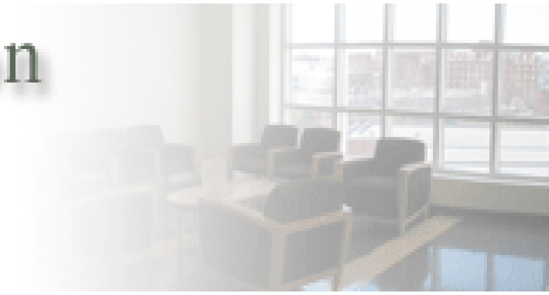

Burgers-type equations with vanishing hyper-viscosity

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OVERVIEW

- Burgers'-type equations

$$u_t + f(u)_x = -\frac{1}{\epsilon} P(\epsilon \partial_x) u, \quad |f''(u)| \geq Const.$$

$\odot$ Main question: the behavior of $u = u^\epsilon$ as $\epsilon \downarrow 0$

- Prototype example

$$u_t + \left(\frac{1}{2}u^2\right)_x = -\frac{1}{\epsilon}\mathcal{F}^{-1}\left[\frac{\epsilon^2\xi^2}{1+\epsilon^2\xi^2}\widehat{u}(t,\xi)\right] = \epsilon\mathcal{F}^{-1}\left[\frac{1}{1+\epsilon^2\xi^2}\widehat{u}(t,\xi)\right]_{xx}$$

From viscosity at low ξ's, $\sim \epsilon u_{xx}$ to relaxation at large ξ's, $\sim -\dfrac{u}{\epsilon}$

$\odot$ 'Chapman-Enskog' asymptotic expansion: ϵ as mean-free path

$$u_t + \left(\frac{1}{2}u^2\right)_x = \epsilon u_{xx} + \epsilon^3 u_{xxxx} + \ldots.$$

$\odot$ Instability of Burnett and super-Burnett equations

Regularized Chapman-Enskog

$$u_t + \Big(\frac{1}{2}u^2\Big)_x = -\frac{1}{\epsilon}\mathcal{F}^{-1}\left[\frac{\epsilon^2\xi^2}{1+\epsilon^2\xi^2}\widehat{u}(t,\xi)\right] = \frac{1}{\epsilon}\Big[Q_\epsilon * u - u\Big], \qquad Q = \frac{1}{2}e^{-|x|}$$

- Radiating gas, convolution model...: $\phi := \epsilon\Big(1-\epsilon^2\partial_x^2\Big)^{-1}u_x = \epsilon Q_\epsilon * u_x$

Rosenau, Schochet-ET, Kawashima-Nishibata, Serre, Lattanzio-Marcati, Liu-ET, ...

$$\begin{aligned} u_t + \Big(\frac{1}{2}u^2\Big)_x &= \phi_x \\ \epsilon^2\phi_{xx} &= \phi - \epsilon u_x \end{aligned}$$

- Finite time breakdown and critical threshold phenomena

- Sharper shock profile (Schochet-ET)

Krushkov and entropic convergence

$$u_t + f(u)_x = \frac{1}{\epsilon}\Big[Q_\epsilon * u - u\Big], \quad 0 \le Q, \ \int Q(y)dy = 1.$$

- The viscous case: Krushkov theory

$$\frac{\partial}{\partial t}\Big|u_2(\cdot,t) - u_1(\cdot,t)\Big| + \frac{\partial}{\partial x}F\Big(u_2(\cdot,t), u_1(\cdot,t)\Big) \le 0$$

- The convolution model: L^1 contraction but no Krushkov pairs;

- Instead – one-sided Lip condition: $u^\epsilon_x(\cdot,t) \le Const$ (ET et. al.)

$$\|u(\cdot,t) - u^\epsilon(\cdot,t)\|_{W^{-s,p}} \le Const.\epsilon^{\frac{sp+1}{2p}}, \ 0 \le s \le 1$$

⊙ Hyperbolic scaling $(t,x) \rightarrow (\frac{t}{\epsilon}, \frac{x}{\epsilon})$

Burgers Poisson equation - dispersive effects

- Euler-Poisson: Fellner & Schmeiser ...

$$
\begin{aligned}
u_t + \left(\frac{1}{2}u^2\right)_x &= \phi_x \\
\epsilon^2 \phi_{xx} &= \phi \pm u
\end{aligned}
$$

- Chapman-Enskog: 'dispersive' expansion: $\epsilon^2 u_{xxx} + \ldots$

- Re:Camassa-Holm

$$
\begin{aligned}
u_t + \left(\frac{1}{2}u^2\right)_x &= \phi_x \\
\phi_{xx} &= \phi + 2\kappa u + u^2 + \frac{1}{2}u_x^2
\end{aligned}
$$

⊙ Vanishing diffusion-dispersion problem (Schonbek...)

$$
u_t + f(u)_x = \epsilon u^\epsilon_{xx} + \delta_\epsilon u^\epsilon_{xxx}, \quad \delta_\epsilon \sim \epsilon^2
$$

⊙ Diffusion dominates if $\delta_\epsilon \ll \epsilon^2$,

Reversing time – hyper-viscosity

$$u_t - f(u)_x = +\frac{1}{\epsilon}\mathcal{F}^{-1}\left[\frac{\epsilon^2\xi^2}{1+\epsilon^2\xi^2}\widehat{u}(t,\xi)\right] = -\epsilon u_{xx} - \epsilon^3 u_{xxxx} + \ldots$$

- Vanishing Kuramoto-Sivashinsky
- Hyper viscosity: $u_t + f(u)_x = -\epsilon^3 u_{xxxx}$
- Lack of monotonicity
- Nonlinear extensions: lubrication, thin film

  Bertozzi et. al., Otto-Westdickenberg, ...

$$u_t + \epsilon^3\Big(\sigma(u)u_{xxx}\Big)_x = 0$$

Burgers-type equation w/vanishing hyper-viscosity

$$\frac{\partial u^\epsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\epsilon(x,t)) = (-1)^{s+1} \epsilon^{2s-1} \frac{\partial^{2s}}{\partial x^{2s}} u^\epsilon(x,t), \qquad f'' > 0.$$

⊙ Krushkov BV theory for monotone viscous case $s = 1$

⊙ Hyper-viscosity case, $s > 1$, lacks monotonicity

⊙ Instead, use compensated compactness theory

⊙ Hyper-viscosity with $s > 1$: weaker entropy dissipation bound than in the viscosity dominated case $s = 1$.

Hyper-viscosity and hyper-dissipation

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x,t)) = \frac{(-1)^{s+1}}{N^{2s-1}} \frac{\partial^{2s}}{\partial x^{2s}} u_N(x,t) =: \mathcal{V}(u_N), \qquad \epsilon \sim \frac{1}{N}.$$

- u_N has a smallest scale of order $1/N$.
- Quadratic entropy dissipation+production:

$$\frac{1}{2}\frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x} \int^{u_N} \xi f'(\xi) d\xi = \frac{(-1)^{s+1}}{N^{2s-1}} u_N \frac{\partial^{2s}}{\partial x^{2s}} u_N =: \mathcal{E}(u_N)$$

- Convergence: H^{-1} compactness of $\mathcal{V}(u_N)$ and $\mathcal{E}(u_N)$
- "Differentiation by parts" (periodic BCs, say):

$$\mathcal{E}(u_N) \equiv \frac{1}{N^{2s-1}} \sum_{\substack{p+q=2s-1 \\ q \geq s}} (-1)^{s+p+1} \overbrace{\frac{\partial}{\partial x}\left[\frac{\partial^p u_N}{\partial x^p} \frac{\partial^q u_N}{\partial x^q}\right]}^{\mathcal{E}_1(u_N)} - \overbrace{\frac{1}{N^{2s-1}} \left(\frac{\partial^s u_N}{\partial x^s}\right)^2}^{\mathcal{E}_2(u_N)}$$

Decomposition into low and high modes

- Entropy dissipation estimate:

$$\|u_N(\cdot,T)\|^2_{L^2} + \frac{1}{N^{2s-1}}\|\partial_x^s u_N\|^2_{L^2(x,t)} \leq \|u_N(\cdot,0)\|^2_{L^2(x)} \leq K_0^2.$$

$$u_N(x,t) = \sum_{|k|\leq N} \widehat{u}_N(k,t)e^{ikx} + \sum_{|k|>N} \widehat{u}_N(k,t)e^{ikx} =: u_N^I(x,t) + u_N^{II}(x,t).$$

- Entropy dissipation of u_N^I becomes *weaker* for $s > 1$:

$$\frac{1}{N^{2s-1}}\|\partial_x^s u_N^I\|^2_{L^2(x,t)} = N \sum_{|k|\leq N} \Big(\frac{|k|}{N}\Big)^{2s} |\widehat{u}(k,t)|^2_{L^2[0,T]} \leq K_0^2$$

⊙ $\|\partial_x^s u_N\|_{L^2} \sim N^{s-1/2}$; interpolate 'gain' on L^2-growth of $\partial_x^q u_N$:

$$\left\|\frac{\partial^q u_N}{\partial x^q}\right\|_{L^2(x,t)} \leq Const.\left\|\frac{\partial^s u_N}{\partial x^s}\right\|^{\frac{q}{s}}_{L^2} \times \left\|u_N\right\|^{1-\frac{q}{s}}_{L^2} \leq Const.N^{q-\delta}, \quad 1 \leq q \leq s$$

Estimates

⊙ Interpolate for $s \le q \le 2s$: $\left\| \dfrac{\partial^q u_N}{\partial x^q}(x,t) \right\|_{L^2(x,t)} \le C_\infty \cdot N^{q-\delta}, \ \ \delta_s = \dfrac{1}{2s}$

⊙ Small scale $\sim \frac{1}{N}$: for $1 \le p < s$, (clear for u_N^I; entropy dissipation $\Longrightarrow$)

$$\left\| \frac{\partial^p u_N}{\partial x^p}(x,t) \right\|_{L^2_t(L^\infty_x)} \le K_\infty \cdot N^p, \qquad K_\infty := \|u_N\|_{L^2_t(L^\infty_x)}$$

$$\begin{aligned} \left\| \left[\frac{\partial^p u_N}{\partial x^p} \frac{\partial^q u_N}{\partial x^q} \right] \right\|_{L^1_t(L^2_x)} & \le \ \|\partial^p_x u_N\|_{L^2_t(L^\infty_x)} \times \|\partial^q_x u_N\|_{L^2(x,t)} \le \\ & \le \ C_{pq} \cdot N^{p+q-\delta}, \qquad p < s \le q < 2s \end{aligned}$$

$$\begin{aligned} \left\| \mathcal{E}_1(u_N) \right\|_{L^1_t(H^{-1}_x)} & \le \frac{1}{N^{2s-1}} \sum_{\substack{p+q=2s-1 \\ q \ge s}} \left\| \frac{\partial}{\partial x} \left[\frac{\partial^p u_N}{\partial x^p} \frac{\partial^q u_N}{\partial x^q} \right] \right\|_{L^1_t(H^{-1}_x)} \le \\ & \le \frac{1}{N^{2s-1}} \sum_{\substack{p+q=2s-1 \\ q \ge s}} C_{pq} \cdot N^{p+q-\delta} \le \frac{C_s}{N^\delta} \to 0, \qquad C_s \sim s^s \end{aligned}$$

THEOREM. Consider the hyper-viscosity solution subject to L^2-bounded initial data, $\|u^\epsilon(\cdot,0)\|_{L^2} \leq K_0$ so that

$$N^{-(s-\frac{1}{4s})}\Big\|\frac{\partial^s}{\partial x^s}u_N(\cdot,0)\Big\|_{L^2} \leq Const.$$

Assume $u^\epsilon(\cdot,t)$ remains uniformly bounded, $\|u^\epsilon(\cdot,t)\|_{L^\infty} < K_\infty$. Then u^ϵ converges to the unique entropy solution of the convex conservation law.

REMARK. The issue of an L^∞ bound for vanishing hyper-viscosity of order $s > 1$ remains an open question.

NOTE. Entropic convergence and consistency with quadratic entropy: Panov, De Lellis et. al. $(\eta(w) = w^2,\ f(u) \in L^2)$

Spectral- and hyper spectral-viscosity

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N \sum_{|k|\leq N} \sigma\Big(\frac{|k|}{N}\Big)\widehat{u}_k(t)e^{ikx}.$$

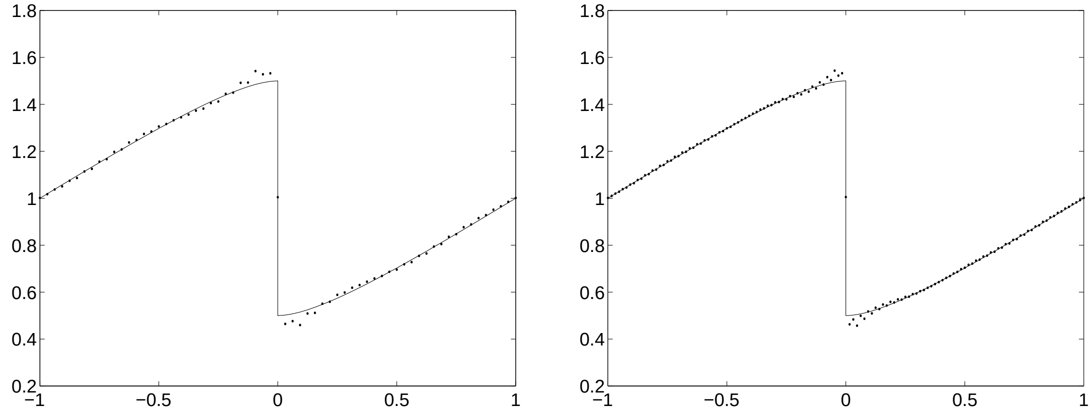
$\sigma(\xi)$ is a symmetric low pass filter satisfying $\sigma(\xi) \geq \Big(|\xi|^{2s} - \frac{1}{N}\Big)_+$

- Spectral viscosity: $s = 1$ viscous free modes $|k| \leq \sqrt{N}$
- Hyper-spectral viscosity: $s > 1$; requires $C_s N^{-\delta} \longrightarrow 0$:
- Spectral viscosity at modes $\Big\{k \mid \theta\frac{N}{(\log N)^{-\mu/2}} \leq |k| \leq N\Big\}$

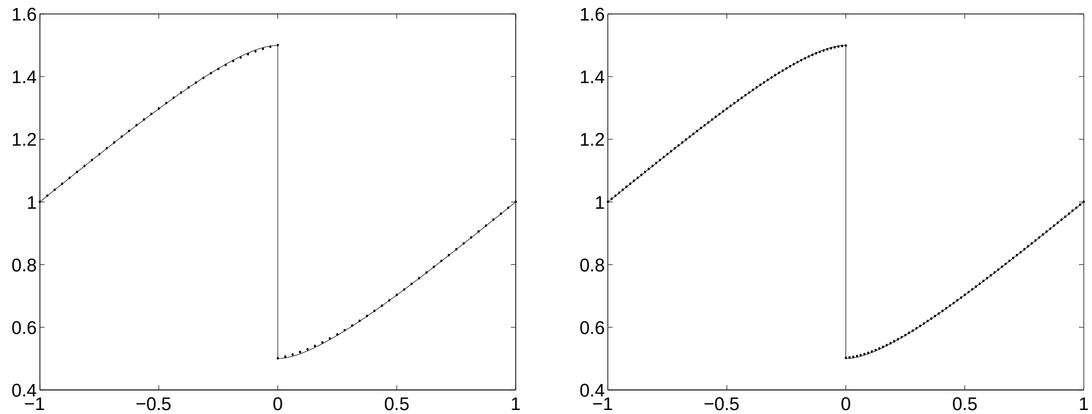
$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N \sum_{m_N\leq|k|\leq N} \Big(\frac{|k|}{N}\Big)^s \widehat{u}_k(t)e^{ikx}, \quad m_N \sim \frac{N}{(\log N)^{-\mu/2}}.$$

⊙ No monotonicity: spurious and Gibbs oscillations are stable and contain high order information

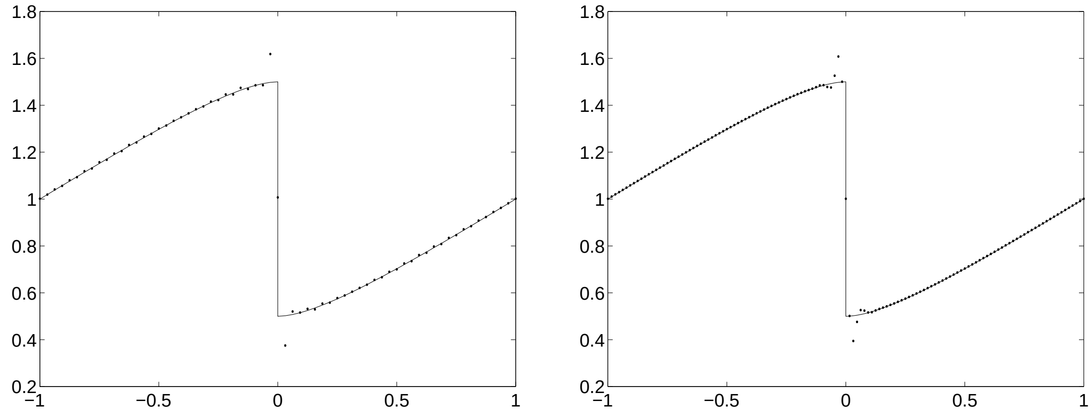
Hyper spectral viscosity: H.-Y Li, H.-P. Ma, ET



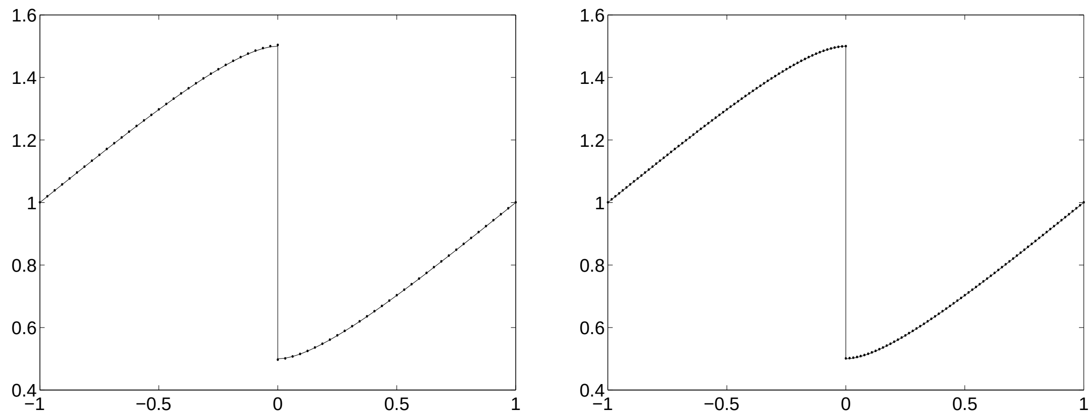
HSV solution based on $s = 1$ and (a) $2N = 64$ modes, (b) $2N = 128$ modes



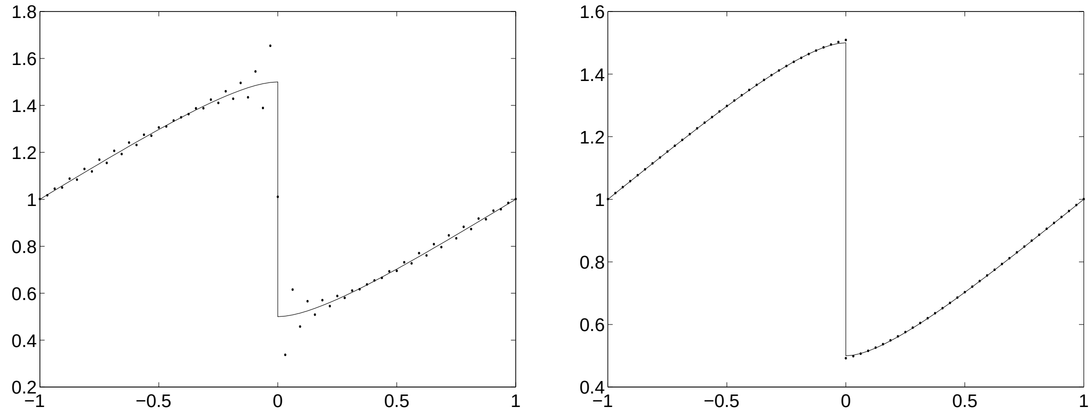
HSV solution after post-processing



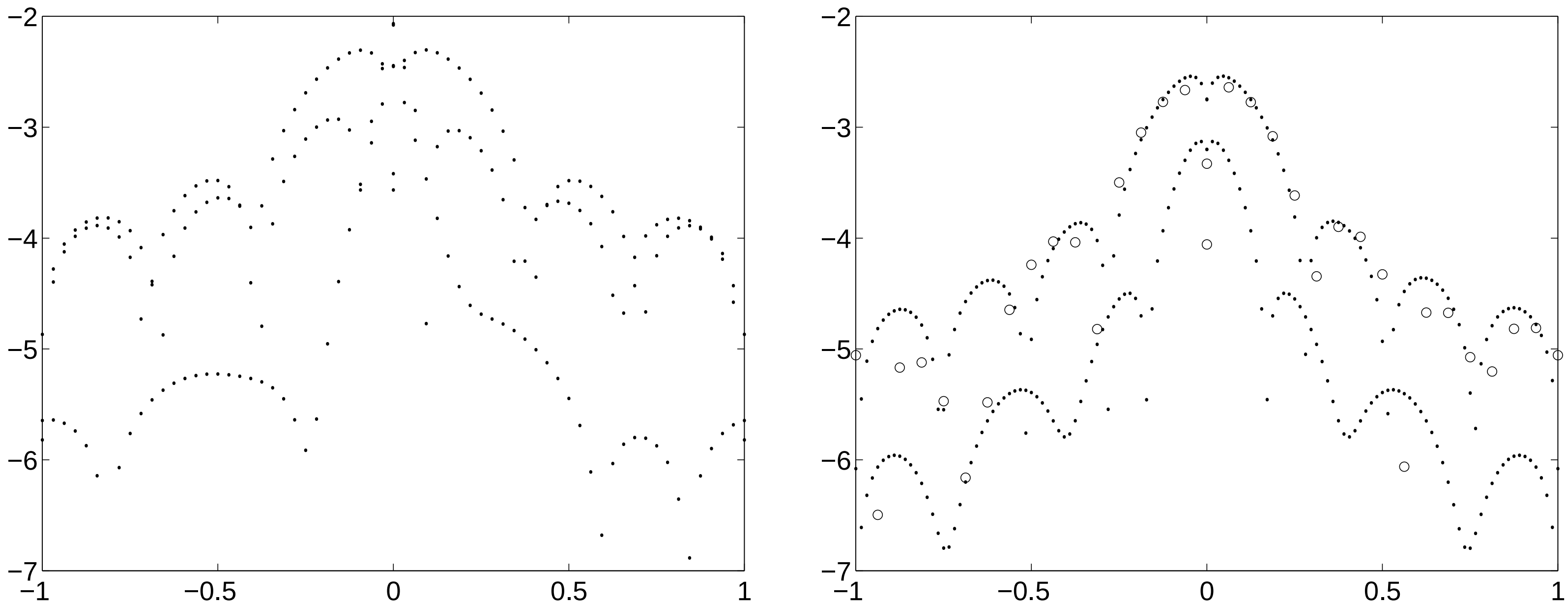
HSV solution based on $s = 2$ and (a) $2N = 64$ modes, (b) $2N = 128$ modes



HSV solution after post-processing



HSV solution based on $s = 3, 2N = 64$ modes before and after post-processing



Log-error plot of the post-processed HSV solutions for

(left) $2N = 64, s = 1, 3, 2$, (right) $2N = 128, s = 1, 2$ (dotted) and $2N = 32, s = 2$ (circled)

Vanishing Kuramoto-Sivashinsky dissipation

$$\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x,t)) = -\frac{1}{N}\frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3}\frac{\partial^4 u_N}{\partial x^4} =: \mathcal{V}(u_N).$$

- Rescaling such that u_N has a smallest scale of order $1/N$,

$$\left\| \frac{\partial^p u_N}{\partial x^p}(x,t) \right\|_{L^2_t(L^\infty_x)} \leq Const. N^p \cdot \| u_N(x,t) \|_{L^2_t(L^\infty_x)}, \quad p < 2$$

⊙ Entropy dissipation:

$$\frac{1}{2}\frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x} \int^{u_N} \xi f'(\xi) d\xi = -\frac{1}{N} u_N \frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3} u_N \frac{\partial^4 u_N}{\partial x^4} := \mathcal{E}(u_N)$$

$$\| u_N(\cdot,t) \|^2_{L^2} + \frac{1}{N^3} \left\| \frac{\partial^2 u_N}{\partial x^2} \right\|^2_{L^2(x,t)} \leq \| u_N(\cdot,0) \|^2_{L^2(x)} + \frac{1}{N} \left\| \frac{\partial u_N}{\partial x} \right\|^2_{L^2(x,t)}$$

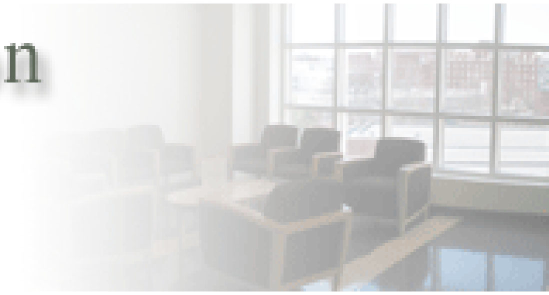
- Giacomelli & Otto: in fact $\| u_N(\cdot,t) \|_{L^2} \downarrow$ as $t \uparrow \infty$

Extensions

- The multidimensional problem:
  2D compensated compactness

- The discrete framework:
  entropic discretizations of nonlinear convection

- Nonlinear hyper-viscosity



THANK YOU