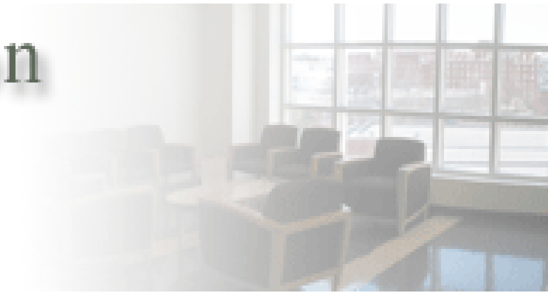

Critical Thresholds in Eulerian Dynamics

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Eulerian dynamics & questions of regularity

- Newton: $$\frac{d^2\mathbf{x}(t)}{dt^2} = \mathbf{F}, \qquad \mathbf{x} = (x_1, \ldots, x_N)^\top \in \mathrm{IR}^N$$
- Eulerian description: $\mathbf{u}(\mathbf{x},t) = \dfrac{d\mathbf{x}}{dt} = (u_1(\mathbf{x},t), \ldots, u_N(\mathbf{x},t))^\top$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \mathbf{F} : \qquad \frac{\partial u_i}{\partial t} + \sum_{k=1}^{N} u_k \frac{\partial u_i}{\partial x_k} = F_i, \qquad i = 1, 2, \ldots, N$$

⊙ velocity $\mathbf{u}(\mathbf{x},t)$ is governed by forcing $\mathbf{F} = \mathbf{F}[\mathbf{u}, \nabla_{\mathbf{x}}\mathbf{u}, \ldots]$

Q.: whether smooth solutions develop singularity in a finite time?

Answer — possible scenarios:

No – global smooth solutions: $\mathbf{u}(\cdot,t)$ remains smooth for all time

Yes – finite time breakdown: shocks, singularities,.. $|\nabla_{\mathbf{x}}\mathbf{u}(\cdot,t_c)| \uparrow \infty$

- Critical threshold phenomena: regularity depends on initial configurations

The prototype example of Euler-Poisson equations

$$\rho_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u}) = 0$$

$$(\rho \mathbf{u})_t + \nabla_{\mathbf{x}} \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = \rho \mathbf{F}$$

⊙ Eulerian dynamics: $\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = \mathbf{F}$

- Density $\rho := \rho(\mathbf{x}, t)$; velocity $\mathbf{u} := \mathbf{u}(\mathbf{x}, t)$; Forcing $\mathbf{F} = \mathbf{F}[\mathbf{u}, \nabla_{\mathbf{x}} \mathbf{u}, ...]$

$$\mathbf{F} = \overbrace{-\kappa \nabla_{\mathbf{x}} \phi}^{\text{Electric field}} + \frac{A}{\rho} \overbrace{\nabla_{\mathbf{x}} p(\rho)}^{\text{pressure}} + \text{relaxation} + \text{dissipation} + \ldots$$

- Poissonian potential $\phi := \phi(\mathbf{x}, t)$: $-\Delta \phi = \rho + \text{background}$

- Applications: semi-conductors, evolution of galaxies, . . .

$\kappa \neq 0$ — a scaled Debye constant:

$\kappa > 0$ repulsive forcing; $\kappa < 0$ attractive forcing

The example of Euler-Poisson equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = -\kappa \nabla_{\mathbf{x}} \phi + \frac{A}{\rho} \nabla_{\mathbf{x}} p(\rho)$$

- $\kappa = 0$: isentropic model of compressible Euler equations finite time blowup — one-dimensional shocks (Lax '72)

- State of the theory (prototype):

Results	Method	initial data
Local regularity $t \in [0, T]$	Energy method	all $(\rho_0 > 0, u_0) \in H^s$
Weak solution $t < \infty$	compactness	all $(\rho_0 > 0, u_0) \in BV$
Global regularity $t < \infty$	energy method	small perturbation
Finite time blowup $t = t_c$	global invariant	large initial data
Critical Threshold	spectral dynamics	'generic' initial data

- A partial list of the experts:

G.-Q. Chen, Donatelli, Engelberg, Gamblin, Y. Guo, T. Luo, Makino, Marcati, Markowich, Natalini, Perthame, Schmeiser, Ukai, D. Wang, Z. Xin, ...

One-dimensional Euler-Poisson equation

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \quad x \in \mathbb{R}, \\ u_t + u u_x &= -\kappa \phi_x \end{aligned}$$

— smooth initial data: $\rho(x,0) = \rho_0(x) > 0, \quad u(x,0) = u_0(x)$

- no pressure; zero background: $-\phi_{xx} = \rho$
- Global smooth solution if $\boxed{u_0'(x) > -\sqrt{2\kappa\rho_0(x)}, \quad \forall x \in \mathbb{R}}$
- Breakdown: if $\exists$ an x s.t. $u_0'(x) \leq -\sqrt{2\kappa\rho_0(x)}$

$\Rightarrow$ regularity breaks down at a finite $t = t_c$: $u(\cdot, t_c) \downarrow -\infty$

- Burgers equation $\kappa = 0$: 'generic' breakdown unless $u_0(x) \uparrow \forall x$
- Critical threshold ($\kappa > 0$):

Global solutions for large set of 'generic' initial configurations

Critical threshold in one-dimensional Euler-Poisson

- Mass equation: $\rho_t + (\rho u)_x = 0$ reads, $\boxed{d := u_x}$

$$(\partial_t + u\partial_x)\rho + u_x\rho = 0 \Longrightarrow \qquad \boxed{\rho' + d\rho = 0} \qquad (1)$$

- ∂_x(Balance equation: $u_t + uu_x = \kappa\phi_x$) reads

$$(\partial_t + u\partial_x)u_x + u_x^2 = \kappa\rho \Longrightarrow \qquad \boxed{d' + d^2 = \kappa\rho} \qquad (2)$$

⊙ Linear stability is of no help: $\lambda \begin{pmatrix} 0 & 0 \\ \kappa & 0 \end{pmatrix} = 0$

⊙ Manipulate: $\rho \times (2) - d \times (1) = \kappa\rho^2 \Longrightarrow \left(\dfrac{d}{\rho}\right)' = \dfrac{\rho d' - d\rho'}{\rho^2} = \kappa$

⊙ Decoupling: $\dfrac{d}{\rho} = \kappa t + \dfrac{u_0'}{\rho_0} \quad \Longrightarrow \quad d' + d^2 = \dfrac{\kappa d}{\kappa t + u_0'/\rho_0}$

- Nonlinear resonance: $u_x = d = \dfrac{u_0' + \kappa\rho_0 t}{1 + u_0' t + \kappa\rho_0 \frac{t^2}{2}}$

- Geometry of characteristics: straight lines ($\kappa = 0$) $\to$ parabolas ($\kappa > 0$)

More on one-dimensional Euler-Poisson $u_t + uu_x = F$

- Adding pressure: $F[u, u_x] = -\kappa\phi_x + \dfrac{A}{\rho}(\rho^\gamma)_x, \ \gamma \geq 1$

<u>Thm</u> (w/Dongming Wei) Global smooth solution iff

$$u_0'(x) \geq -\sqrt{2K\rho_0(x)} + \sqrt{A\gamma}\frac{|\rho_0'(x)|}{\left(\sqrt{\rho_0(x)}\right)^{3-\gamma}}, \qquad K = K(\kappa) \sim \kappa.$$

Poisson and pressure compete: global regularity vs. breakdown

- Adding non-zero background: $-\phi_{xx} = \rho - c$: $\ |u_0'(x)| \leq \sqrt{\kappa\big(2\rho_0(x) - c\big)}$

- Adding relaxation: $u_t + uu_x = -\kappa\phi_x - \dfrac{u}{\varepsilon}$

weak vs. strong(= monotonic) relaxation depending on ε vs. $1/\sqrt{\kappa}$

$\odot$ Semi-classical limit NLSP: $i\epsilon\psi_t^\epsilon = -\frac{\epsilon^2}{2}\Delta_x\psi^\epsilon - \kappa\left(\Delta_x^{-1}(|\psi^\epsilon|^2 - c)\right)\psi^\epsilon$

- WKB ansatz $\psi^\epsilon = A_0^\epsilon e^{iS^\epsilon/\epsilon}$: $\qquad u := \nabla S^\epsilon, \ \rho := |A^\epsilon|^2$

$$\rho_t + \nabla\cdot(\rho u) = 0, \qquad u_t + u\cdot\nabla u = \kappa\nabla\Delta_x^{-1}(\rho - c) + \frac{\epsilon^2}{2}\left[\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right]$$

- Classical limit with 1D sub-critical data: $|S_0''(x)| \leq \sqrt{\kappa(2|A_0(x)|^2 - c)}$

Plan of this talk

I. Multidimensional models: spectral dynamics

II. 2D example: Poisson forcing
— Critical threshold for 2D restricted Euler-Poisson
— 2D viscosity

III. 2D examples cont'd: Rotation forcing
— Rotation prevents finite time breakdown
— Near periodic solutions for shallow-water eq's

The 2D example of Viscosity forcing

IV. 3D and 4D examples: Pressure forcing
— The 3D restricted Euler equations and ...
— A surprising 4D scenario of critical threshold

Joint works with Bin Cheng(Maryland), S. Engelberg (Jerusalem), Hailiang Liu (Iowa State), Dongming Wei (Maryland)

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I. The multidimensional case — Spectral Dynamics

- $N = 1$ Key issue: control of the scalar $d = u_x$
- Critical Threshold phenomena for multidimensional systems:
  Velocity $\mathbf{u} = (u_1, \ldots, u_N)^\top$; Forcing $\mathbf{F} = \{F_i[\mathbf{u}, \nabla_\mathbf{x}\mathbf{u}, ...]\}_{i=1}^N$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla_\mathbf{x}\mathbf{u} = \mathbf{F}[\mathbf{u}, \nabla_x \mathbf{u}, \ldots]$$

Key point: balance of nonlinearities: $\mathbf{F} = \mathbf{F}[\mathbf{u}, \nabla_\mathbf{x}\mathbf{u}, ...]$ vs. $\mathbf{u} \cdot \nabla_\mathbf{x}\mathbf{u}$

Key issue: control of the matrix $D := \left(\frac{\partial u_i}{\partial x_j}\right), \ i, j = 1, 2, \ldots, N$

$$D_t + \mathbf{u} \cdot \nabla_\mathbf{x} D + D^2 = \nabla_\mathbf{x}\mathbf{F}, \qquad \nabla_\mathbf{x}\mathbf{F} = \left(\frac{\partial F_i}{\partial x_j}\right)_{i,j=1,\ldots,N}$$

- Spectral dynamics: $\lambda(D)$ an eigenvalue w/eigenpair $\langle \ell, r \rangle = 1$

$$\partial_t \lambda_i + \mathbf{u} \cdot \nabla_\mathbf{x} \lambda_i + \lambda_i^2 = \langle \nabla_\mathbf{x}\mathbf{F} \ell_i, r_i \rangle \qquad i = 1, 2, \ldots, N$$

— Difficult interaction of eigenstructure–forcing $\cdots \ \langle \nabla_\mathbf{x}\mathbf{F}\ell, r \rangle$

II. Multidimensional Euler-Poisson: $\mathbf{F} = -\kappa\nabla\phi, \ -\Delta\phi = \rho$

- Poisson forcing: $\nabla_{\mathbf{x}}\mathbf{F} = -\kappa\partial_i\partial_j\phi = \kappa\partial_i\partial_j\Delta^{-1}[\rho] =: \kappa R[\rho]$

$$R[\rho] = \partial_i\partial_j\Delta^{-1}\rho = \frac{\rho}{N}\delta_{ij} + \underbrace{\fint_{\mathrm{IR}^N} \frac{|x-y|^2\delta_{ij} - N(x_i-y_i)(x_j-y_j)}{|x-y|^{N+2}}\rho(y)dy}$$

- Restricted Euler-Poisson: $R[\rho] = \frac{\rho}{N}I_{N\times N} + \ldots\ldots\ldots\ldots \rightarrow \frac{\rho}{N}I_{N\times N}$

Retaining the local part of the global term $R[\rho]$; more later...

- Spectral dynamics – scalar forcing: $\langle\nabla_{\mathbf{x}}\mathbf{F}\ell_i, r_i\rangle = \kappa\langle R[\rho]\ell_i, r_i\rangle \rightarrow \kappa\frac{\rho}{N}$

$$\partial_t\lambda_i + \mathbf{u}\cdot\nabla_{\mathbf{x}}\lambda_i + \lambda_i^2 = \kappa\frac{\rho}{N}, \quad i = 1, \cdots, N$$

... and ρ is determined by mass equation: $\rho_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\rho + \rho\nabla_{\mathbf{x}}\cdot\mathbf{u} = 0$:

$$\partial_t\rho + \mathbf{u}\cdot\nabla_{\mathbf{x}}\rho + \rho\sum_{j=1}^{N}\lambda_j = 0$$

⊙ Turn to the 2D $N = 2$-case....

Critical threshold in 2D Restricted Euler-Poisson (REP)

⊙ spectral dynamics along particle path:

$$\lambda_i' + \lambda_i^2 = \kappa \frac{\rho}{N}, \qquad \{\cdot\}' := \partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}$$

$$(1): \quad \lambda_1' + \lambda_1^2 = \kappa \frac{\rho}{2} \qquad (2): \quad \lambda_2' + \lambda_2^2 = \kappa \frac{\rho}{2}$$

⊙ Take the difference – let $\eta := \lambda_2 - \lambda_1$ be the spectral gap –

- (#2) – (#1) $\longrightarrow$: $\quad \eta' + \eta \times (\lambda_1 + \lambda_2) = 0$
- mass eq.:$\rho_t + \mathbf{u} \cdot \nabla_x \rho + \rho \cdot div_x \mathbf{u} = 0 \rightarrow$: $\rho' + \rho \times (\lambda_1 + \lambda_2) = 0$

$$\left(\frac{\eta}{\rho}\right)' = 0$$

⊙ 2D spectral invariant: $\dfrac{\lambda_2 - \lambda_1}{\rho} = Const.$ along particle path

Critical threshold in 2D Restricted Euler-Poisson (REP)

Thm(w/H. Liu)

The solution of 2D REP remains smooth for all time iff

$$d_0(x) > g\Big(\rho_0(x), \eta_0(x)\Big) \quad \forall x \in \mathbb{R}^2$$

- Critical surface: $g(\rho, \eta) := sgn(\eta^2 - 2k\rho)\sqrt{\eta^2 - 2\kappa\rho + 2\kappa\rho\ ln\left(\frac{2\kappa}{\eta^2}\right)}$

⊙ Dependence on the spectral gap $\eta := \lambda_1 - \lambda_2$, $d := \lambda_1 + \lambda_2$

⊙ Example: Solutions of the 2D REP remains smooth for all time if both $\lambda_i(0)$ are complex: $Im(\lambda_i(\alpha, 0)) \neq 0$, $i = 1, 2$.

- Non-zero background $-\Delta\phi = \rho - c$:

  Critical threshold consists of union of several critical surfaces

OPEN QUESTIONS

- Q. What happens with the full Euler-Poisson $\nabla_{\mathbf{x}}\mathbf{F} = R[\rho]$?

⊙ On the transport of the Riesz matrix $R[\rho]$

- Q. Adding pressure – competition with Poisson forcing

- Q. Who plays the role of spectral gap in 3D?

⊙ 3D REP spectral invariant: $\dfrac{(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1)}{\rho^2} = Const.$

III. 2D example: rotation prevents finite time breakdown

$$2D: \qquad \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \frac{1}{\alpha}J\mathbf{u}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

- Spectral dynamics: $\lambda_i = \lambda_i(D), \quad D = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix}$

- Forcing $\mathbf{F} = \dfrac{1}{\alpha}J\mathbf{u}$ is local but non-isotropic: $\langle \nabla_{\mathbf{x}}\mathbf{F}\ell, r\rangle \propto \langle JD\ell, r\rangle$

$$\begin{aligned}(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\lambda_1 + \lambda_1^2 &= \frac{\lambda_1}{\alpha} \times \langle \ell_1, \ell_2\rangle \\ (\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\lambda_2 + \lambda_2^2 &= -\frac{\lambda_2}{\alpha} \times \langle \ell_1, \ell_2\rangle\end{aligned}$$

⊙ $\langle \ell_1, \ell_2\rangle = \frac{\omega}{\eta}$, $\quad \eta := \lambda_2 - \lambda_1$ is the spectral gap; $\quad$ ($\omega = 0 \leftrightarrow D$ symmetric)

· Difference $(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\eta + d\eta = -\frac{d\omega}{\alpha\eta}$ ...

III. 2D example: rotation prevents finite time breakdown

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- Spectral dynamics: $\lambda_i = \lambda_i(D), \quad D = \begin{pmatrix} \partial_1 u_1 & \partial_2 u_1 \\ \partial_1 u_2 & \partial_2 u_2 \end{pmatrix}$
- Forcing $\mathbf{F} = \dfrac{1}{\alpha}J\mathbf{u}$ is local but non-isotropic; $\langle \nabla_{\mathbf{x}}\mathbf{F}\ell, r\rangle \propto \langle JD\ell, r\rangle$

$$(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\lambda_1 + \lambda_1^2 = \frac{\lambda_1}{\alpha}\times\langle \ell_1, \ell_2\rangle$$
$$(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\lambda_2 + \lambda_2^2 = -\frac{\lambda_2}{\alpha}\times\langle \ell_1, \ell_2\rangle$$

$\odot$ $\langle \ell_1, \ell_2\rangle = \frac{\omega}{\eta}$, $\quad \eta := \lambda_2 - \lambda_1$ is the spectral gap; $\quad (\omega = 0 \leftrightarrow D$ symmetric)

· Difference $(\partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}})\eta + d\eta = -\frac{d\omega}{\alpha\eta}$ and sum $d := \lambda_1 + \lambda_2$ ...

$$(1) \quad \eta' + d\eta = -\frac{d\omega}{\alpha\eta} \qquad (2) \quad d' + \frac{d^2+\eta^2}{2} = -\frac{\omega}{\alpha} \qquad (3) \quad \omega' + d\omega = \frac{d}{\alpha}$$

Critical thresholds for 2D rotation:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} = \frac{1}{\alpha}J\mathbf{u}$$

⊙ Two spectral invariants: $\varphi = 1 - \alpha\omega, \ \ \varphi' = d\varphi$

$$(1) \quad \frac{2\alpha\omega + \alpha^2\eta^2 - 1}{2\alpha\omega - \alpha^2\omega^2 - 1} = \text{Const.} > 0, \qquad (2) \quad \frac{d^2 - \eta^2}{1 - \alpha\omega} = \text{Const.}$$

<u>Thm</u> (w/H. Liu) Rotation prevents finite time breakdown for

subcritical data: $\boxed{2\alpha\omega_0 + \alpha^2\eta_0^2 < 1}$

⊙ if $\eta_0^2 > 0$: global solution if $\alpha < \alpha^c_+ := -\omega_0 + \sqrt{\omega_0^2 + \eta_0^2}$;

⊙ if $\eta_0^2 < 0$: global solution if $\alpha < \alpha^c_-$ or $\alpha > \alpha^c_+$

⊙ The flow map is $2\pi\alpha$ periodic in time ... Lagrangian point of view

⊙ Conservation: $E(t) := \int \rho(\cdot,t)|\mathbf{u}(\cdot,t)|^2 dx = E_0, \quad \rho_t + \nabla_x(\rho\mathbf{u}) = 0$

Adding 'pressure': the 2D rotational shallow-water eq's

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \overbrace{g\nabla_{\mathbf{x}}h}^{\text{gravitation}} = \overbrace{fJ\mathbf{u}}^{\text{rotation}}; \quad \overbrace{h_t + \nabla_{\mathbf{x}}(h\mathbf{u}) = 0}^{\text{mass quation}};$$

- scaling — Froude #: $\beta = \frac{U}{\sqrt{gH}}$ Rossby #: $\alpha = \frac{U}{fL}$

$$\begin{aligned} h_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}h + \Big(\frac{1}{\beta} + h\Big)\nabla_{\mathbf{x}}\mathbf{u} &= 0 \\ \mathbf{u}_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}\mathbf{u} + \frac{1}{\beta}\nabla_{\mathbf{x}}h &= \frac{1}{\alpha}J\mathbf{u} \end{aligned}$$

- Assumption – rotation dominated flows: $\delta := \frac{\alpha}{\beta^2} \ll 1$

<u>Thm</u>(w/Bin Cheng) For sub-critical initial data:

there exists a smooth, near periodic solution $t \lesssim |\log(\delta)|$:

$$\|\mathbf{u}_{\alpha,\beta}(\cdot,t) - \mathbf{u}^{periodic}_{\alpha,0}(\cdot,t)\|_{H^s} \lesssim \delta\frac{e^{Ct}-1}{1-\delta e^{Ct}\|\mathbf{u}_0\|_{H^{s+3}}}.$$

⊙ Rotation delays finite-time breakdown; (no smallness of $\alpha \ll 1$)

Babin, Constantin, Chemin, Gallagher, Mahalov, Majda, Nicolaenko, Saint-Raymond, ...

2D Burgers': $\mathbf{u}^\epsilon_t + \mathbf{u}^\epsilon \cdot \nabla_{\mathbf{x}} \mathbf{u}^\epsilon = \epsilon \Delta \mathbf{u}^\epsilon, \quad \mathbf{u} = (u_1, u_2)^\top$

- Once more — it is the spectral gap:

$$\|\eta(\nabla_{\mathbf{x}} \mathbf{u}^\epsilon)(\cdot,t)\|_{L^1} \leq \|\eta(\nabla_{\mathbf{x}} \mathbf{u}^\epsilon)(\cdot,0)\|_{L^1}$$

- $\|u^\epsilon(\cdot,t)\|_{BV} \leq Const_0 \Longrightarrow \exists \lim \mathbf{u}^\epsilon = \bar{\mathbf{u}}$

$$\frac{\partial}{\partial t} u_1^\epsilon + u_1^\epsilon \frac{\partial}{\partial x_1} u_1^\epsilon + u_2^\epsilon \frac{\partial}{\partial x_2} u_1^\epsilon = \epsilon \Delta u_1^\epsilon$$

$$\frac{\partial}{\partial t} u_2^\epsilon + u_1^\epsilon \frac{\partial}{\partial x_1} u_2^\epsilon + u_2^\epsilon \frac{\partial}{\partial x_2} u_2^\epsilon = \epsilon \Delta u_2^\epsilon$$

Q. What is the dynamics of $\bar{\mathbf{u}}$?

A1. $\mathbf{u}_0 = \nabla_{\mathbf{x}} S_0$: $\bar{\mathbf{u}} = \nabla_x \Big(\text{viscosity sln. of 2-D Eikonal } S_t + |\nabla S|^2 = 0\Big)$:

L^1 spectral gap, $\eta(\partial_i \partial_j S)$: $\left\| \sqrt{(\Delta S)^2 - 4(S_{xx}S_{yy} - S_{xy}^2)}(\cdot,t) \right\|_{L^1_{loc}(R^2)} \downarrow$

- General $\mathbf{u}_0$: a proper weak formulation for the limit?

IV. Euler and Restricted Euler

- Incompressible Euler equations: $\mathbf{u}_t + \mathbf{u} \cdot \nabla_x \mathbf{u} = -\nabla_x p$

⊙ It's this pressure again....$-\Delta p = div\ \mathbf{u} \cdot \nabla_{\mathbf{x}} \mathbf{u} = trace(\nabla_{\mathbf{x}} \mathbf{u})^2$

$$\nabla_x \mathbf{F} = -\partial_i \partial_j p = \partial_i \partial_j \Delta^{-1}[trace(D^2)] = R[trace(D^2)]$$

- Full Euler equations: $D_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} D + D^2 = R[trace(D^2)]$
- Restricted Euler model: Léorat, 1975, Vieillefosse, 1982:

$$R[trace(D^2)] \to \frac{trace(D^2)}{N} I_{N \times N} : \quad D_t + \mathbf{u} \cdot \nabla_{\mathbf{x}} D + D^2 = \frac{trace D^2}{N} I_{N \times N}.$$

⊙ Retains incompressibility: $(\partial_t + \mathbf{u} \cdot \nabla_{\mathbf{x}}) trace D = 0$

- Why this model?– Vieillefosse, Cantwell, Shraimann, Pumir, Siggia, Pelz,

— localized model of Euler/Navier-Stokes equations

—describe the local (blow-up?) topology of Euler eq's

(Beale-Kato-Majda - $\|\omega(\cdot, t)\|_{L^1([0,T_c-],L^\infty)} \uparrow \infty$)

— capture certain statistical features of physical flow

— restricted model for incompressible MHD

Spectral Dynamics for restricted Euler model

The nonlinear dependence: $\lambda = \lambda(D)$

- Spectral dynamics: $D' + D^2 = \frac{trace(D^2)}{N} I_{N\times N} \qquad ' \equiv \partial_t + \mathbf{u}\cdot\nabla_{\mathbf{x}}$

$$\lambda_i' + \lambda_i^2 = \frac{1}{N}\sum_{k=1}^{N}\lambda_k^2, \qquad i = 1, \cdots, N$$

- Spectral invariants: $(\lambda_i - \lambda_j)' + (\lambda_i - \lambda_j)(\lambda_i + \lambda_j) = 0$

$$\left(\sum \ln(\lambda_i - \lambda_j)\right)' = -\sum(\lambda_i + \lambda_j) \stackrel{?}{=} 0$$

⊙ Incompressibility: $\sum_{i=1}^{N} \lambda_i(t) = 0$

Q. Seek $\prod_{(i,j)\in\mathcal{I}} (\lambda_i(t) - \lambda_j(t)) =$ Const. $(i,j) \in \mathcal{I}$ such that ...

$$\sum_{(i,j)\in\mathcal{I}} (\lambda_i + \lambda_j) \;\propto\; \sum_k \lambda_k \;\ldots\; = 0$$

Ans. $\#\{\mathcal{I}\} \geq \left[\frac{N}{2}\right]$ independent spectral invariants.

3D finite time breakdown

- 3D spectral invariant: $(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)(\lambda_3 - \lambda_1) = Const.$

indeed $\{(1,2),(2,3),(3,1)\} \in \mathcal{I}$:

$$\lambda_1 + \lambda_2 + \lambda_2 + \lambda_3 + \lambda_3 + \lambda_1 = 2(\lambda_1 + \lambda_2 + \lambda_3) = 0$$

Thm 3D Global solutions iff $\Lambda_0 := (\lambda_{10}, \lambda_{20}, \lambda_{30}) = (1,1,-2) \times a(x)$

Dilation: $\Lambda_0 \times a(x)$; permutation: $\Lambda_0 = (1,-2,1),(-2,1,1) \times a(x)$

Finite time breakdown at finite time, t_c, where $\lambda_i \sim \frac{1}{t-t_c}$.

- 3D RE blow-up is generic except for one point projection
- Vieillefosse, ...

Critical thresholds for 4D restricted Euler

- Two spectral invariants: $(\lambda_1 - \lambda_2)(\lambda_3 - \lambda_4)$ & $(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)$

  $\{(1,2),(3,4)\}$ and $\{(1,3),(2,4)\} \in \mathcal{I}$: $\quad \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$

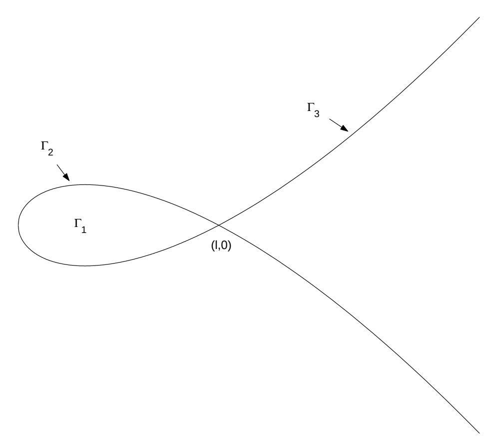
  Thm (w/Hailiang Liu and Dongming Wei)

  Global smooth solutions iff $\Lambda_0 := (\lambda_{10}, \lambda_{20}, \lambda_{30}, \lambda_{40}) \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$,

  Γ_3: real eigenvalues $\Lambda_0 = (-1+s, -1, -1, 3-s) \times a(x)$, $0 \leq s \leq 4$

  Γ_2: 1 complex pair + 2 real e.v. $\Lambda_0 = (r+i, r-i, -r, -r) \times a(x)$

  Γ_1: 2 complex pairs$\Lambda_0 = (r+bi, r-bi, -r+ci, -r-ci) \times a(x)$, $bc \neq 0$



$m_2 - m_4$ space

OPEN QUESTIONS

Q. What does the restricted model tell us about the full Euler equations?

THANK YOU

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