; $\sup_{\text{supp} f(t)} |\mathbf{v}-\mathbf{v}'| \stackrel{t\to\infty}{\longrightarrow} 0$

Flocking hydrodynamics

$$\begin{aligned} \text{mass :} \quad & \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) &= 0 \\ \text{momentum :} \quad & (\rho \mathbf{u})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u} + P) &= \mathcal{R}, \quad \mathcal{R} = \mathcal{R}(\mathbf{u}) \\ \text{energy :} \quad & (\rho E)_t + \nabla_{\mathbf{x}} \cdot (\rho E \mathbf{u} + P\mathbf{u} + q) &= \mathcal{S}, \quad \mathcal{S} = \mathcal{S}(\mathbf{u}, \mathbf{u}) \end{aligned}$$

- non-local action $\mathcal{R}(\mathbf{u}) = \alpha \int_{\mathbb{R}^d} a(\mathbf{x}, \mathbf{y}) \big(\mathbf{u}(t, \mathbf{y}) - \mathbf{u}(t, \mathbf{x}) \big) \rho(t, \mathbf{x}) \rho(t, \mathbf{y}) d\mathbf{y}$

stress tensor $P = (p_{ij})$: $\quad p_{ij} = \int_{\mathbb{R}^d} (\mathbf{v}_i - \mathbf{u}_i)(\mathbf{v}_j - \mathbf{u}_j) f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}$

- Derivation of P — empirical; closure of moments:
- Closure – in terms of local Maxwellian: $\quad P = P(f) \mapsto P(M_f)$
- Mono-phase CS[4]: $\quad M_{\{\rho, \mathbf{u}\}}(\mathbf{v}) = \rho(t, \mathbf{x}) \delta(\mathbf{v} - \mathbf{u}(t, \mathbf{x})) \ \rightsquigarrow \ P_{ij} \equiv 0$
- Other closures...[4b]: $\quad M_{\{\rho, \mathbf{u}\}}(\mathbf{v}) = \rho(t, \mathbf{x}) \frac{1}{(2\pi)^{d/2}} e^{-\frac{|\mathbf{v} - \mathbf{u}(\mathbf{x})|^2}{2}} \ \rightsquigarrow \ P_{ij} \equiv \rho \mathbb{I}$

Brownian effect: $\quad f_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \nabla_{\mathbf{v}} \cdot Q(f, f) = \frac{1}{\epsilon} \nabla_{\mathbf{v}} \cdot ((\mathbf{v} - \mathbf{u}) f) + \frac{1}{\epsilon} \nabla_{\mathbf{v}} f$

Flocking hydrodynamics. Alignment with non-local means[5]

Mono-phase model $P \equiv 0^{4b}$: $\begin{cases} \rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0 \\ \mathbf{u}_t + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \alpha(\mathbf{x})\left(\overline{\mathbf{u}}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})\right) \end{cases}$

- Tendency to align w/mean $\overline{\mathbf{u}}_{\text{mean}}(t,\mathbf{x}) = \dfrac{\overline{\rho\mathbf{u}}}{\overline{\rho}}, \quad \overline{\mathbf{w}} := \displaystyle\int_{\mathbf{y}} \phi(|\mathbf{x}-\mathbf{y}|)\mathbf{w}(\mathbf{y})d\mathbf{y}$

  Cucker-Smale: $\alpha(\mathbf{x}) \equiv \phi(|\cdot|) * \rho$; RHS $= \phi * (\rho\mathbf{u}) - \mathbf{u}\phi * \rho$;

  Motsch-ET: $\alpha(\mathbf{x}) \equiv 1$; RHS $= \dfrac{\phi * (\rho\mathbf{u})}{\phi * \rho} - \mathbf{u}$

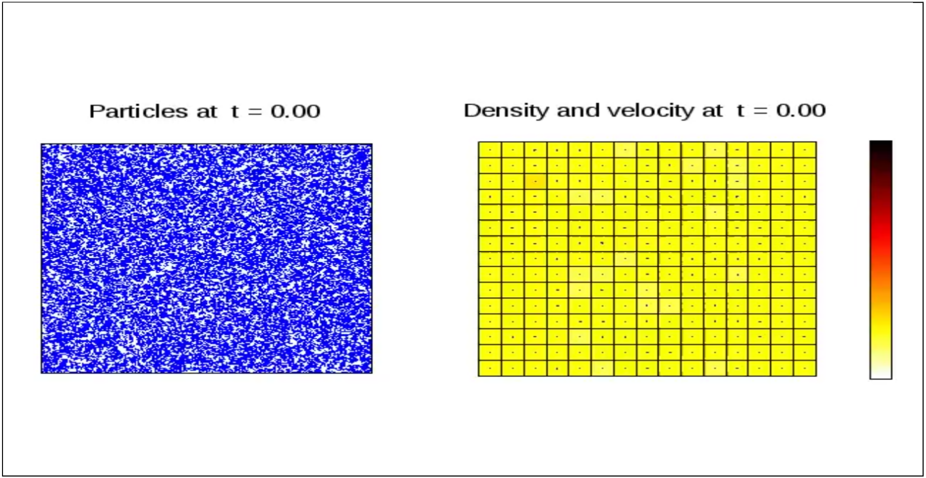
- Projected mean $\overline{\mathbf{u}}_{\text{mean}}(t,\mathbf{x}) := \dfrac{\overline{\rho\chi_{\mathbf{x},\mathbf{y}}\mathbf{u}}}{\overline{\rho}}, \quad \chi_{\mathbf{x},\mathbf{y}} := \dfrac{\langle \mathbf{u}(\mathbf{x}), \mathbf{u}(\mathbf{y})\rangle}{|\mathbf{u}(\mathbf{y})|^2}$

- $\phi(|\mathbf{x}|) \mapsto \varepsilon^{-(\mathrm{d}+2)}\phi\left(\dfrac{|\mathbf{x}|}{\varepsilon}\right) \overset{\epsilon\to 0}{\rightsquigarrow} (\rho\mathbf{u})_t + \mathrm{div}(\rho\mathbf{u}\otimes\mathbf{u}) = C\mathrm{div}(\rho\nabla\mathbf{u})$
- Singular influence: fractional Laplacian[4b] $\phi(|\mathbf{x}|) \sim |\mathbf{x}|^{-\mathrm{d}-2\theta}$
- Non-local means: smooth $\phi \in C^1$ — local vs. global

[5] Ha & ET(2008); Carrillo et. al.(2012); [5b] Caffarelli-Vasseur, Kiselev-Nazarov, Constantin-Vicol, ... [5c] Mellet, Vasseur, C. Yu

Agent-base model vs. hydrodynamic description

Vicsek model: agent-base model vs. hydrodynamic description



Classical solutions must flock (with C. Tan)

$$\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$$ subject to compactly supported ρ_0

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} = \alpha(x)\,(\overline{\mathbf{u}} - \mathbf{u}), \quad \overline{\mathbf{u}}(t,\mathbf{x}) := \int_{\mathbf{y}} a(\mathbf{x},\mathbf{y})\big(\mathbf{u}(t,\mathbf{y})\rho(t,\mathbf{y})\,\mathrm{d}\mathbf{y}$$

Theorem[6]. Set diameter $\big[\mathbf{u}(t)\big] := \sup_{\mathbf{x},\mathbf{y} \in \operatorname{Supp}\rho(t)} \big|\mathbf{u}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{y})\big|$

If $\mathbf{u} \in C^1$ $\rightsquigarrow$ $\dfrac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{u}(t)\big] \leqslant -\alpha \min_{\mathbf{x},\mathbf{x}'} \eta(\mathbf{x},\mathbf{x}')\,\big[\mathbf{u}(t)\big]$:

$\eta(\mathbf{x},\mathbf{x}')$ is coefficient of ergodicity $:= \displaystyle\int_{\operatorname{Supp}\rho(t)} \min\big\{a(\mathbf{x},\mathbf{y}), a(\mathbf{x}',\mathbf{y})\big\}\,\mathrm{d}\mathbf{y}$

$\eta(\mathbf{x},\mathbf{x}') \geqslant \phi(\big[\mathbf{x}(t)\big])$ $\rightsquigarrow$ $\dfrac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{u}(t)\big] \leqslant -\alpha\phi(\big[\mathbf{x}(t)\big])\big[\mathbf{u}(t)\big]$, $\dfrac{\mathrm{d}}{\mathrm{d}t}\big[\mathbf{x}(t)\big] \leqslant \big[\mathbf{u}(t)\big]$

Then — Unconditional flocking[6b]: $\displaystyle\int^{\infty} \phi(s)\mathrm{d}s = \infty$ implies

(i) $\big[\operatorname{Supp}(\rho)\big](t,\cdot) \leqslant S_\infty < \infty$; (ii) $\big[\mathbf{u}(t)\big]\Big|_{\operatorname{Supp}(\rho)} \xrightarrow{t\to\infty} 0$

Proof — Lagrangian...

- Does $\mathbf{u}(t,\cdot) \in C^1$? "expects" conditional regularity $\mapsto$ critical thresholds

[6] ET & C. Tan, Proc. Roy. Soc. A (2014); [6b] S.-Y. Ha & J.G. Liu, CMS (2009)

Critical threshold - 1D alignment


- 1D alignment $\qquad \rho_t + (\rho u)_x = 0$

$$u_t + uu_x = \int \phi(|x-y|)\big(u(t,y) - u(t,x)\big)\rho(t,y)\mathrm{d}y$$

$$\rho' := (\partial_t + u\partial_x)\rho = -\rho \mathrm{d}, \qquad \{\cdot\}' := \partial_t + \partial_x \ \text{ and } \ \mathrm{d} := u_x$$

$$\mathrm{d}' + {\color{red}\mathrm{d}^2} = -(\phi * \rho)' - \mathrm{d}(\phi * \rho)$$

- Differentiate along the particle path: set $\{\cdot\}' := \partial_t + \partial_x$ and $\mathrm{d} := u_x$

$$\rho = -\rho d {\color{red}\rho' = -\rho \mathrm{d}}$$

$$\mathrm{d}' + {\color{red}\mathrm{d}^2} = \overbrace{\phi * (\rho u)_x}^{-(\phi * \rho)_t} - u(\phi * \rho)_x - u_x(\phi * \rho) = -(\partial_t + u\partial_x)\phi * \rho - d(\phi * \rho)$$

- Riccati balanced by alignment: $\quad (\mathrm{d} + \phi * \rho)' = -{\color{red}\mathrm{d}}({\color{red}\mathrm{d}} + \phi * \rho)$

- Sub-critical data[7] — Global smooth solution iff $\ {\color{red}u_0'(x) + \phi * \rho_0(x) \geqslant 0}$

[7] Y.-P. Choi, J. Carrillo, ET, C. Tan (2015)

Alignment and Poisson forcing

$$u_t + uu_x = \int \phi(|x-y|)(u(t,y) - u(t,x)\rho(t,y)\mathrm{d}y - \kappa\psi_x, \ -\psi_{xx} = \rho$$

$$\rho_t + (\rho u)_x = 0$$

- Critical threshold:

  Global smooth solution if $u_0'(x) > -\phi * \rho_0(x) + \sigma_+(x)$

  Finite time breakdown if $\exists x$ s.t. $u_0'(x) > -\phi * \rho_0(x) - \sqrt{2\kappa\rho_0(x)}$

Systems: spectral dynamics[8]

- $$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \nabla_{\mathbf{x}}\Psi[\mathbf{u},\nabla_{\mathbf{x}}\mathbf{u},\rho\ldots], \qquad \rho_t + \nabla_{\mathbf{x}}\cdot(\rho\mathbf{u}) = 0$$
- Key issue: control of the matrix $D := \left(\frac{\partial u_i}{\partial x_j}\right), \ i,j = 1,2,\ldots,d$

  satisfies a Riccati equation

$$D_t + \mathbf{u}\cdot\nabla_{\mathbf{x}} D + D^2 = \partial^2_{\mathbf{x}}\Psi, \qquad \partial^2_{\mathbf{x}}\Psi = \left(\frac{\partial^2\Psi}{\partial x_i \partial x_j}\right)_{i,j=1,\ldots,d}$$

- Spectral dynamics: $\lambda(D)$ an eigenvalue w/eigenpair $\langle \ell, r\rangle = 1$

$$\boxed{\partial_t\lambda_i + \mathbf{u}\cdot\nabla_{\mathbf{x}}\lambda_i + \lambda_i^2 = \langle\, \ell_i, \partial^2_{\mathbf{x}}\Psi\, r_i\,\rangle \qquad i = 1,2,\ldots,d}$$

— Difficult interaction of eigenstructure–forcing $\cdots\ \langle\, \ell, \partial^2_{\mathbf{x}}\Psi\, r\,\rangle$

[8] Liu & ET, Spectral dynamics of the velocity gradient field in restricted flows (2002)

Critical thresholds in 2D Eulerian dynamics

- 2D rotation $\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \dfrac{1}{\alpha}\mathbf{u}^\top$:

  Global smooth solution iff $2\alpha\omega_0 + \alpha^2\eta_0^2 < 1, \quad \eta := \lambda_2\left(\frac{\partial u_i}{\partial x_j}\right) - \lambda_1\left(\frac{\partial u_i}{\partial x_j}\right)$
- Rotation prevents finite-time breakdown...shallow water equations[9]
- Restricted dynamics: $\quad R_iR_j(\star) \mapsto \frac{1}{d}(\star)\mathbb{I}_d$

$$D_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}D + D^2 = \partial_{\mathbf{x}}^2\Delta_{\mathbf{x}}^{-1}\mathrm{trace}(D^2) = R_iR_j(\mathrm{trace}(D^2)) \rightsquigarrow \frac{1}{d}\mathrm{trace}(D^2)\mathbb{I}_d$$

- Euler-Poisson: $D_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}D + D^2 = \kappa\partial_{\mathbf{x}}^2\Delta_{\mathbf{x}}^{-1}(\rho) = \kappa R_iR_j(\rho) \rightsquigarrow \dfrac{1}{d}\rho\mathbb{I}_d$

The solution of 2D REP remains smooth for all time iff...

Dependence on divergence $d := \lambda_1 + \lambda_2$ and the spectral gap $\eta := \lambda_2 - \lambda_1$

$$\left\{\begin{array}{l} \eta_0(x) \leqslant 0 \quad \text{and} \quad \left\{\begin{array}{ll} d_0(x) \geqslant 0, & \text{if } \rho_0(x) = 0 \\ d_0(x) \text{ arbitrary}, & \text{if } \rho_0(x) > 0 \end{array}\right\} \\ \text{OR} \\ \rho_0(x) > 0, \eta_0(x) > 0 \quad \text{and} \quad d_0(x) > g(\rho_0(x), \eta_0(x)) \end{array}\right\} \quad \forall x \in \mathbb{R}^2$$

- Critical surface: $\quad g(\rho,\eta) := sgn(\eta^2 - 2\kappa\rho)\sqrt{\eta^2 - 2\kappa\rho + 2\kappa\rho\ ln\left(\frac{2\kappa}{\eta^2}\right)}$

[9] B. Cheng & ET (2008); [9b] H. Liu, ET, D. Wei (2010)

Critical thresholds in 2D flocking hydrodynamics

- What about the 2D case? Critical threshold in 2D self-alignment

$$
\begin{aligned}
\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) &= 0 \quad \text{subject to } \underline{\text{compactly supported}}\ \rho_0 \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla_{\mathbf{x}})\mathbf{u} &= \alpha(\mathbf{x})\left(\overline{\mathbf{u}}(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})\right)
\end{aligned}
$$

- Ricatti for $D := \{\partial_i \mathbf{u}_j\}$ is studied in terms of its spectral dynamics[10] ...

  $\{\partial_i \mathbf{u}_j\}(t,\cdot)$ remain bounded (and hence flock) if
  - (i) the initial divergence $\text{div}_{\mathbf{x}}\mathbf{u}(0)$ is not too negative;
  - (ii) the spectral gap $\eta := \lambda_2(D) - \lambda_1(D)$ is not too large:

$$
\text{Critical threshold}: \quad \text{div}_{\mathbf{x}}\mathbf{u}_0 > \sigma_+(\eta_0) \quad \rightsquigarrow \quad \nabla_{\mathbf{x}}\mathbf{u}(t,\mathbf{x})\Big|_{\text{Supp}(\rho)} < \infty
$$

- Flocking hydrodynamics with tendency ...
- Why the spectral gap?

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[10] ET & C. Tan, Critical thresholds in flocking hydrodynamics... (2014)

THANK YOU