Computation of entropic measure-valued solutions for Euler equations

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Prologue. Perfect derivatives and conservative differences

$$\underbrace{u_x}_{\downarrow} \quad \rightsquigarrow \quad \int_a^b u_x\,dx = \text{boundary terms of } u$$

$$\frac{u_{\nu+1}-u_{\nu-1}}{2\Delta x} \quad \rightsquigarrow \quad \sum\left(\frac{u_{\nu+1}-u_{\nu-1}}{2\Delta x}\right)\Delta x = \text{boundary terms}$$

$$\underbrace{uu_x}_{\downarrow} \quad \rightsquigarrow \quad \int_a^b \overbrace{uu_x}^{\frac{1}{2}(u^2)_x}\,dx = \text{boundary terms of } u$$

$$u_\nu\left(\frac{u_{\nu+1}-u_{\nu-1}}{2\Delta x}\right) \rightsquigarrow \overbrace{\sum\left(\frac{u_\nu u_{\nu+1}-u_{\nu-1}u_\nu}{2\Delta x}\right)}^{\text{telescoping sum}}\Delta x = \text{quadratic boundary terms}$$

- but ... $\underbrace{u^3u_x}_{\downarrow} \quad \rightsquigarrow \quad \int_a^b \overbrace{u^3u_x}^{\frac{1}{4}(u^4)_x}\,dx = \text{quartic boundary terms of } u$

$$u_\nu^3\left(\frac{u_{\nu+1}-u_{\nu-1}}{2\Delta x}\right) \rightsquigarrow \sum\overbrace{\left(\frac{u_\nu^3u_{\nu+1}-u_{\nu-1}u_\nu^3}{2\Delta x}\right)}^{\text{no perfect deriv.}}\Delta x \quad \mapsto \quad \text{no cancellation}$$

Prologue. Perfect derivatives and conservative differences

$(\star)$ Set $u_x \approx \frac{3}{4\Delta x}\left(\frac{u_{\nu+1}^4 - u_\nu^4}{u_{\nu+1}^3 - u_\nu^3} - \frac{u_\nu^4 - u_{\nu-1}^4}{u_\nu^3 - u_{\nu-1}^3}\right)$

$$\sum \frac{3}{4\Delta x}\left(\frac{u_{\nu+1}^4 - u_\nu^4}{u_{\nu+1}^3 - u_\nu^3} - \frac{u_\nu^4 - u_{\nu-1}^4}{u_\nu^3 - u_{\nu-1}^3}\right)\Delta x \quad \mapsto \quad \text{boundary terms of } u$$

$$\sum \frac{3}{4\Delta x} u_\nu^3 \left(\frac{u_{\nu+1}^4 - u_\nu^4}{u_{\nu+1}^3 - u_\nu^3} - \frac{u_\nu^4 - u_{\nu-1}^4}{u_\nu^3 - u_{\nu-1}^3}\right)\Delta x =$$

$$= -\sum \frac{3}{4\Delta x}\left(u_{\nu+1}^3 - u_\nu^3\right)\left(\frac{u_{\nu+1}^4 - u_\nu^4}{u_{\nu+1}^3 - u_\nu^3}\right)\Delta x \;\mapsto\; \text{boundary terms of } u^4$$

- How do we come up with $(\star)$?
- Can we conserve general multipliers $\eta'(u)u_x = \eta(u)_x$
- Applications — entropic solution for nonlinear conservation laws

Strong and weak entropic solutions

- Euler equations with pressure law $p := (\gamma - 1)\left(E - \frac{\rho}{2}(u^2 + v^2)\right)$:

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho u \\ \rho v \\ E\end{bmatrix} + \frac{\partial}{\partial x_1}\begin{bmatrix}\rho u \\ \rho u^2 + p \\ \rho u v \\ u(E+p)\end{bmatrix} + \frac{\partial}{\partial x_2}\begin{bmatrix}\rho v \\ \rho u v \\ \rho v^2 + p \\ v(E+p)\end{bmatrix} = 0$$

- Nonlinear conservation laws: $\mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0; \quad \mathbf{u} := (\rho, \rho u, \rho v, E)^\top$

balance of conservative variables $\mathbf{u}$ over spatial variables $\mathbf{x} = (x_1, x_2, ...)$

- Strong solutions – pointwise values $\mathbf{u}(\mathbf{x}, t)$
- Weak solutions – give up the certainty in pointvalues

Instead – observe the cell averages $\overline{\mathbf{u}}(\mathbf{x}, t) := \frac{1}{\Delta \mathbf{x}} \int_{\mathbf{x}-\Delta\mathbf{x}/2}^{\mathbf{x}+\Delta\mathbf{x}/2} \mathbf{u}(\mathbf{y}, t) d\mathbf{y}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}(\mathbf{x}, t) + \frac{1}{\Delta \mathbf{x}}\left[\int_{\tau=t}^{t+\Delta t} \mathbf{f}\left(\mathbf{u}(\mathbf{x} + \frac{\Delta \mathbf{x}}{2}, \tau)\right) d\tau - \int_{\tau=t}^{t+\Delta t} \mathbf{f}\left(\mathbf{u}(\mathbf{x} - \frac{\Delta \mathbf{x}}{2}, \tau)\right) d\tau\right] = 0$$

- Compute cell averages and numerical fluxes: $\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) + \frac{\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}}{\Delta \mathbf{x}} = 0$

Weak solutions and entropy stability

$$\left\langle \eta'(\mathbf{u}),\ \mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) \right\rangle = 0 \rightsquigarrow \overbrace{\eta(\mathbf{u})_t}^{\text{entropy}} + \overbrace{\left\langle \eta'(\mathbf{u}), \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) \right\rangle}^{\text{perfect derivatives?}} = 0$$

- $\eta(\mathbf{u})$ is an entropy iff $\boxed{\left\langle \eta'(\mathbf{u})\ ,\ \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) \right\rangle = \nabla_{\mathbf{x}} \cdot F(\mathbf{u})}$
- Entropy stability: $\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot F(\mathbf{u}) \begin{cases} = 0 \\ \leqslant 0 \end{cases}$
- Euler equations – specific entropy $S = \ln(p\rho^{-\gamma})$:

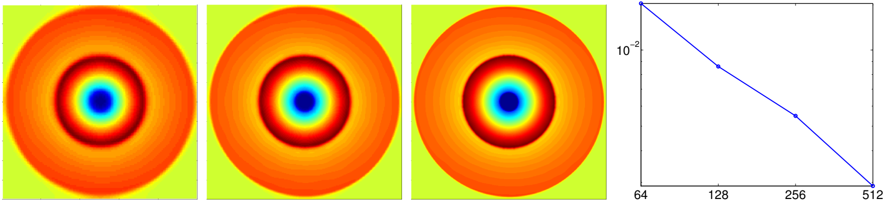
$$\overbrace{(-\rho S)_t}^{\eta(\mathbf{u})} + \nabla_{\mathbf{x}} \cdot \overbrace{(-\rho \mathbf{u} S)}^{F(\mathbf{u})} \begin{cases} = 0 \\ \leqslant 0 \end{cases}$$

- Computation – impose the entropy stability requirement

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(\overline{\mathbf{u}}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta \mathbf{x}} \begin{cases} = 0 \\ \leqslant 0 \end{cases}$$

2D radial shock tube problem

- Radial initial data $\mathbf{u}_0(\mathbf{x}) = \begin{cases} \mathbf{u}_L, & |x| \leqslant r_0 \\ \mathbf{u}_R, & |x| > r_0 \end{cases} + \epsilon(\delta\mathbf{u}_0)(\mathbf{x})$
- Perturbation of steady-state $\epsilon = 0$: $\begin{bmatrix} (\delta u_0)(\mathbf{x}) \\ (\delta v_0)(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \sin(2\pi x_1) \\ \sin(2\pi x_2) \end{bmatrix}$
- Convergence: Fix $\epsilon = 0.01$ keep mesh refinement...ρ_ϵ at $t = 0.24$:



(a) 128^2 (b) 256^2 (c) 512^2 (d) $\|u^{\Delta x} - u^{\Delta x/2}\|_{L^1}$

Figure: Approximate density of the perturbed Sod problem ($\epsilon = 0.01$) computed with TeCNO2 scheme at $t = 0.24$

- Stability as $\epsilon \downarrow 0 \ldots$

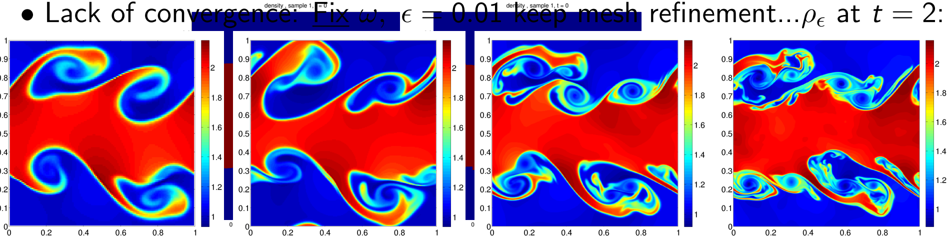
- Initial state of three layers $\mathbf{u}_0(\mathbf{x}) = \begin{cases} \mathbf{u}_L, & H_1 \leqslant x_2 < H_2 \\ \mathbf{u}_R, & 0 \leqslant x_2 \leqslant H_1 \text{ or } H_2 \leqslant x_2 \leqslant 1 \end{cases}$

  <u>fixed</u> $\mathbf{u}_{L,R}$; perturbed interface $H_j(\mathbf{x}, \omega) = \dfrac{j}{2} - \dfrac{1}{4} + \epsilon Y_j(\mathbf{x}, \omega), \quad j = 1, 2$

- $\epsilon = 0 : \quad \mathbf{u}_L \mathbf{1}(x_2)_{[\frac{1}{4}, \frac{3}{4}]} + \mathbf{u}_R \mathbf{1}(x_2)_{[0, \frac{1}{2}] \cup [\frac{3}{4}, 1]}$ is an entropic steady state
- Perturbed state — $\mathbf{u}_0(\mathbf{x}, \omega) = \mathbf{u}_L \mathbf{1}(x_2)_{[H_1, H_2]} + \mathbf{u}_R \mathbf{1}(x_2)_{[0, H_1] \cup [H_2, 1]}$:

$$Y_j(\mathbf{x}, \omega) = \sum_n a_n(\omega) \cos(b_n(\omega) + 2n\pi x_1) \quad \sum a_n = 1 \quad \text{so that} \quad |\epsilon Y_j| \leqslant \epsilon$$

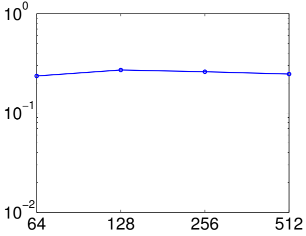
- Lack of convergence: <u>Fix</u> ω, $\epsilon = 0.01$ keep mesh refinement... ρ_ϵ at $t = 2$:



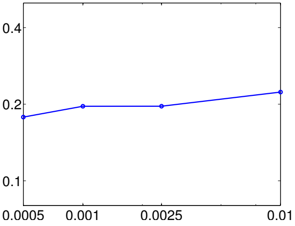
(a) 128^2 (b) 256^2 (c) 512^2 (d) 1024^2

KH initial data: amplitude perturbation (left) and phase perturbation (right) with $\epsilon = 0.01$

The notion of entropic weak solution is not enough ($d > 1$)



(e) L^1 rates vs. number of gridpoints with $\epsilon = 0.01$



(f) L^1 error with respect to ϵ, at a fixed 1024^2 mesh

Figure: L^1 differences in density ρ at time $t = 2$ for the Kelvin-Helmholtz problem

- These are NOT numerical artifacts but...
  contemplate lack of uniqueness [De Lellis, Szekelyhidi, E. Chiodaroli, O. Kreml, et. al.]

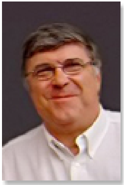
"Just because we cannot prove that compressible flows with prescribed initial values exist doesn't mean that we cannot compute them" [Lax 2007]

- The question is what information is encoded in our computations?

Answer: Young measures and measure valued solutions

- Recall weak convergence: $\mathbf{u}_n \rightharpoonup \mathbf{u}_\infty$ and $\mathbf{f}(\mathbf{u}_n) \rightharpoonup \mathbf{f}_\infty$
  What is the relation $\mathbf{f}_\infty$ and $\mathbf{f}(\mathbf{u}_\infty)$?
- $\mathbf{f}_\infty(\mathbf{x}, t)$ depends linearly and positively on $\mathbf{f}$:

  hence $\mathbf{f}_\infty(\mathbf{x}, t) = \langle \nu_{\mathbf{x},t}, \mathbf{f} \rangle$ in terms of measure $\nu = \nu_{\mathbf{x},t}$
- Complete description in terms of Young measure $\nu_{\mathbf{x},t}$



Luc Tartar

Certainty: strong convergence with Dirac Mass $\nu_{\mathbf{x},t} = \delta_{\mathbf{u}_\infty(\mathbf{x},t)} \rightsquigarrow \mathbf{f}_\infty = \mathbf{f}(\mathbf{u}_\infty)$

- Entropy measure valued (EMV) solutions: $\mathbf{u}(\mathbf{x}, t) \rightsquigarrow \nu_{\mathbf{x},t}$:

$$\mathbf{u}_t + \nabla_{\mathbf{x}} \cdot \mathbf{f}(\mathbf{u}) = 0 \quad \rightsquigarrow \quad \partial_t \langle \nu_{\mathbf{x},t}, \mathrm{id} \rangle + \nabla_{\mathbf{x}} \cdot \langle \nu_{\mathbf{x},t}, \mathbf{f} \rangle = 0$$

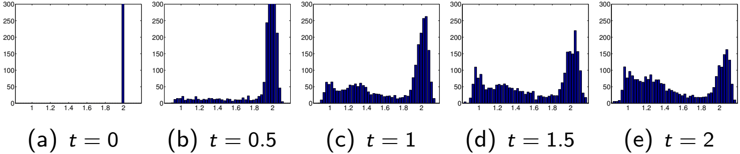
$$\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot F(\mathbf{u}) \leqslant 0 \quad \rightsquigarrow \quad \partial_t \langle \nu_{\mathbf{x},t}, \eta \rangle + \nabla_{\mathbf{x}} \cdot \langle \nu_{\mathbf{x},t}, F \rangle \leqslant 0$$

- $\nu_{\mathbf{x},t}$ is the (weighted) probability of having the value $\langle \nu_{\mathbf{x},t}, \mathrm{id} \rangle$ at $(\mathbf{x}, t)$
- Measure valued solutions –
  give up the certainty in a given solution
  Instead – observe averages in configuration space

$\rightsquigarrow$ Back to weak solutions — $\nu_{\mathbf{x},t} = \delta_{\mathbf{u}(\mathbf{x},t)}$
(certainty of strongly convergent averages)

Ron DiPerna

Young measures as quantifiers of uncertainty



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 1.5$ (e) $t = 2$

Figure: The approximate PDF for density $\rho_{\nu_{\mathbf{x},t}}$ at $\mathbf{x} = (0.5, 0.7)$.

- More about this later ...
- how do we compute these Entropic Measure-Valued solutions (EMVs)?

Computing Entropy Measure Valued (EMV) solutions

- (Formally) arbitrarily high-order of accuracy;
- Entropy stability — as a selection principle
- Essentially non-oscillatory in the presence of discontinuities;
- Convergence for linear systems;
- How to realize (the configuration space of) EMVs?
- Computationally efficient;

The class of the entropy Conservative Non-Oscillatory (TeCNO) schemes[1] which compute "faithful" samples required for the ensemble of EMVs[1b,1c]

[1] Fjordholm, Mishra, ET, Arbitrarily high-order accurate entropy stable schemes ...., SINUM 2012

[1b] Fjordholm, Kappeli, Mishra & ET, Construction of approximate entropy measure valued solutions for hyperbolic systems of conservation laws, FoCM (2015)

[1c] FMT, On the computation of measure-valued solutions, Acta Numerica (2016)

Entropy inequality: PDEs $\rightarrow$ numerical approximations

$$\left\langle \eta'(\mathbf{u}),\ \mathbf{u}_t + \mathbf{f}(\mathbf{u})_x \right\rangle = 0 \quad \overset{\left\langle \eta'(\mathbf{u}),\ \mathbf{f}(\mathbf{u})_x \right\rangle = F(\mathbf{u})_x}{\Longrightarrow} \quad \eta(\mathbf{u})_t + F(\mathbf{u})_x \begin{cases} = 0 \\ \leqslant 0 \end{cases}$$

- Semi-discrete approximations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) + \frac{1}{\Delta x}\left[\mathbf{f}_{\nu+\frac{1}{2}} - \mathbf{f}_{\nu-\frac{1}{2}}\right] = 0 \quad \mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}(\overline{\mathbf{u}}_{\nu-p+1}, \ldots, \overline{\mathbf{u}}_{\nu+p})$$

- Entropy conservative discretization — given η find $\mathbf{f}^*$ s.t.

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) + \frac{\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}}{\Delta x} = 0 \ \overset{?}{\rightsquigarrow}\ \frac{\mathrm{d}}{\mathrm{d}t}\eta(\overline{\mathbf{u}}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x} \begin{cases} = 0 \\ \leqslant 0 \end{cases}$$

Does $\boxed{\left\langle \eta'(\overline{\mathbf{u}}_\nu)\ ,\ \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right\rangle \overset{?}{\rightsquigarrow} F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}$

so that $\Longrightarrow \quad \sum_\nu \eta(\overline{\mathbf{u}}_\nu(t))\Delta x \begin{Bmatrix} = \\ \leqslant \end{Bmatrix} \sum_\nu \eta(\overline{\mathbf{u}}_\nu(0))\Delta x$

Entropy variables and entropy conservative schemes

- Fix an entropy $\eta(\mathbf{u})$. Set Entropy variables: $\mathbf{v} \equiv \mathbf{v}(\mathbf{u}) := \eta'(\mathbf{u})$.

⊙ Convexity of $\eta(\cdot)$ implies that $\mathbf{u} \longleftrightarrow \mathbf{v}$ is 1-1: $\mathbf{v}_\nu = \eta'(\overline{\mathbf{u}}_\nu) \ldots$

- Entropy conservation: $\left\langle \eta'(\overline{\mathbf{u}}_\nu)\,,\, \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right\rangle \stackrel{?}{=} F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}$

$$\overbrace{\left\langle \mathbf{v}_\nu\,,\, \mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}} \right\rangle}^{\text{perfect difference}} \quad \text{if and only if} \quad \overbrace{\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu\,,\, \mathbf{f}^*_{\nu+\frac{1}{2}} \right\rangle}^{\text{perfect difference}}:$$

- Entropy conservative flux: $$\boxed{\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu, \mathbf{f}^*_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)}$$

- Entropy flux potential: $\psi(\mathbf{v}) := \left\langle \mathbf{v}\,,\, \mathbf{f}(\mathbf{v}) \right\rangle - F(\mathbf{u}(\mathbf{v}))$

  Abbreviation — entropy conservative schemes: $\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{f}^*_{\nu+\frac{1}{2}} \right\rangle = \Delta\psi_{\nu+\frac{1}{2}}$

#1. Scalar examples: $f^*_{\nu+\frac{1}{2}} = \dfrac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu}$

- Toda flow: $u_t + (e^u)_x = 0$ with Exp entropy pair: $(e^u)_t + (e^{2u})_x = 0$

$\eta(u) = v(u) = e^u,\ F(u) = \frac{1}{2}e^{2u}$ and potential $\psi(v) := vf - F = \frac{1}{2}v^2$

$\rightsquigarrow$ Entropy-conservative flux:

$$f^*_{\nu+\frac{1}{2}} = \frac{\psi(v_{\nu+1}) - \psi(v_\nu)}{v_{\nu+1} - v_\nu} = \frac{\frac{1}{2}v^2_{\nu+1} - \frac{1}{2}v^2_\nu}{v_{\nu+1} - v_\nu} = \frac{1}{2}(v_\nu + v_{\nu+1}) = \frac{1}{2}\left[e^{u_\nu} + e^{u_{\nu+1}}\right]$$

Toda flow: $\dfrac{d}{dt}\overline{u}_\nu(t) = -\dfrac{e^{u_{\nu+1}(t)} - e^{u_{\nu-1}(t)}}{2\Delta x} \rightsquigarrow \dfrac{d}{dt}\sum e^{\overline{u}_\nu(t)}\Delta x = Const.$

- Inviscid Burgers: $u_t + (\frac{1}{2}u^2)_x = 0$ quadratic entropy $(\dfrac{1}{2}u^2)_t + (\dfrac{1}{3}u^3)_x = 0$

$\eta(u) = \dfrac{u^2}{2},\ v(u) = u,\ F(u) = \dfrac{u^3}{3}$ and potential $\psi(v) := vf - F = \frac{1}{6}u^3$

$\rightsquigarrow$ Entropy conservative "$\frac{1}{3}$-rule":

$$\frac{d}{dt}\overline{u}_\nu(t) = -\frac{2}{3}\left[\frac{u^2_{\nu+1} - u^2_{\nu-1}}{4\Delta x}\right] - \frac{1}{3}\left[u_\nu \frac{u_{\nu+1} - u_{\nu-1}}{2\Delta x}\right] \rightsquigarrow \sum \overline{u}^2_\nu(t)\Delta x = Const.$$

#2. Linear equations

- Linear transport: How to discretize u_x such that $u_t + u_x = 0$ conserves the balance of quartic perfect derivatives $\left(u^4\right)_t + \left(u^4\right)_x = 0$?
- Conservative differences of u_x and $4u^3 u_x = \left(u^4\right)_x$:

$$\eta(u) = u^4,\ v = 4u^3,\ f(u) = u,\ F = u^4 \text{ and } \psi(u) = 3u^4$$

$$\rightsquigarrow \qquad f^*_{\nu+\frac{1}{2}} = \frac{\Delta\psi_{\nu+\frac{1}{2}}}{\Delta v_{\nu+\frac{1}{2}}} = \frac{3}{4}\frac{u^4_{\nu+1} - u^4_\nu}{u^3_{\nu+1} - u^3_\nu}, \qquad u_x \approx \frac{f^*_{\nu+\frac{1}{2}} - f^*_{\nu-\frac{1}{2}}}{\Delta x}$$

$(\star)$ Indeed — set $u_{x|x=x_\nu} \approx \frac{3}{4\Delta x}\left(\frac{u^4_{\nu+1} - u^4_\nu}{u^3_{\nu+1} - u^3_\nu} - \frac{u^4_\nu - u^4_{\nu-1}}{u^3_\nu - u^3_{\nu-1}}\right)$ then

$$\sum \frac{3}{4\Delta x}\left(\frac{u^4_{\nu+1} - u^4_\nu}{u^3_{\nu+1} - u^3_\nu} - \frac{u^4_\nu - u^4_{\nu-1}}{u^3_\nu - u^3_{\nu-1}}\right)\Delta x \quad \mapsto \quad \text{boundary terms of } u$$

$$\sum \frac{3}{4\Delta x} u^3_\nu \left(\frac{u^4_{\nu+1} - u^4_\nu}{u^3_{\nu+1} - u^3_\nu} - \frac{u^4_\nu - u^4_{\nu-1}}{u^3_\nu - u^3_{\nu-1}}\right)\Delta x =$$

$$= -\sum \frac{3}{4\Delta x}\left(u^3_{\nu+1} - u^3_\nu\right)\left(\frac{u^4_{\nu+1} - u^4_\nu}{u^3_{\nu+1} - u^3_\nu}\right)\Delta x \ \mapsto \text{quartic boundary terms}$$

#3. Second and higher-order accuracy

- The u^4 entropy conservative flux is 2nd-order accurate:

$$f^{*}_{\nu+\frac{1}{2}} = \frac{3}{4}\frac{u^4_{\nu+1} - u^4_{\nu}}{u^3_{\nu+1} - u^3_{\nu}}: \quad \frac{f^{*}_{\nu+\frac{1}{2}} - f^{*}_{\nu-\frac{1}{2}}}{2\Delta x} \text{ is 2nd-order approximation of } u_x$$

$$f^{*}_{\nu+\frac{1}{2}} = \frac{\Delta\psi}{\Delta u} = \frac{3}{4}\frac{u^4_{\nu+1} - u^4_{\nu}}{u^3_{\nu+1} - u^3_{\nu}} \approx \frac{3}{4}\cdot\frac{8u^3_{\nu+\frac{1}{2}}\frac{\Delta x}{2}u_x}{6u^2_{\nu+\frac{1}{2}}\frac{\Delta x}{2}u_x} \approx u_{\nu+\frac{1}{2}}$$

- The general case: express the flux in the "viscosity form"

"viscosity form": $f^{*}_{\nu+\frac{1}{2}} = \frac{1}{2}(f_{\nu} + f_{\nu+1}) - \frac{1}{2}D^{*}_{\nu+\frac{1}{2}}(u_{\nu+1} - u_{\nu})$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{u}_{\nu}(t) = -\overbrace{\left[\frac{f(u_{\nu+1}) - f(u_{\nu-1})}{2\Delta x}\right]}^{\text{2nd order centered differencing}} + \frac{1}{2\Delta x}\overbrace{\left[D^{*}_{\nu+\frac{1}{2}}\Delta u_{\nu+\frac{1}{2}} - D^{*}_{\nu-\frac{1}{2}}\Delta u_{\nu-\frac{1}{2}}\right]}^{\Delta x^2 (Du_x)_x}$$

- 2nd-order: $f^{*}_{\nu+\frac{1}{2}} \quad \rightsquigarrow \quad D^{*}_{\nu+\frac{1}{2}} := \frac{1}{8}\left(\int_{\xi=-1}^{1}(1-\xi^2)f''(u_{\nu+\frac{1}{2}})\mathrm{d}\xi\right)\cdot\Delta u_{\nu+\frac{1}{2}}$

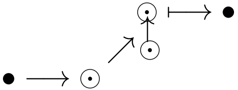
$\rightsquigarrow$ The entropy-conservative flux is 2nd-order accurate

$\rightsquigarrow$ Entropy conservative fluxes to any order of accuracy (LeFloch & Rohde 2000)

#4. Systems: $\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_{\nu}\,,\, \mathbf{f}^{*}_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_{\nu})$

- Choice of path: N linearly independent directions $\left\{\mathbf{r}^j\right\}_{j=1}^{N}$

⊙ Intermediate states $\left\{\mathbf{v}^j_{\nu+\frac{1}{2}}\right\}_{j=1}^{N}$:



Starting with $\mathbf{v}^1_{\nu+\frac{1}{2}} = \mathbf{v}_\nu$, and followed by $(\Delta\mathbf{v}_{\nu+\frac{1}{2}} \equiv \mathbf{v}_{\nu+1} - \mathbf{v}_\nu)$

$$\mathbf{v}^{j+1}_{\nu+\frac{1}{2}} = \mathbf{v}^{j}_{\nu+\frac{1}{2}} + \left\langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle \mathbf{r}^j,\ j = 1, 2, \ldots, N \quad (\mathbf{v}^{N+1}_{\nu+\frac{1}{2}} = \mathbf{v}_{\nu+1})$$

- [ET 2003] The conservative scheme $\dfrac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) = -\dfrac{1}{\Delta x}\left[\mathbf{f}^{*}_{\nu+\frac{1}{2}} - \mathbf{f}^{*}_{\nu-\frac{1}{2}}\right]$

$$\boxed{\mathbf{f}^{*}_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^{j}_{\nu+\frac{1}{2}}\right)}{\left\langle \boldsymbol{\ell}^j, \Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle} \boldsymbol{\ell}^j}$$

is entropy conservative: $\left\langle \mathbf{v}_{\nu+1} - \mathbf{v}_{\nu}\,,\, \mathbf{f}^{*}_{\nu+\frac{1}{2}} \right\rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_{\nu})$

#5. Entropy conservative Euler scheme (with W.-G. Zhong)

$$\mathbf{f}^{*}_{\nu+\frac{1}{2}} = \sum_{j=1}^{N} \frac{\psi\left(\mathbf{v}^{j+1}_{\nu+\frac{1}{2}}\right) - \psi\left(\mathbf{v}^{j}_{\nu+\frac{1}{2}}\right)}{\left\langle \ell^j, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \right\rangle} \ell^j$$

- Entropy function: $\eta(\mathbf{u}) = -\rho S$
- Euler entropy variables: $\mathbf{v}(\mathbf{u}) = \eta_{\mathbf{u}}(\mathbf{u}) = \begin{bmatrix} -E/e - S + \gamma + 1 \\ q/\theta \\ -1/\theta \end{bmatrix}$
- Euler entropy flux potential $\psi(\mathbf{v}) = \langle \mathbf{v}\,, \mathbf{f} \rangle - F(\mathbf{u}) = (\gamma - 1)m$
- path in phase-space:
- $\mathbf{v}^0 = \mathbf{v}_\nu, \quad \mathbf{v}^{j+1} = \mathbf{v}^j + \langle \ell^j, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \rangle \mathbf{r}^j, \quad \mathbf{v}^4 = \mathbf{v}_{\nu+1}$

  $\left\{\mathbf{r}^j\right\}_{j=1}^{3}$: three linearly independent directions in **v**-space (Riemann path)

  $\left\{\ell^j\right\}_{j=1}^{3}$: the corresponding orthogonal system

  $\left\{m^j\right\}_{j=1}^{3}$: intermediate values of the momentum along the path
- $$\mathbf{f}^{*}_{\nu+\frac{1}{2}} = (\gamma - 1) \sum_{j=1}^{3} \frac{m^{j+1} - m^j}{\langle \ell^j\,, \Delta \mathbf{v}_{\nu+\frac{1}{2}} \rangle} \ell^j$$

#6. Affordable entropy conservative fluxes for Euler eqs

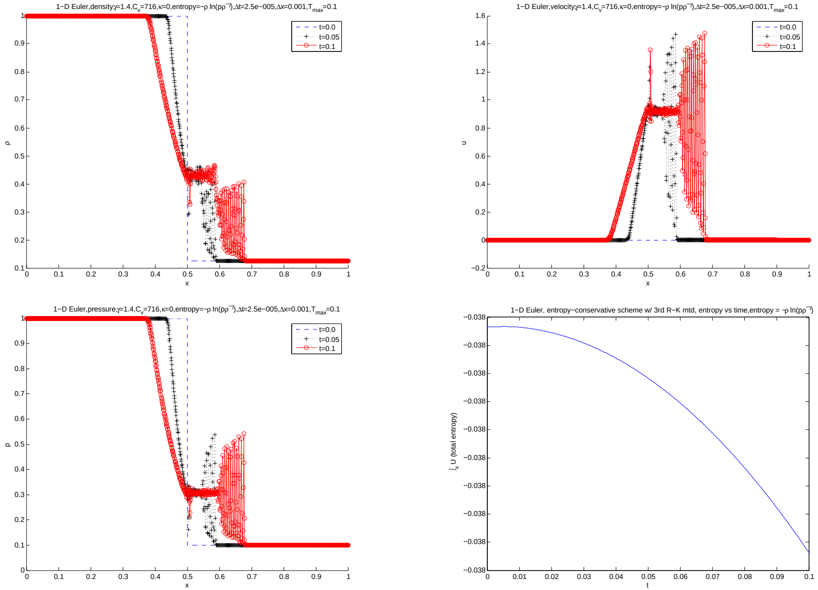
- [Ismail & Roe & (2009)] 2D Euler eqs $\mathbf{u}_t + \mathbf{f}_\mathbf{x} + \mathbf{g}_\mathbf{y} = 0$:

Express an entropy conservative flux $\mathbf{f}^*_{\nu+\frac{1}{2}} := (\mathbf{f}^1, \mathbf{f}^2, \mathbf{f}^3, \mathbf{f}^4)^\top$

in terms of $\mathbf{z} := \sqrt{\frac{\rho}{p}} \begin{bmatrix} 1 \\ u \\ v \\ p \end{bmatrix}$ and the averages $\begin{cases} \overline{z}_{\nu+\frac{1}{2}} & := & \frac{1}{2}(z_\nu + z_{\nu+1}) \\ z^{ln}_{\nu+\frac{1}{2}} & := & \dfrac{\Delta z_{\nu+\frac{1}{2}}}{\Delta \log(z)_{\nu+\frac{1}{2}}} \end{cases}$

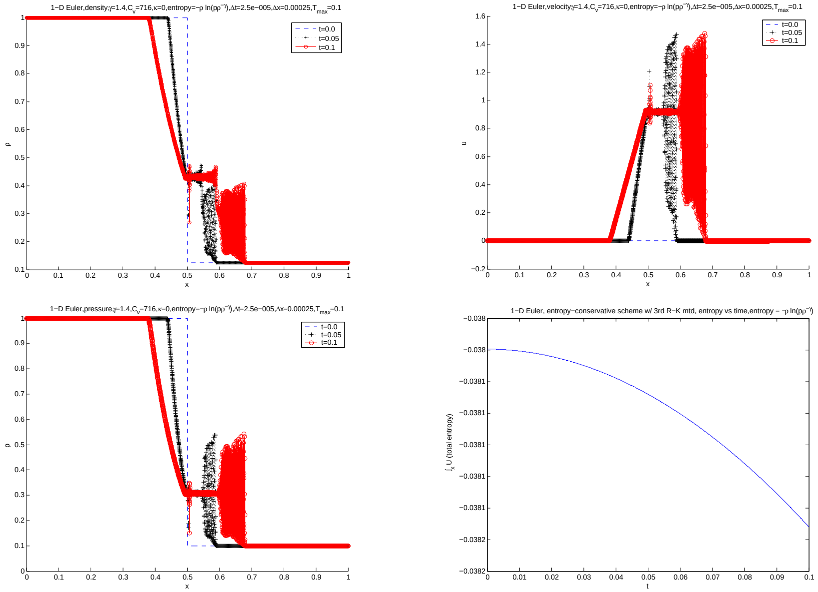
$$
\begin{aligned}
\mathbf{f}^1_{\nu+\frac{1}{2}} &= (\overline{z}_2)_{\nu+\frac{1}{2}} (z_4)^{ln}_{\nu+\frac{1}{2}} \\
\mathbf{f}^2_{\nu+\frac{1}{2}} &= \frac{(\overline{z}_4)_{\nu+\frac{1}{2}}}{(\overline{z}_1)_{\nu+\frac{1}{2}}} + \frac{(\overline{z}_2)_{\nu+\frac{1}{2}}}{(\overline{z}_1)_{\nu+\frac{1}{2}}} \mathbf{f}^1_{\nu+\frac{1}{2}} \\
\mathbf{f}^3_{\nu+\frac{1}{2}} &= \frac{(\overline{z}_3)_{\nu+\frac{1}{2}}}{(\overline{z}_1)_{\nu+\frac{1}{2}}} \mathbf{f}^1_{\nu+\frac{1}{2}} \\
\mathbf{f}^4_{\nu+\frac{1}{2}} &= \frac{1}{2(\overline{z}_1)_{\nu+\frac{1}{2}}} \left(\frac{\gamma+1}{\gamma-1} \frac{1}{(z_1)^{ln}_{\nu+\frac{1}{2}}} \mathbf{f}^1_{\nu+\frac{1}{2}} + (\overline{z}_2)_{\nu+\frac{1}{2}} \mathbf{f}^2_{\nu+\frac{1}{2}} + (\overline{z}_3)_{\nu+\frac{1}{2}} \mathbf{f}^3_{\nu+\frac{1}{2}} \right)
\end{aligned}
$$

satisfies the compatibility relation $\langle \mathbf{v}_{\nu+1} - \mathbf{v}_\nu, \mathbf{f}^*_{\nu+\frac{1}{2}} \rangle = \psi(\mathbf{v}_{\nu+1}) - \psi(\mathbf{v}_\nu)$



Entropy conservative results for Euler's Sod problem:

density, velocity, pressure & entropy. 1000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$



Entropy conservative density, velocity, pressure & entropy w/4000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln \left(p\rho^{-\gamma} \right)$

- Does not “enforce” physical solution with numerical viscosity

The question of entropy stability – dissipation

- Entropy conservation: $\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot F(\mathbf{u}) = 0$

- Entropy decay due to shock discontinuities (Lax):

$$\eta(\mathbf{u})_t + \nabla_{\mathbf{x}} \cdot F(\mathbf{u}) \leqslant 0$$

- Entropy decay is balanced by perfect derivatives:

$$\int \eta(\mathbf{u}(\mathbf{x}, t_2))d\mathbf{x} \leqslant \int \eta(\mathbf{u}(\mathbf{x}, t_1))d\mathbf{x}, \quad t_2 > t_1$$

- Q. How much entropy decay "$\leqslant$" is enough?

- Entropic measure-valued solutions:

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) + \frac{\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}}{\Delta x} = 0\epsilon(D\mathbf{v}_x)''_x\,; \quad \frac{\mathrm{d}}{\mathrm{d}t}\eta(\overline{\mathbf{u}}_\nu(t)) + \frac{F_{\nu+\frac{1}{2}} - F_{\nu-\frac{1}{2}}}{\Delta x} =\!\!\leqslant 0.$$

- ϵ dictates the small scale of dissipation ...

#7. Entropy balance in Navier-Stokes (NS) eq's

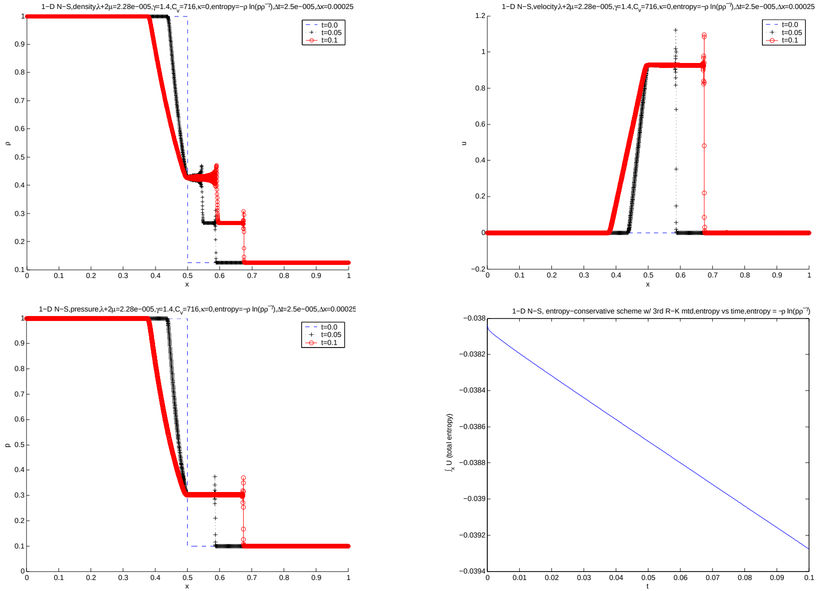
- A semi-discrete scheme of NS equations $\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \epsilon\big(D(\cdot)\mathbf{v}_x\big)_x$

$$\Big\langle \overbrace{\eta'(\overline{\mathbf{u}}_\nu)}^{\mathbf{v}_\nu}, \frac{\mathrm{d}}{\mathrm{d}t}\overline{\mathbf{u}}_\nu(t) + \frac{1}{\Delta x}\Big(\mathbf{f}^*_{\nu+\frac{1}{2}} - \mathbf{f}^*_{\nu-\frac{1}{2}}\Big) = \epsilon\Big(D_{\nu+\frac{1}{2}}\frac{\Delta \mathbf{v}_{\nu+\frac{1}{2}}}{\Delta x} - D_{\nu-\frac{1}{2}}\frac{\Delta \mathbf{v}_{\nu-\frac{1}{2}}}{\Delta x}\Big)\Big\rangle$$

- Recall — $\mathbf{f}^*_{\nu+\frac{1}{2}}$ is entropy-conservative: $\Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, \mathbf{f}^*_{\nu+\frac{1}{2}}\Big\rangle = \Delta\psi_{\nu+\frac{1}{2}}$
- Euler fluxes conserve entropy: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\sum_\nu \eta(\overline{\mathbf{u}}_\nu(t))\,\Delta x = -\sum \Delta\psi_{\nu+\frac{1}{2}} = 0$
- NS dissipate: $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\sum_\nu \eta(\overline{\mathbf{u}}_\nu(t))\,\Delta x = -\epsilon\sum_\nu \Big\langle \Delta \mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\Delta \mathbf{v}_{\nu+\frac{1}{2}}\Big\rangle \leqslant 0$

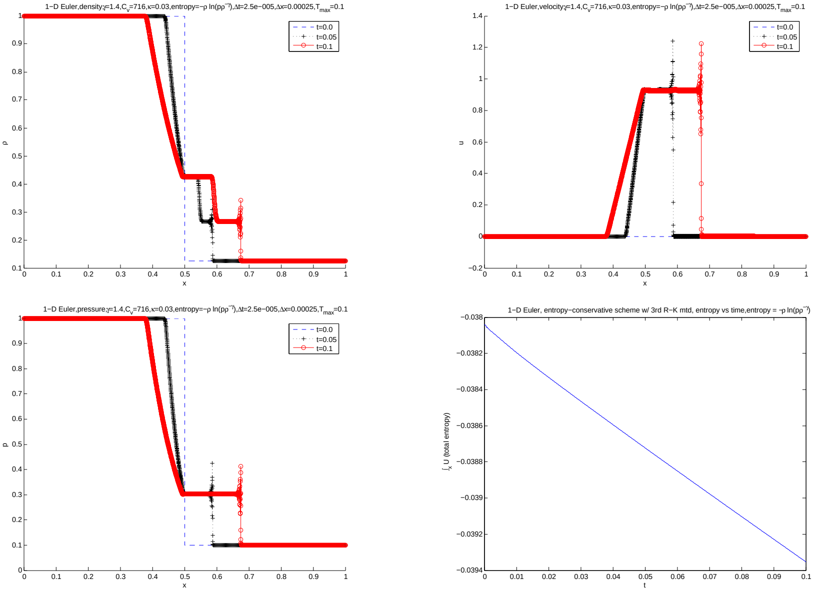
$\star$ $\displaystyle\frac{\mathrm{d}}{\mathrm{d}t}\sum_\nu(-\rho_\nu S_\nu)\Delta x =$

$$\overbrace{-(\lambda+2\mu)\sum_\nu\Big(\frac{\Delta q_{\nu+\frac{1}{2}}}{\Delta x}\Big)^2\overline{\Big(\frac{1}{\theta}\Big)}_{\nu+\frac{1}{2}}\Delta x}^{\textit{viscosity}}\ \overbrace{-\kappa\sum_\nu\Big(\frac{\Delta\theta_{\nu+\frac{1}{2}}}{\Delta x}\Big)^2\widetilde{\Big(\frac{1}{\theta}\Big)}^2_{\nu+\frac{1}{2}}\Delta x}^{\textit{heat conduction}} \leqslant 0$$



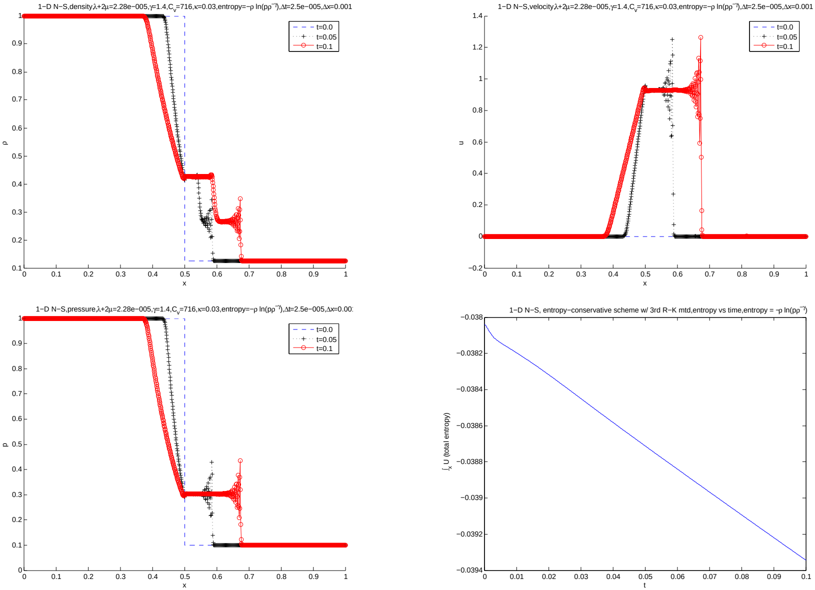
Navier-Stokes equations for Sod's problem:

viscosity but no heat conduction; 4000 spatial grids. $\eta(\mathbf{u}) = -\rho \ln \left(p\rho^{-\gamma} \right)$



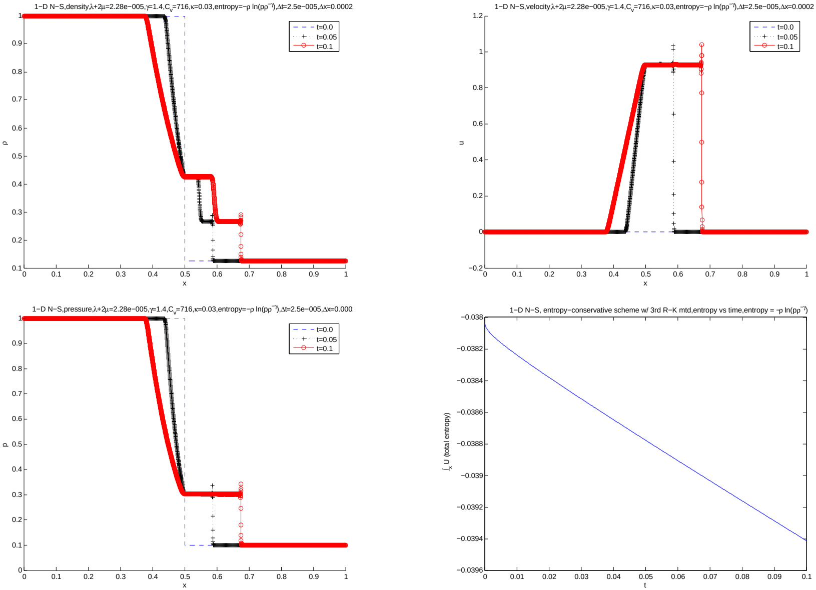
Entropy conservative results for Sod's problem:

heat conduction but no viscosity; 4000 spatial grids. $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$



Navier-Stokes equations Sod problem:

both viscosity and heat conduction; 1000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln \left(p\rho^{-\gamma}\right)$



Navier-Stokes equations Sod problem:

viscosity and heat conduction; 4000 spatial grids, $\eta(\mathbf{u}) = -\rho \ln\left(p\rho^{-\gamma}\right)$

#8. TeCNO: Arbitrarily high-order entropy stable schemes

- Set numerical viscosity $\mathbf{f}_{\nu+\frac{1}{2}} = \mathbf{f}^{*}_{\nu+\frac{1}{2}} - D_{\nu+\frac{1}{2}}(\mathbf{v}_{\nu+1} - \mathbf{v}_{\nu})$ such that

  $\mathbf{f}^{*}_{\nu+\frac{1}{2}}$ is high order and $\mathbf{f}_{\nu+\frac{1}{2}}$ entropy stable: $\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\Delta\mathbf{v}_{\nu+\frac{1}{2}} \right\rangle < 0$

- <u>Any</u> $D_{\nu+\frac{1}{2}} > 0$ will do for entropy dissipation; but ...

  To overcome first order accuracy: $D_{\nu+\frac{1}{2}} \sim \left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^{p-1}$; non-oscillatory?

- A class of **arbitrarily** high-order entropy-stable schemes:

  $(\star)$ $$\mathbf{f}_{\nu+\frac{1}{2}} := \mathbf{f}^{*}(\mathbf{v})_{\nu+\frac{1}{2}} - D_{\nu+\frac{1}{2}}\langle\!\langle \mathbf{v} \rangle\!\rangle_{\nu+\frac{1}{2}}, \quad \langle\!\langle \mathbf{v} \rangle\!\rangle_{\nu+\frac{1}{2}} := \mathbf{v}^{-}_{\nu+1} - \mathbf{v}^{+}_{\nu}$$

  Entropy stability requires $\left\langle \Delta\mathbf{v}_{\nu+\frac{1}{2}}, D_{\nu+\frac{1}{2}}\langle\!\langle \mathbf{v} \rangle\!\rangle_{\nu+\frac{1}{2}} \right\rangle < 0$...?

- Interface values $\mathbf{v}^{\pm}$: reconstructed by ENO (Harten, Engquist, Osher, Shu,..)

(i) ENO reconstruction is highly accurate: $D_{\nu+\frac{1}{2}}\langle\!\langle \mathbf{v} \rangle\!\rangle_{\nu+\frac{1}{2}} \sim \left|\Delta\mathbf{v}_{\nu+\frac{1}{2}}\right|^{p}$

(ii) It is **E**ssentially **N**on-**O**scillatory (hence the acronym...) ; and...

(iii) [FMT 2012] The sign property: $\text{sign}\langle\!\langle v \rangle\!\rangle_{\nu+\frac{1}{2}} = \text{sign}\,\Delta v_{\nu+\frac{1}{2}}$ $\rightsquigarrow$ $(\star)$!

Back to computation of entropy measure valued solutions

- EMV solutions $\nu_{\mathbf{x},t}$: $\partial_t \langle \nu_{\mathbf{x},t}, \mathrm{id} \rangle + \nabla_{\mathbf{x}} \cdot \langle \nu_{\mathbf{x},t}, \mathbf{f} \rangle = 0$
- How to compute a realization of the Young measure $\nu_{\mathbf{x},t}$?
  Every Young measure can be realized as a law of random variable:

$$\exists(\Omega, \Sigma, \mathbb{P}) : \quad \nu_{\mathbf{x},t}(E) = \mathbb{P}(\mathbf{u}(\mathbf{x}, t, \omega) \in E)$$

- **Initial data**: Realize the law of $\sigma_{\mathbf{x}} = \nu_{(\mathbf{x},t=0)}$ by a random field $\mathbf{u}_0(\mathbf{x}, \omega)$
  "Classical" case – atomic initial delta: $\sigma_{\mathbf{x}} = \delta_{\mathbf{u}_0(\mathbf{x})}$
  Beyond classical case – uncertainty in initial measurement
- Compute the entropy measure valued solutions:
  To be well-defined, we need to make sure $\{\mathbf{u}(\cdot, t, \omega)\}$ form
  a measurable ensemble in the right space $(\Omega, \Sigma, \mathbb{P}) \mapsto (X, \mathcal{B}(X))$
  ⊙ Scalar case – $X = L^1$ (Risebro, Schwab, Weber, ...)
  ⊙ Systems – ?

Example Back to 2D Euler — realize steady KH as MC average:

$$\delta_{\mathbf{u}_0}(\mathbf{x}) = \mathbb{E}\mathbf{u}_0(\mathbf{x}, \omega) \approx \frac{1}{M} \sum_{k=1}^{M} \mathbf{u}_0^k(\mathbf{x}, \omega), \quad \mathbf{u}_0(\mathbf{x}, \omega) = \sum a_n \cos(b_n + 2n\pi x_1)$$

#9. Computation of EMV solutions cont'd

- Evolution: compute the ensemble of entropic solutions

$$\mathbf{u}_0^{\Delta x,k}(\cdot,\omega) \mapsto \mathbf{u}^{\Delta x,k}(\cdot,t,\omega)$$

- Realize the EMVs by propagating the law of MC simulation

$$\nu^{\Delta x}_{\mathbf{x},t} = \frac{1}{M}\sum_{k=1}^{M} \delta_{\mathbf{u}^{\Delta x,k}(\mathbf{x},t,\omega)}, \quad \langle \nu^{\Delta x}_{\mathbf{x},t}, g(\lambda)\rangle = \frac{1}{M}\sum_k g(\mathbf{u}^{\Delta x,k}(\mathbf{x},t,\omega))$$

- This is an approximate entropic MV solution: for all $\phi(\mathbf{x},t) \in C_0^\infty$

$$\sum_{\mathbf{x}_\nu,t^n} \int_{\mathbb{R}_+\times\mathbb{R}^d} \phi_t \langle \nu^{\Delta x}_{\mathbf{x},t}, \mathrm{id}\rangle + \nabla_{\mathbf{x}}\phi \cdot \langle \nu^{\Delta x}_{\mathbf{x},t}, \mathbf{f}\rangle \, \Delta t \, d\mathbf{x}\, dt + \sum_{\mathbf{x}_\nu} \int_{\mathbb{R}^d} \phi(\mathbf{x}_\nu,0)\langle \sigma_{\mathbf{x}}, \mathrm{id}\rangle \, \Delta\mathbf{x} \, d\mathbf{x} \rightsquigarrow 0 = 0$$

and for all convex entropy pairs (η, F) with $0 \leqslant \phi \in C_0^\infty$

$$\sum_{\mathbf{x}_\nu,t^n} \int_{\mathbb{R}_+\times\mathbb{R}^d} \phi_t \langle \nu^{\Delta x}_{\mathbf{x},t}, \eta\rangle + \nabla_{\mathbf{x}}\phi \cdot \langle \nu^{\Delta x}_{\mathbf{x},t}, F\rangle \, \Delta t \, d\mathbf{x}\, dt + \sum_{\mathbf{x}_\nu} \int_{\mathbb{R}^d} \phi(\mathbf{x}_\nu,0)\langle \sigma_{\mathbf{x}}, \mathrm{id}\rangle \, \Delta\mathbf{x} \, d\mathbf{x} \rightsquigarrow 0 \leqslant 0$$

- Entropic $\nu^{\Delta x}_{\mathbf{x},t}$ with weak BV-bound $\rightsquigarrow$ weak$*$–lim $\nu^{\Delta x}_{\mathbf{x},t} = \nu_{\mathbf{x},t}$ is an EMV

On entropic measure-valued solutions ...

- Entropic $\nu^{\Delta x}_{\mathbf{x},t}$ with weak BV-bound $\rightsquigarrow$ weak$*$–lim $\nu^{\Delta x}_{\mathbf{x},t} = \nu_{\mathbf{x},t}$ is an EMV
- The scalar case — Wasserstein W_1-stability: for all ξ's in $\mathbb{R}$:

$$\int_{\mathbb{R}^d} \langle \nu_{\mathbf{x},t}, |u(\mathbf{x},t) - \xi| \rangle d\mathbf{x} \leqslant \int_{\mathbb{R}^d} \langle \sigma_{\mathbf{x}}, |u_0(\mathbf{x}) - \xi| \rangle d\mathbf{x}.$$

  In particular, if $\sigma = \delta_{u_0}$ at $t = 0$ then $\nu_{\mathbf{x},t} = \delta_{u(\mathbf{x},t)}$

- No uniqueness for non-atomic initial data
- Systems — weak-strong W_2-stability: if $\mathbf{u} \in W^{1,\infty}$ classical solution

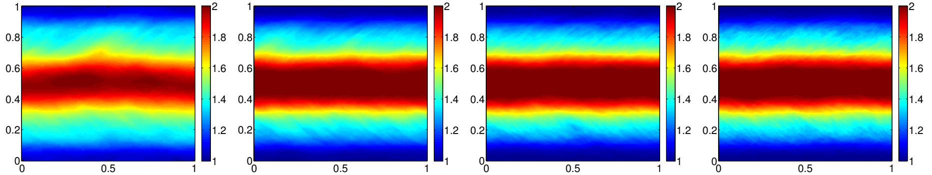
$$\int_{\mathbb{R}^d} \langle \nu_{\mathbf{x},t}, |\mathbf{u}(\mathbf{x},t) - \boldsymbol{\xi}|^2 \rangle d\mathbf{x} \leqslant \int_{\mathbb{R}^d} \langle \sigma_{\mathbf{x}}, |\mathbf{u}_0(\mathbf{x}) - \boldsymbol{\xi}|^2 \rangle d\mathbf{x}.$$

- There is no uniqueness but compute the observables —

  the mean: $\langle \nu_{\mathbf{x},t}, \lambda \rangle \quad \mapsto \quad \mathbb{E}\mathbf{u}^{\Delta x}(\mathbf{x}, t, \omega)$

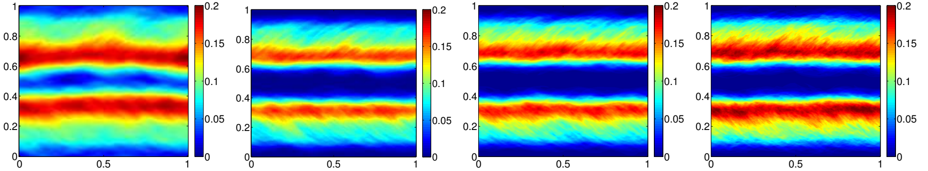
  the variance: $\langle \nu_{\mathbf{x},t}, \lambda^2 \rangle \quad \mapsto \quad \mathbb{E}\left(\mathbf{u}^{\Delta x}(\mathbf{x}, t, \omega) - \mathbb{E}\mathbf{u}^{\Delta x}\right)$, etc.,

#10. The computed observables



(a) 128^2 (b) 256^2 (c) 512^2 (d) 1024^2

Figure: Approximate sample means of the density for the Kelvin-Helmholtz problem at $t = 2$ and different mesh resolutions. All results are with 400 Monte Carlo samples.



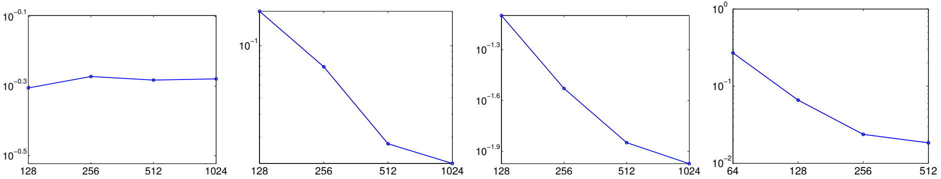
(a) 128^2 (b) 256^2 (c) 512^2 (d) 1024^2

Figure: Approximate sample variances of the density for the Kelvin-Helmholtz problem at $t = 2$ and different mesh resolutions. All results are with 400 Monte Carlo samples.

Comments on the computation of EMVs

- MC slow convergence $\|\mathbb{E}\mathbf{u}^{\Delta x}(\cdot, t, \omega) - \mathbb{E}\mathbf{u}^{\Delta x, M}(\cdot, t, \omega)\| \lesssim M^{-1/2}$

  Key point: Each ensemble was computed with $M = 400$ samples...!

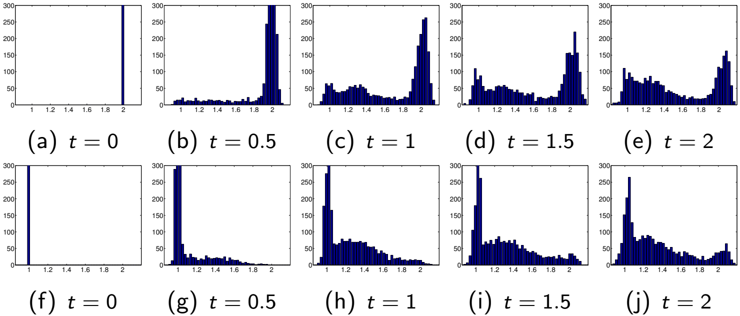


(a) "Pointwise" (b) Mean (c) Variance (d) W_1-rate

Figure: Kelvin-Helmholtz. (a) The Cauchy rates at $t = 2$ for the density (y-axis) for a single sample of the Kelvin-Helmholtz problem, vs. (b)+(c) different mesh resolutions (x-axis) (d) Cauchy rates in the W_1 distance at $t = 2$ for the density (y-axis) with respect to different mesh resolutions (x-axis)

- Richtmeyer-Meshkov instability: similar computations; other methods ...

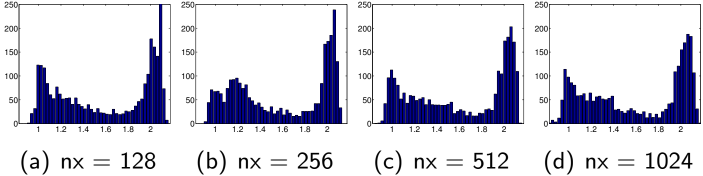
Computing EMVs: Young measures quantify uncertainty



(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 1.5$ (e) $t = 2$

(f) $t = 0$ (g) $t = 0.5$ (h) $t = 1$ (i) $t = 1.5$ (j) $t = 2$

Figure: The approximate PDF for density ρ at $x = (0.5, 0.7)$ (first row) and $x = (0.5, 0.8)$ (second row) on a grid of 1024^2 mesh points.



(a) $\mathrm{nx} = 128$ (b) $\mathrm{nx} = 256$ (c) $\mathrm{nx} = 512$ (d) $\mathrm{nx} = 1024$

Figure: The approximate PDF for density ρ at the points $x = (0.5, 0.7)$ (first row) and $x = (0.5, 0.8)$ (second row) a series of meshes.

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- FMT[2], "ENO reconstruction and ENO interpolation are stable", FoCM 2012
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[2]Fjordholm, Mishra Tadmor; [2b]R. Kappli

THANK YOU