



And Mathematical Modeling

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On a new scale of
borderline regularity spaces for Euler equations

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Euler's Equations

$$\begin{cases} u_t + \nabla_x \cdot (u \otimes u) = -\nabla_x p, & u = (u_1, \dots, u_d) \\ \operatorname{div} u = 0 \\ \text{initial and boundary data} \end{cases}$$

Weak solutions

P1. Finite Energy: L^2_{loc} -energy – $u(x, t) \in L^\infty([0, T]; L^2_{\text{loc}}(\mathbb{R}^d))$.

P2. Balance Law: $\forall \varphi \in C_c^\infty([0, T) \times \mathbb{R}^d; \mathbb{R}^d)$ with $\operatorname{div} \varphi = 0$:

$$\int_0^T \int_{\mathbb{R}^d} \varphi_t \cdot u + D\varphi (u \otimes u) \, dx dt + \int_{\mathbb{R}^d} \varphi(x, 0) \cdot u(x, 0) \, dx = 0.$$

P3. Incompressibility: $\operatorname{div} u = 0$ in \mathcal{D}' .

• Weak regularity in time (Lopes & Schochet) $u \in \operatorname{Lip}((0, T); H^{-L}_{\text{loc}}(\mathbb{R}^d; \mathbb{R}^d))$.

⊙ Assume the initial data in H^{-L} , $L > 1$ sense

• Existence. Passing to a limit with a sequence of **approximate solutions**.

Approximate Solutions

P1. L^2_{loc} -Energy bound: $\{u^\varepsilon\} \hookrightarrow L^\infty([0, T]; L^2_{\text{loc}}(\mathbb{R}^d))$.

P2. Weak Consistency: $\forall \varphi \in C_c^\infty([0, T] \times \mathbb{R}^d)$ with $\text{div } \varphi = 0$:

$$\int_0^T \int_{\mathbb{R}^d} \varphi_t \cdot u^\varepsilon + D\varphi (u^\varepsilon \otimes u^\varepsilon) dx dt + \int_{\mathbb{R}^d} \varphi(x, 0) \cdot u^\varepsilon(x, 0) dx \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

P3. (Approximate) Incompressibility: $\text{div } u^\varepsilon = 0$ in \mathcal{D}' ($\rightarrow 0$ in H_{loc}^{-1}).

⊙ In practice, H^{-s} consistency: $\varphi \in H_c^s([0, T] \times \mathbb{R}^d)$...

⊙ Energy-bound implies $\{u^\varepsilon\} \hookrightarrow \text{Lip}((0, T); H_{\text{loc}}^{-L}(\mathbb{R}^d))$, $L(s, n) > 1$.

EXAMPLES • *Mollification of initial data:* $u_0^\varepsilon = K_\varepsilon * \omega_0$, $K_\varepsilon := \eta_\varepsilon * K$.

• *Navier-Stokes approximate solutions.*

• *Vortex blob approximations*

• *Discrete methods:* High-resolution difference, Spectral and FEM methods

Existence of Weak Solutions

- ⊙ Energy bound $\implies u^\varepsilon \rightharpoonup u$ in $L^\infty([0, T], L^2_{\text{loc}}(\mathbb{R}^d))$
- ⊙ Weak regularity in time: $\{u^\varepsilon\} \hookrightarrow \text{Lip}((0, T), H^{-1}_{\text{loc}}(\mathbb{R}^d))$
- Main issue: passing to limit in quadratic terms: $u \otimes u \dots$
- ⊙ Either $u^\varepsilon \longrightarrow u$ in $L^\infty([0, T], L^2_{\text{loc}}(\mathbb{R}^d)) \implies u$ is a weak solution;
- ⊙ Or no strong convergence: $\int_E |u|^2 dxdt < \liminf \int_E |u^\varepsilon|^2 dxdt$
 \implies Energy concentrates on sets with non-zero reduced defect measure

$$\mu(E) := \limsup_\varepsilon \int_{E \subset [0, t] \times \mathbb{R}^d} |u^\varepsilon - u|^2 dxdt > 0$$

- (DiPerna-Majda). The phenomena of concentration-cancellation.

$$u_i^\varepsilon u_j^\varepsilon \rightharpoonup u_i u_j, \quad i \neq j.$$

H^{-1} Stability

- Characterize lack of concentrations (and hence existence)

- Typically, formulated in terms of vorticity $\omega_{ij}^\varepsilon = \frac{\partial u_i^\varepsilon}{\partial x_j} - \frac{\partial u_j^\varepsilon}{\partial x_i} \in \mathbb{A}^d$

Definition [H^{-1} -stability]: *The sequence $\{u^\varepsilon\}$ is H^{-1} -stable if $\{\omega^\varepsilon\}$ is a precompact in $C((0, T); H_{loc}^{-1}(\mathbb{R}^d; \mathbb{A}^d))$.*

⊙ No growth conditions at infinity ⊙ $u^\varepsilon \cdot \hat{n} = 0$ for bounded domains

Statement of main result (*M. Lopes, H. Lopes-Nussenzveig, T.*).

If $\{u^\varepsilon\}$ is H^{-1} -stable, then a subsequence converges strongly to a weak solution u in $L^\infty([0, T]; L_{loc}^2(\mathbb{R}^d))$.

- H^{-1} -stability as a criterion which excludes concentrations.

Proof $\operatorname{div} u^\varepsilon \hookrightarrow C([0, T], H_{loc}^{-1}(\mathbb{R}^d))$ and $\operatorname{curl} u^\varepsilon \hookrightarrow C([0, T], H_{loc}^{-1}(\mathbb{R}^d))$

div-curl lemma $\implies u^{\varepsilon_k} \cdot u^{\varepsilon_k} \rightharpoonup \bar{u} \cdot \bar{u}$, No concentration: $u^{\varepsilon_k} \longrightarrow \bar{u}$, $L^2([0, T], L_x^2)$

- Passing information from ω^ε to u^ε

$$\operatorname{div} u^\varepsilon = 0 \quad (\overset{\text{comp}}{\hookrightarrow} H^{-1}) \quad \operatorname{curl} u^\varepsilon = \omega^\varepsilon \quad \overset{\text{comp}}{\hookrightarrow} \text{“nice space”}$$

1. Biot-Savart Kernel (the 2D case): $u^\varepsilon = K * \omega^\varepsilon$, $K(x) \sim \frac{x^\perp}{|x|^2}$

⊙ CZ + Sobolev imbedding $L^p(\mathbb{R}^2) \rightarrow W^{1,p}(\mathbb{R}^2) \hookrightarrow L^2(\mathbb{R}^2)$.

* Delicate as $p \downarrow 1$.

2. Stream-function formulation: $\Delta \psi^\varepsilon := \omega^\varepsilon$, $u^\varepsilon = \nabla^\perp \psi^\varepsilon$

⊙ Elliptic Regularity (delicate as $p \downarrow 1$).

* For $W^{2,p}$ regularity of ψ^ε – requires growth control at infinity

3. Our approach – generalized Div-Curl Lemma (Tartar-Murat)

* Sharp local condition – simplifies & generalize previous results

- Greatly simplify previous results

- Generalization – unbounded domains, $d > 2$ dimensions

- Crystallize new regularity spaces...

A Retrospect of L^p Scales of Regularity Spaces

- Lebesgue $L^p(\mathbb{R}^d)$: $\left| \int \omega \varphi dx \right| \leq \text{Const.} \|\varphi\|_{L^{p'}}, \quad \forall \varphi \in L^{p'}$
- Lorentz - $L^{p,\infty}(\mathbb{R}^d)$: $\varphi \mapsto \chi_E, \quad \forall E's \in \mathbb{R}^d$

$$\int_E |\omega| dx \leq \text{Const.} |E|^{1/p'}, \quad \text{arbitrary sets } E's,$$

- Morrey - $M^p(\mathbb{R}^d)$: $\varphi \mapsto \chi_B, \quad \forall \text{arbitrary balls } B \in \mathbb{R}^d$

$$\int_{B_R} |\omega| dx \leq \text{Const.} |R|^{d/p'},$$

- Logarithmic refinements: $L^p(\log L)^\alpha, L^{p,\infty}(\log L)^\alpha, M^{p,\alpha}, \dots$

$$L^p(\log L)^\alpha := \{ \omega \mid \int |\omega|^p (\log^+ |\omega|)^\alpha dx \leq \text{Const.} \}$$

$$M^{p,\alpha} := \{ \omega \mid R^{-d/p'} |\log R|^\alpha \int_{B_R(x_0)} |\omega| dx \leq \text{Const.}, \quad R \downarrow 0 \}$$

The 2D problem – scalar vorticity transported

- Transport equation $\omega_t + u \cdot \nabla_x \omega = 0$, $\omega = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$
- $H^{-1}(\mathbb{R}^{d=2})$ -compactness: Critical $p_{crit} = \frac{2d}{d+2} = 1$

⊙ Lebesgue (Yudovich, DiPerna-Majda)– **borderline \mathcal{BM}_c (vortex sheets)**

$$\omega_0 \in L^p_c(\mathbb{R}^2), p > 1 \implies \omega^\varepsilon(\cdot, t) \in L^p_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

⊙ Orlicz (Morgulis, Chae)– **propagation of compactness** in borderline $L(\log L)^{\frac{1}{2}}$

$$\omega_0 \in L(\log L)_c^\alpha(\mathbb{R}^2), \alpha \geq 1/2 \implies \omega^\varepsilon(\cdot, t) \in L(\log L)^\alpha_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

⊙ Lorentz (P. L. Lions)– **propagation of compactness** in borderline $L^{(12)}$

$$\omega_0 \in L^{(1q)}(\mathbb{R}^2), q \leq 2 \implies \omega^\varepsilon(\cdot, t) \in L^{(1q)}_{loc} \hookrightarrow H^{-1}_{loc}(\mathbb{R}^2)$$

- $L^{(12)}$ – **largest rearrangement invariant borderline case in $H^{-1}(\mathbb{R}^2)$**

⊙ ... beyond rearrangement invariant spaces ...

Beyond Rearrangement Invariant Spaces

- **Morrey spaces:** $\widetilde{M}^{(p;\alpha)} := \{\omega \mid \sup_x \int_{B_R(x)} |\omega| \leq \text{Const.} R^{\frac{d}{p}} |\log R|^{-\alpha}\}$

Assertion (R. DeVore & T. Tao). $\widetilde{M}^{p;\alpha}(\Omega)$, $\Omega \subset \mathbb{R}^d$, is compactly imbedded in $H^{-1}(\Omega)$ if either: (i) $p > \frac{d}{2}$ or (ii) $p = \frac{d}{2}$ and $\alpha > 1$.

- Two-dimensional Morrey space (DiPerna-Majda)

- ⊙ $\widetilde{M}^{(1,\alpha)}(\mathbb{R}^2) : \int_{B_R} |\omega^\varepsilon| \leq C |\log R|^{-\alpha}$, $\alpha > 1 \implies$ no concentration

$$\omega^\varepsilon(\cdot, t) \in L^\infty([0, T], \widetilde{M}^{(1,\alpha)}(\mathbb{R}^2)) \xrightarrow{\text{comp}} H_{\text{loc}}^{-1}(\mathbb{R}^2), \quad \alpha > 1$$

- ⊙ Positive vorticity (Delort, Majda): $\omega^\varepsilon(\cdot, t) \in \mathcal{BM}_c^+ \implies \widetilde{M}_c^{(1;\frac{1}{2})}(\mathbb{R}^2)$

- ⊙ **Q.** Is $\widetilde{M}^{(1,\frac{1}{2})}(\mathbb{R}^2)$ borderline regularity space for concentration-cancellation?

- ⊙ **Q.** On the **borderline gap** $\widetilde{M}^{(1,\alpha)}(\mathbb{R}^2)$, $\frac{1}{2} < \alpha \leq 1$.

* Uniqueness: L^∞ Borderline – Besov $B_{2/s,1}^s$ (Vishik)

No Concentration – the multiD ($d > 2$) case

- No concentration for $\omega^\varepsilon(\cdot, t) \in L^\infty([0, T], X)$

Lebesgue : $X = L^p_c(\mathbb{R}^d), \quad p > \frac{2d}{d+2} \longmapsto L^p \hookrightarrow H^{-1}(\mathbb{R}^d)$

(since $H^1 \hookrightarrow L^p_c(\mathbb{R}^d), \quad p < p^* = \frac{2d}{d-2}$)

Morrey : $X = M^p(\mathbb{R}^d), \quad p > \frac{d}{2} \longmapsto M^p \hookrightarrow H^{-1}(\mathbb{R}^d)$

Q1. On the **borderline gap** $\frac{6}{5} < p < \frac{3}{2}$ for the $d = 3$ -D case?

- The 3D Navier-Stokes - $M^{3/2}$ existence (Giga-Miyakawa)

$$\frac{1}{R} \int_{B_R(x_0)} |\omega| dx \leq Const.$$

⊙ Comparison of $L^{6/5}$ and $M^{3/2}$ - measures of singular support (CKN)

⊙ Identify borderline regularity: $p = \frac{2d}{d+2} = \frac{6}{5}$

* Uniqueness & Energy loss – Brenier, Shnirelman,...

Borderline regularity – the multiD ($d \geq 2$) case

Theorem Assume **borderline regularity**: $\omega^\varepsilon(\cdot, t) \in L_c^{\frac{2d}{d+2}}(\mathbb{R}^d)$. Then there is no concentration with **'super-critical'** energy bound

$$u^\varepsilon(\cdot, t) \in L^\infty([0, T], L^{p>2}(\mathbb{R}^d))$$

Proof (by Murat Lemma). By interpolation of $X_r := W^{-1,r}(\mathbb{R}^d)$

$$\left\{ \begin{array}{l} L_c^{\frac{2d}{d+2}}(\mathbb{R}^d) \xrightarrow{\text{comp}} X_q, \quad q < 2 \\ \{\omega^\varepsilon\} \quad \text{in} \quad X_p, \quad p > 2 \end{array} \right. \implies \omega^\varepsilon(\cdot, t) \xrightarrow{\text{comp}} X_2 = H^{-1}(\mathbb{R}^d).$$

⊙ Example ($d = 2$). The critical regularity $\omega_0 \in \mathcal{BM}_c(\mathbb{R}^2)$:

$u^\varepsilon(\cdot, t) \in L^\infty([0, T], L^{p>2}(\mathbb{R}^2)) \implies$ no concentration (DiPerna-Majda)

⊙ Example ($d = 3$). The critical regularity $\omega_0 \in L_c^{6/5}(\mathbb{R}^3)$:

$u^\varepsilon(\cdot, t) \in L^\infty([0, T], L^{p>2}(\mathbb{R}^3)) \implies$ no concentration.

Q2. What can we say about $L^{6/5}$ as a regularity space for $\omega^\varepsilon(\cdot, t)$?

Candidates for regularity spaces

- ◉ **Lebesgue** L^p : $\left\{ \omega \mid \left| \int_x \omega \varphi dx \right| \leq \text{Const.} \|\varphi\|_{L^{p'}} \right\}$
- ◉ **Lorentz** wk - $L^{p,\infty}$: $\left\{ \omega \mid \varphi = \chi_E, \text{ arbitrary } E's \right\}$
- ◉ **Morrey** M^p : $\left\{ \omega \mid \varphi = \chi_B, \text{ arbitrary } B's \right\}$

$$\|\omega\|_{M^p} = \sup_B \frac{1}{|B|^{1/p'}} \int_B |\omega| dx \leq \infty$$

- **A new scale** V^{pq} : $\left\{ \omega \mid \varphi = \chi_{\cup B_j}, \text{ arbitrary covering } B's \right\}$

$$V^{pq} := \sup_{\{B_j\} \subset \mathcal{B}} \left\{ \frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx \right\}_{\ell^q} < \infty, \quad \text{arbitrary } \{B_j\}'s \subset \mathcal{B}$$

The new* scale of regularity spaces

$$\checkmark^{pq} : \quad \sum_j \left(\frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx \right)^q \leq \text{Const.}, \quad q \geq p$$

$$\checkmark^{pp} : \quad \sum_j \left(\frac{1}{|B_j|} \int_{B_j} |\omega| \right) \chi_{B_j}(x) \in L^p \longmapsto L^p \quad (\text{Riesz})$$

$$\checkmark^{p\infty} : \quad \frac{1}{|B|^{1/p'}} \int_B |\omega| dx \leq \text{Const.} \longmapsto M^p \quad (\text{Morrey})$$

- $L^{p\infty}$ - total mass on arbitrary sets; M^p - on arbitrary balls
- \checkmark^{pq} - ℓ^q -weighted distribution of L^p mass on arbitrary coverings

* Brudnyi \checkmark -spaces (1970): $\checkmark_{L^1}^0(\tau^\lambda, q)$ - smoothness of order $\lambda > 0$

⊙ \checkmark^{pq} (Nonlinearity, 2001) corresponds to $\lambda = d\left(\frac{1}{q} - \frac{1}{p}\right) \leq 0$

A New Scale of Regularity Spaces

- \mathcal{V}^{pq} -dyadic vs. \mathcal{V}^{pq} (w/ DeVore) - they are the same:

$$\sup_{\cup B_j = \Omega} \left(\sum_j \left(R_j^{-d/p'} \int_{B_j} |\omega(x)| dx \right)^q \right)^{1/q} \leq Const, \quad 1 \leq p \leq q \leq \infty.$$

- Logarithmic refinement: $\mathcal{V}^{pq,\alpha} = \mathcal{V}^{pq}(\log \mathcal{V})^\alpha$

$$\|\omega\|_{\mathcal{V}^{pq}(\log \mathcal{V})^\alpha(\Omega)} := \sup_{R_j < R_0} \|\{R_j^{d/p} |\log R_j|^\alpha \bar{\omega}_j\}\|_{\ell^q}, \quad q > p.$$

- ⊙ Example: $\|\omega\|_{\mathcal{V}^{pq}(\log \mathcal{V})^\alpha(\Omega)} < \infty$: Covering Ω by a dyadic lattice \mathcal{C}_{jk}

$$\sum_j \left(\int_{\mathcal{C}_{jk}} |\omega(x)| dx \right)^q \leq 2^{-kNq/p'} |1 + k_+|^{-\alpha q}, \quad \mathcal{C}_{jk}(\cdot) := 2^{-k} \mathcal{C}(\cdot + j).$$

- ⊙ weak- $L^p \leftrightarrow$ arbitrary sets; $M^p \leftrightarrow$ arbitrary balls

- \mathcal{V}^{pq} -spaces: \leftrightarrow collection of disjoint balls: bridges the gap

- ⊙ $\mathcal{V}^{pq} = (L^p, M^p)_{\theta, q}$, $\theta = \frac{p}{q} \leq 1$, $\mathcal{V}^{pp} = L^p \dots \mathcal{V}^{p\infty} = M^p$

Readers' digest

$$\vee^{pq} : \quad \sum_j \left(\frac{1}{|B_j|^{1/p'}} \int_{B_j} |\omega| dx \right)^q < \infty$$

- Cover $\Omega \subset \mathbb{R}^d$ by dyadic covering of cubes $C_{jk} := 2^{-k}C(\cdot + j)$

$$\vee^{pq}(\Omega) : \quad \sum_j \left(\int_{C_{jk}} |\omega| dx \right)^q \leq 2^{-kdq/p'}$$

- Comparison of borderline regularity spaces

$$p = \frac{3}{2}, \quad M^{\frac{3}{2}}(\mathbb{R}^3) : \quad \frac{1}{R} \int_{B_R(x_0)} |\omega| dx \leq \text{Const.}$$

$$p = \frac{6}{5}, \quad \vee^{\frac{6}{5}2}(\mathbb{R}^3) : \quad \sum_j \frac{1}{R_j} \left(\int_{B_j} |\omega| dx \right)^2 \leq \text{Const.}$$

- difference in Hausdorff dim (sing support ω)

Compact Embeddings of v 's in H^{-1}

Statement of compactness. $\check{V}^{p2,\alpha} \xrightarrow{\text{comp}} H_{\text{loc}}^{-1}(\mathbb{R}^d)$ if

$$(i) \ p > \frac{2d}{d+2} \quad \text{or} \quad (ii) \ p = \frac{2d}{d+2}, \ \alpha > 1/2.$$

⊙ The 2D borderline case: $\check{V}_c^{12,\alpha}(\mathbb{R}^2) \xrightarrow{\text{comp}} H_{\text{loc}}^{-1}(\mathbb{R}^2), \quad \alpha > \frac{1}{2}$

$$\check{V}^{12}(\log \check{V})^{1/2}(\Omega) = \left\{ \omega \mid \sup_{\cup B_j = \Omega} \sum_j |\log R_j| \left(\int_{B_j} |\omega| \right)^2 \leq \text{Const.} \right\}, \ \Omega \subset \mathbb{R}^2.$$

⊙ The 3D borderline case: $\check{V}_c^{p2}(\mathbb{R}^3) \xrightarrow{\text{comp}} H_{\text{loc}}^{-1}(\mathbb{R}^3), \quad p > \frac{6}{5}$

$$\|\omega\|_{\check{V}_{\frac{6}{5}}^2(\Omega)}^2 = \sup_{\cup B_j = \Omega} \sum_j \frac{1}{R_j} \left(\int_{B_j} |\omega| \right)^2 \leq \text{Const.}, \ \Omega \subset \mathbb{R}^3$$

Proof (of v -compact imbedding).

Measure the H^{-1} size of f^ε in terms of its wavelet expansion

$$f^\varepsilon = \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}^+} \sum_{j \in \mathbb{Z}^d} \hat{f}_{jk}^\varepsilon \psi_{jk}, \quad \psi_{jk} := 2^{kd/2} \psi(2^k x - j)$$

Using the $v^{p^2}(\log v)^\alpha$ -bounds

$$\begin{aligned} \sum_{j \in \mathbb{Z}^d} |\hat{f}_{jk}^\varepsilon|^2 &\leq 2^{kd} \sum_{j \in \mathbb{Z}^d} \left(\int_{\mathcal{C}_{jk}} |f^\varepsilon(x)| dx \right)^2 \\ &\leq \text{Const} \cdot 2^{kd} \|f^\varepsilon\|_{v^{p^2, \alpha}}^2 \cdot 2^{-2kd/p'} |1 + k_+|^{-2\alpha}. \end{aligned}$$

we conclude: if $(p - \frac{2d}{d+2})_+ + (\alpha - 1/2)_+ > 0$

$$\begin{aligned} \left\| \sum_{k > k_0} \sum_{j \in \mathbb{Z}^d} \hat{f}_{jk}^\varepsilon \psi_{jk} \right\|_{H^{-1}}^2 &= \sum_{\psi \in \Psi} \sum_{(j,k) \in (\mathbb{Z}^d, \mathbb{Z}^+)} |\hat{f}_{jk}^\varepsilon|^2 \|\psi_{jk}\|_{H^{-1}}^2 \\ &\leq \text{Const} \cdot \sum_{k > k_0} 2^{k(d-2d/p'-2)} |1 + k_+|^{-2\alpha} \rightarrow 0. \end{aligned}$$

Concentration-Cancelation in 2D Euler's equations

• 2D pseudo-energy $H(\omega) := -\frac{1}{2\pi} \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \log |x - y| d\omega(x) d\omega(y) \leq H_0$

⊙ V-scale classification of 2D regularity: $X_\alpha = \tilde{V}^{12}(\log \tilde{V})_\epsilon^\alpha(\mathbb{R}^2)$

Theorem. {i} **No concentration** if $\omega^\epsilon \in X_\alpha$, $\alpha > 1/2$;

{ii} **Concentration-Cancelation** if $\omega^\epsilon \in X_\alpha$, $\alpha \in (0, \frac{1}{2}]$.

⊙ Extension of Delort's result for one-signed measures: $\mathcal{BM}^+(\mathbb{R}^2) \subset X_{1/2}$

Q1. Is $X_\alpha = \tilde{V}^{12}(\log \tilde{V})_\epsilon^\alpha(\mathbb{R}^2)$ an invariant regularity space for 2D Euler?

$$\omega_0 \in X_\alpha \implies \eta_\epsilon * \omega_0 \mapsto \omega^\epsilon(\cdot, t) \in X_\alpha ?$$

⊙ Propagation of compactness in borderline regularity: $X = L(\log L)^{1/2}$, $L^{(12)}$, ...

$$\{\omega_\epsilon^\epsilon\} \subset X_{1/2} \not\hookrightarrow H^{-1}(R^2) \quad \text{but does} \quad \eta_\epsilon * \omega_0 \mapsto \omega^\epsilon(\cdot, t) \hookrightarrow H^{-1}(R^2)$$

Q2. ... No concentration phenomena for one-signed measures?

$$\{\omega^\epsilon\} \subset X_{1/2} \not\hookrightarrow H^{-1}(R^2) : \omega_0^\epsilon = \frac{1}{\epsilon^2 \sqrt{|\log \epsilon|}} \omega \left(\frac{|x|}{\epsilon} \right), \text{ but } \dots \eta_\epsilon * \omega_0 \mapsto \omega^\epsilon(\cdot, t) \hookrightarrow H^{-1}(R^2) ?$$

Concentration-Cancelation in 3D Euler's equations

- 3D Coulomb energy $H(\omega(x, t)) := \frac{1}{8\pi} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\langle \omega(x, t), \omega(y, t) \rangle}{|x - y|} dx dy \equiv H_0$
- ⊙ Split between long-range and short-range $H(\omega) =: H_{ie}(\omega) + H_{si}(\omega)$
- ⊙ Long-range interaction energy - bounded from below...

$$H_{ie}(\omega^\varepsilon(x, t)) = \frac{1}{8\pi} \sum_{j \neq k} \int \int_{C_j \times C_k} \frac{\langle \omega^\varepsilon(x, t), \omega^\varepsilon(y, t) \rangle}{|x - y|} dx dy \geq -Const_{ie}.$$

\implies Upper-bound short-range self-induced energy:

$$H_0 + Const_{ie} \geq H_{si}(\omega^\varepsilon(\cdot, t)) = \frac{1}{8\pi} \sum_j \int \int_{C_j \times C_j} \frac{\langle \omega^\varepsilon(x, t), \omega^\varepsilon(y, t) \rangle}{|x - y|} dx dy \geq \dots?$$

Q1. Seeking $\sqrt{\frac{6}{5}}^2$ -bound:

$$\langle \omega^\varepsilon(x), \omega^\varepsilon(y) \rangle \geq (1 - \theta^2) |\omega^\varepsilon(x)| \cdot |\omega^\varepsilon(y)|$$

$$\sum_j \int \int_{C_j \times C_j} \frac{\langle \omega^\varepsilon(x, t), \omega^\varepsilon(y, t) \rangle}{|x - y|} dx dy \geq (1 - \theta^2) \sum_j \frac{1}{2R_j} \left(\int_{C_j} |\omega^\varepsilon(x, t)| dx \right)^2$$

No concentration in 3D Euler's equations

- A weak alignment condition (Constantin-Fefferman-Majda):

$$\left| \frac{\omega^\varepsilon(x, t)}{|\omega^\varepsilon(x, t)|} - \frac{\omega^\varepsilon(y, t)}{|\omega^\varepsilon(y, t)|} \right|_{|x-y| \leq \delta} \leq \sqrt{2}\theta. \quad \theta < 1.$$

Theorem A weak alignment condition implies

$$\left| \frac{\omega^\varepsilon(x, t)}{|\omega^\varepsilon(x, t)|} - \frac{\omega^\varepsilon(y, t)}{|\omega^\varepsilon(y, t)|} \right|_{|x-y| \leq \delta} < \sqrt{2} \quad \implies \|\omega^\varepsilon(\cdot, t)\|_{\dot{V}^{\frac{6}{5}^2}(\Omega)} \leq \text{Const}_T.$$

Q2. Propagation of compactness of borderline regularity $X = \dot{V}^{\frac{6}{5}^2}(\mathbb{R}^3)$?!

Theorem Assume a weak alignment condition. Then as long as $u^\varepsilon \in L^\infty([0, T], L^{p>2}(\mathbb{R}^3))$ there is no 3D concentration: $u^\varepsilon \rightharpoonup u$



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THANK YOU

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