Effects of a Saturating Dissipation in Burgers-Type Equations

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Abstract

We propose and study a new variant of the Burgers equation with dissipation fluxes that saturate as the gradients become unbounded. If the upstream-downstream transition is above a critical threshold, the corresponding Riemann problem admits a weak solution wherein part of the transit is accomplished by a jump. It is shown that the solution to a Cauchy problem with sufficiently small compact or periodic initial data preserves its initial smoothness. © 1997 John Wiley & Sons, Inc.

1 Introduction

The model problem studied in this work,

$$u_t + f(u)_x = \nu Q(u_x)_x\,, \quad \nu > 0\,, \tag{1.1}$$

is an attempt to advance our understanding of the interaction between nonlinear convection and nonlinear diffusion with a saturating dissipation flux. The model problem (1.1) extends the Burgers equation in two ways; $f(u)$ is assumed to be an arbitrary smooth function and the flux function $Q(s)$ satisfies

$$|Q(s)| \le 1\,,\ Q'(s) > 0 \text{ for all } s;\ Q'(s) \to 0 \text{ as } |s| \to \infty\,. \tag{1.2}$$

Recall that in the Burgers equation, a toy model of the Navier-Stokes equations, the flux function is linear in gradients so that the response to a sharp interface may become unbounded and fail to represent the physical reality. The same difficulty occurs in the Navier-Stokes equations. The failure of a typical continuum equation to describe faithfully high-gradient phenomena is due to the fact that the derivation of a continuum model from a microscopic system is based very explicitly on the assumption of small gradients. This is a typical state of affairs in many equations of mathematical physics. Exactly when diffusion is needed to counterbalance the steepening due to convection, diffusion is least capable of reacting properly. The excessive amount of diffusion available at high gradients is an accidental by-product of the expansion in small

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gradients and bears no resemblance to the behavior of the original system in the ultraviolet regime. The high-gradients falsetto is a direct consequence of truncation that turns functional expansions into polynomials, and this brings in the disastrous behavior at infinity. Clearly, postponing the truncation one more order, as is so often done by physicists, not only does not improve the overall response of the system, but in many cases replaces a poor model with an ill-posed problem! In addition, a local expansion in gradients tends to eliminate important global constraints embedded in the original problem such as the Hamiltonian structure or an upper bound on the speed of propagation.

When diffusion is based on linear flux-gradient relations, there is an immediate response to a sharp interface accompanied by an infinite flux. It is physically clear that the rate of growth of the flux function must be finite. Depending on the problem at hand, different strategies have to be utilized to achieve this goal. For instance, in dilute gases governed by the Boltzmann equation, the moment (Chapman-Enskog) expansion, instead of being truncated, is resumed approximately. In the resulting system transport coefficients become wavelength dependent with heat and momentum fluxes that saturate at short wavelengths [8]. In fast processes the conventional ordering that begets the Navier-Stokes equations is replaced with a new ordering that places temporal and spatial changes on an equal footing. This leads to a hyperbolic diffusion with acoustic speed serving as a natural upper bound that tempers the response of the system to large gradients.

The convection-dissipation model [10] studied in the present work is an extension of the dissipation flux model proposed and analyzed in [9, 11]. A typical flux function was found to be [9]

$$Q(s) = \frac{s}{\sqrt{1+s^2}}\,.$$

Equilibrium states constructed on the basis of such dissipation fluxes support discontinuous interfaces [9]. It was found in [11] that it takes a finite time for such a flux to resolve an initially imposed, perfectly sharp interface.

The model equation (1.1) is thus a natural candidate for studying the interaction between saturating dissipation and convection. In particular, we examine when this interaction generates smooth patterns. It is demonstrated in Section 2 that if the downstream state is below a critical threshold, the upstream-downstream transition is smooth. However, above this threshold part of the upstream-downstream transition must be accomplished via a discontinuous jump. Such states will be referred to as *supercritical*. The critical threshold is determined by the saturation level of the particular dissipation flux. We also demonstrate numerically that both the continuous and discontinuous kink solutions are attractors. Kinklike initial states are seen to converge in time to a kink solution. Its nature depends solely on the total upstream-downstream disparity of the initial state.

In Section 3 and Section 4 we introduce a weak solution and consider the Cauchy problem associated with (1.1) subject to periodic or compactly supported initial datum

$$u(x,0) = u_0(x)\,. \tag{1.3}$$

In Section 3 we prove the existence of the solution to the problem (1.1), (1.3) by the vanishing viscosity method; that is, we consider the equation

$$u^\delta_t + f(u^\delta)_x = \nu Q(u^\delta_x)_x + \delta u^\delta_{xx}\,, \quad \delta > 0\,, \tag{1.4}$$

with the same initial data (1.3). Its smooth solution depends on the (small) parameter δ, and the solution of (1.1),(1.3) will be obtained as a limit of u^δ by letting $\delta \downarrow 0$. This approach provides a convenient way to define weak solutions to equation (1.1).

We also prove the uniqueness (Section 3) and the existence (Section 5) of the smooth (classical) solution of (1.1) with a sufficiently small, smooth initial datum. In this context it is necessary to call attention to a gap in our understanding of the Cauchy problem, namely, while for a sufficiently small and smooth initial datum we can ascertain the existence of a classical solution, if condition (3.5) is violated, it is unclear whether the solution remains smooth or steepens and breaks down within a finite or infinite time. This open status of the Cauchy problem should be contrasted with our understanding of the Riemann problem, where numerical experiments clearly indicate the emergence of sub- and supercritical states (see Section 2). A complete proof of this fact is still not available.

In Section 4 we consider equation (1.1) with ν assumed to be a small parameter,

$$u^\nu_t + f(u^\nu)_x = \nu Q(u^\nu_x)_x\,, \quad \nu > 0\,. \tag{1.5}$$

We study the behavior of solutions $u^\nu(x,t)$ of (1.5),(1.3) as $\nu \downarrow 0$. We prove that in this case $u^\nu(x,t)$ converges to the entropy solution of the scalar conservation law

$$u_t + f(u)_x = 0 \tag{1.6}$$

with initial data (1.3). The proof is straightforward: We obtain an error estimate in the $W^{-1}(L^\infty)$-norm, which also allows one to estimate the rate of convergence in the L^p-norms. For $2 \leq p < \infty$ these estimates are better than the estimates that were obtained for the standard vanishing viscosity approximation of (1.6),

$$u_t + f(u)_x = \varepsilon u_{xx} \tag{1.7}$$

(see [12]). Thus if one considers (1.5) as a differential approximation of (1.6), it is better than the usual vanishing viscosity approach. This is quantified more precisely in Section 4.

2 Traveling Waves

Study of traveling waves provides perhaps the simplest way to examine the convective-dissipative interaction. Throughout this section we let $\nu = 1$ and $f(u) = u^2$, and we begin with $Q(s) = s/\sqrt{1+s^2}$, that is,

$$u_t + (u^2)_x = \left[\frac{u_x}{\sqrt{1+u_x^2}} \right]_x, \tag{2.1}$$

The $Q(s)$ used in (2.1) is typical of the flux functions we shall consider and serves as a motivation to study the general case. Let $u = 0$ and $u = u_1$ be the upstream and downstream values, respectively, and let $z = x - \lambda t$; then one integration yields

$$-\lambda u + u^2 = \frac{u_z}{\sqrt{1+u_z^2}}, \tag{2.2}$$

and u_z vanishes at $u = 0$. For u_z to vanish at u_1 we need $u_1 = \lambda$ as well, which relates the downstream amplitude with the speed λ of the wave. We now re-express equation (2.2) in terms of u_z to obtain

$$u_z^2 = \frac{u^2(u-u_1)^2}{1 - u^2(u-u_1)^2}. \tag{2.3}$$

Equation (2.3) contains the needed information, for as long as the denominator does not vanish, there is a continuous trajectory connecting upstream with downstream. At the critical value of $\lambda = 2$, the denominator vanishes at $u = 1$ and the profile has a vertical slope at this point. For $\lambda > 2$, no continuous upstream-downstream transit is possible; part of it must be accomplished via a discontinuous jump. This is a genuine subshock layer. In this region, other physical mechanisms that may otherwise be negligible become crucial. A typical solution with a discontinuous jump is displayed in Figure 2.1.

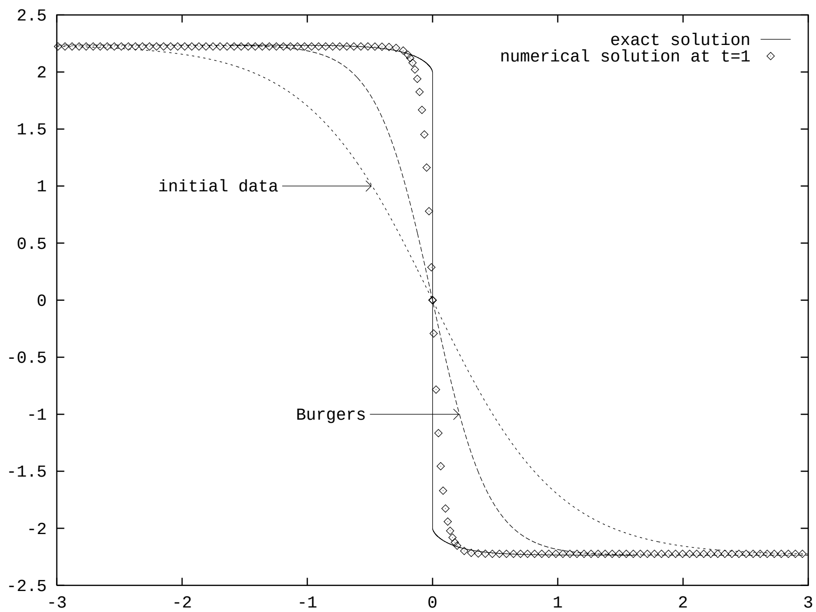

Figure 2.1. A supercritical downstream converges in time to a discontinuous kink. For comparison, a solution to the corresponding Burgers equation is also shown.

The total amount of the jump across the subshock is easily calculated and found to be

$$[u] = \sqrt{u_1^2 - 4}\,. \tag{2.4}$$

For downstream states close to the critical value, the subshock is weak but may become arbitrarily large with a further increase in the value of the downstream state.

Since in the purely diffusive problem (i.e., $f(u) = 0$) the rate of saturation was found to determine whether an initial discontinuity can be sustained for a finite time [11], it is also of interest to examine the interaction between inertia and an arbitrary saturating function Q. To this end we reconsider problem (2.1) using an arbitrary Q limited only by (1.2) and take $f(u) = u^2$. As before, we seek steadily progressing waves with upstream and downstream states being $u = 0$ and $u = u_1$, respectively. Again $u_1 = \lambda$, but instead of (2.2), we obtain

$$-\lambda u + u^2 = Q(u_z)\,. \tag{2.5}$$

To find the highest permissible speed that supports a continuous trajectory, we note that the dissipative flux Q is bounded, while the inertial flux (the LHS of (2.5)) is not. Balancing the two fluxes determines the largest permissible wave speed. In equation (2.5) both fluxes assume a negative value; therefore we compare their corresponding minima attained when $Q = Q_0$ and $u = \lambda/2$. Consequently, $-\lambda^2/4 = Q_0$ implies

$$\lambda = \sqrt{-4Q_0}\,, \tag{2.6}$$

which provides the upper bound on the speed of continuous, progressing waves. The actual amount of the jump in a supercritical state depends, of course, on the particular choice of Q.

Thus it is the saturation level of the flux function that matters most, with the saturation details being of lesser importance. Note, however, that we also need the flux function to be monotone to ensure stability. For if anywhere $Q'(s) < 0$, convexity of the elliptic part is lost and instability may set in. This depends on the details of the flux function [3].

We conclude this section with a number of numerical experiments intended to demonstrate the role of the traveling solutions as attractors. It is convenient to impose symmetric upstream-downstream states at $u = \pm u_1$ so that the resulting traveling wave becomes stationary and thus easily traced numerically. In Figures 2.1 and 2.2 we consider the supercritical and subcritical cases, respectively, and demonstrate how the kink solution is approached in time by an initially imposed kink. The supercritical and subcritical downstream states in the figures are $\sqrt{5}$ and $\sqrt{5}/10$, respectively.

The same procedure is used in Figures 2.3 and 2.4, but now the initial datum takes the form of a step function. The symmetric choice of the upstream and downstream states dictates that instead of (2.4) the jump condition should be

$$[u] = 2\sqrt{u_1^2 - 1}\,.$$

On the basis of the examples above and many others, one concludes that the ultimate outcome of the evolution depends only on whether the upstream-downstream disparity of initial data is sub- or supercritical. Although we lack a rigorous proof to quantify the supercritical affairs, numerical experiments demonstrate very clearly that both subcritical and supercritical solutions emerge as global attractors to wide classes of initial data. For comparison, we also display the kink form of the analogous Burgers equation. The shape of the resulting profile is very similar to the shape of the subcritical kink.

Remark. For the numerical studies of the subcritical cases we have used a simple first-order Lax-Friedrichs-type difference scheme. However, in the

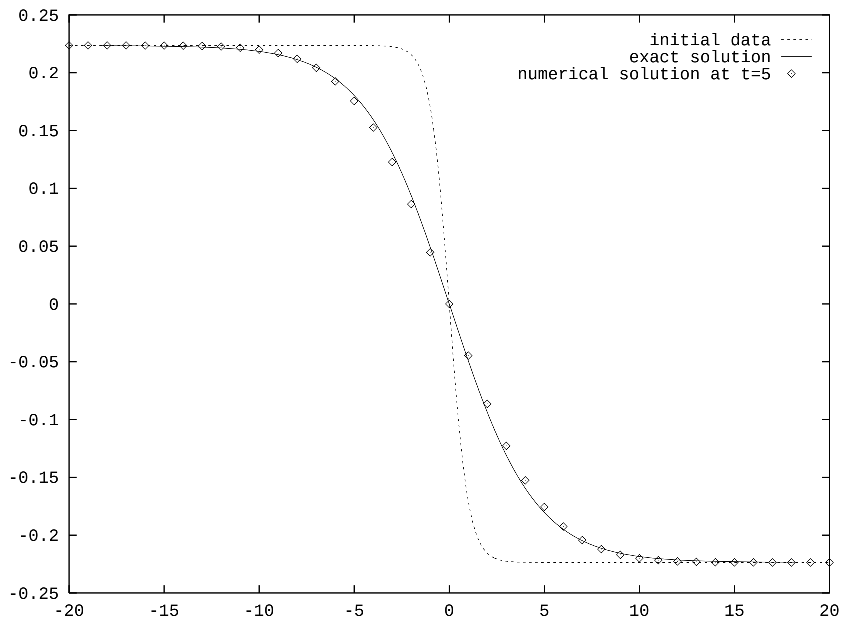

Figure 2.2. A subcritical downstream state is shown to converge in time to a smooth kink solution that for all practical purposes is indistinguishable from the solution of the corresponding Burgers equation.

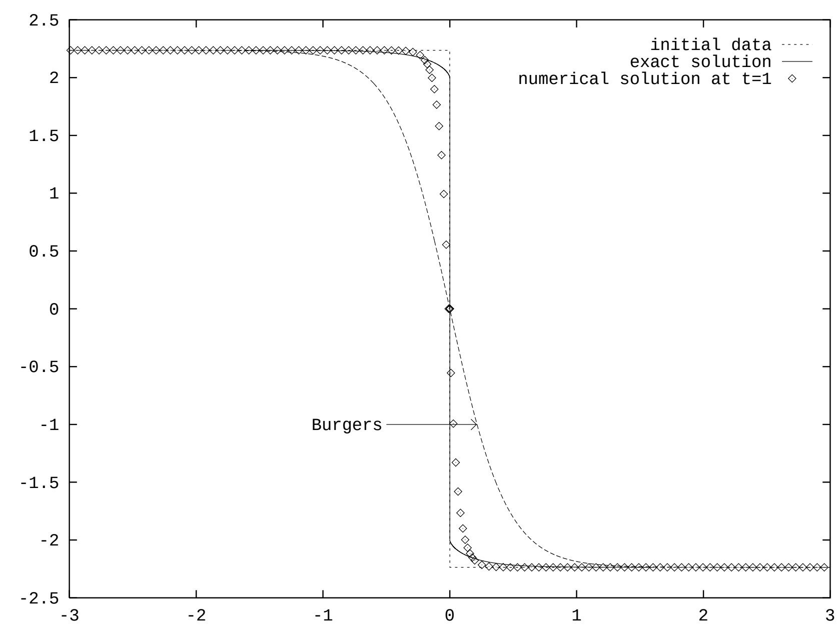

Figure 2.3. The same as in Figure 2.1 but with a discontinuous initial datum.

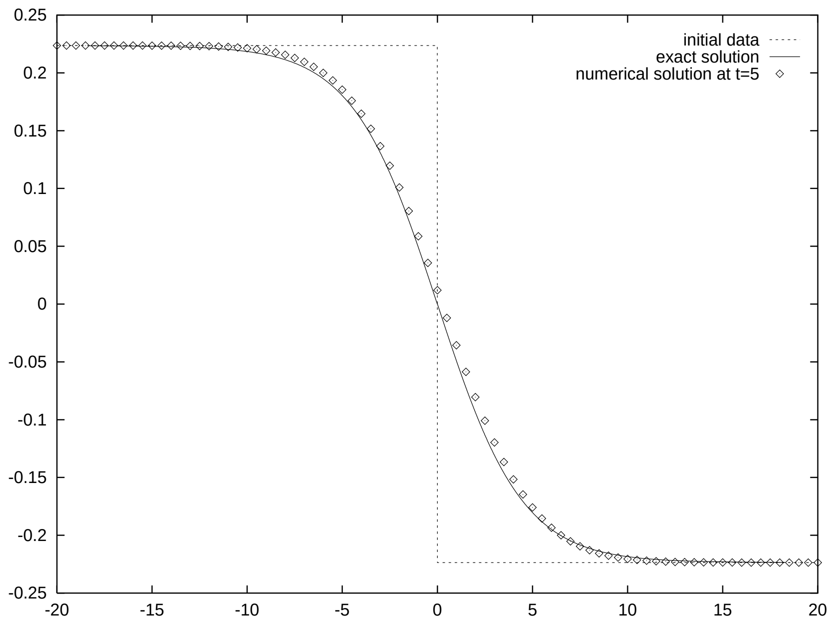

Figure 2.4. The same as in Figure 2.2 but with a discontinuous initial datum.

presence of the discontinuity, the first-order scheme does not provide sufficient resolution of the shock. Therefore, in the supercritical case the numerical solution was obtained using a second-order scheme based on a nonoscillatory central difference scheme due to Nessyahu and Tadmor [6].

3 Weak and Classical Solutions

To define a weak solution of equation (1.1), we consider its δ-regularization (1.4). This is a strictly parabolic regularization ($\delta > 0$) and consequently, by standard arguments (which we omit), problem (1.4),(1.3) admits a unique global classical solution. This brings us to the following definition:

DEFINITION 3.1 Let u^δ be a solution of the regularized problem (1.4),(1.3). Then we define a *weak solution* of the problem (1.1),(1.3) as $\lim_{\delta\downarrow 0} u^\delta$; that is,

$$u^\delta \xrightarrow{L^1} u \quad \text{as } \delta \downarrow 0\,. \tag{3.1}$$

To validate this definition it is necessary to prove the existence of the limit (3.1).

THEOREM 3.2 *Consider equation* (1.4) *subject to* $W^1(L^1)$*-initial data,* (1.3), *and assume that condition* (1.2) *holds. Then there exists a sequence* δ_n *such that* $\delta_n \downarrow 0$ *and* u^{δ_n} *converges in the* L^1*-norm as* $n \to \infty$.

PROOF: We differentiate (1.4) with respect to x, then multiply by $\operatorname{sgn}(u_x^\delta)$ and integrate over the x-domain. We obtain the following equation:

$$\begin{aligned}\int_x |u_x^\delta|_t dx + \int_x f(u^\delta)_{xx} \operatorname{sgn}(u_x^\delta) dx \\ = \nu \int_x Q(u_x^\delta)_{xx} \operatorname{sgn}(u_x^\delta) dx + \delta \int_x u_{xxx}^\delta \operatorname{sgn}(u_x^\delta) dx\,.\end{aligned} \tag{3.2}$$

The second term in the LHS of (3.2) is equal to zero, and its RHS is nonpositive due to assumption (1.2). Hence,

$$\|u_x^\delta(\cdot,t)\|_{L^1} \le ||u_0'(\cdot)||_{L^1} \quad \text{for all } t > 0\,. \tag{3.3}$$

This means that $u^\delta \in W^1(L^1(x))$, which is compactly imbedded in $L^1(x)$. Therefore Theorem 3.2 follows from compactness arguments. ∎

Remark. Note that one can relax the assumption on the initial datum and assume only its L^∞-boundedness.

We now study the question of existence and uniqueness of the smooth (classical) solution of (1.1),(1.3). Uniqueness of the weak solution of (1.1),(1.3) will be presented shortly.

THEOREM 3.3 *Consider the problem* (1.1),(1.3) *with* Q *satisfying* (1.2). *Let the range of* $Q(s)$ *be denoted by*

$$Q : \Re \longrightarrow [a,b]\,, \tag{3.4}$$

where $a < 0$ *and* $b > 0$. *If* $u_0(x) \in C^3$, *and if it is sufficiently small so that*

$$\nu||Q(u_0')||_{L^\infty} + 2||f(u_0)||_{L^\infty} \le \alpha < \nu \cdot \min(-a,b)\,, \tag{3.5}$$

then there exists a unique global classical solution of (1.1),(1.3), $u(x,t) \in C^{2,1}(x,t)$.

Remark. It is a challenging task to understand what happens if condition (3.5) does not hold. Let us rewrite equation (1.1) in the following nondivergent form:

$$u_t + f(u)_x = \nu Q'(u_x) u_{xx}\,. \tag{3.6}$$

Recall that $Q'(s) > 0$ for all s, and $Q'(s) \to 0$ as $|s| \to \infty$. Thus the RHS of (3.6) is "almost" viscous, because it vanishes only as $|u_x| \to \infty$. To understand the difficulty in analyzing the problem, we note that in our problem the competition between dissipation and convection is far more intricate than, say, the classical Burgers equation. The nonlinearity of the dissipative part tends to induce a cascading effect; if, for any reason, convection is enhanced, the resulting increase in gradients depresses dissipation, which in turn causes a further increase of the gradients. Will smoothness be ultimately lost? And if it is, will it occur in finite time?

PROOF: The existence of the classical solution will be shown in Section 5. We now turn to the uniqueness part of the theorem. Let $u^1(x,t)$ and $u^2(x,t)$ be two classical solutions of equation (1.1) with the same initial data (1.3), that is,

$$u^1_t + f(u^1)_x = \nu Q(u^1_x)_x\,, \qquad u^1(x,0) = u_0(x)\,; \tag{3.7}$$

$$u^2_t + f(u^2)_x = \nu Q(u^2_x)_x\,, \qquad u^2(x,0) = u_0(x)\,. \tag{3.8}$$

Subtracting (3.8) from (3.7), we obtain that $u^1(x,t) - u^2(x,t)$ satisfies the following equation:

$$(u^1 - u^2)_t + \Big[f(u^1) - f(u^2)\Big]_x = \nu\Big[Q'(\xi)\cdot(u^1 - u^2)_x\Big]_x\,, \tag{3.9}$$

where

$$u^1(x,0) - u^2(x,0) \equiv 0 \tag{3.10}$$

and $\xi = \xi(x,t)$ is between $u^1(x,t)$ and $u^2(x,t)$. Next we multiply (3.9) by $\mathrm{sgn}(u^1 - u^2)$ and integrate over x. Then the positivity of Q' implies that for all t

$$\frac{d}{dt}||u^1(\cdot,t) - u^2(\cdot,t)||_{L^1} \le 0\,, \tag{3.11}$$

which in turn yields the L^1-contraction of the solution operator for (1.1):

$$||u^1(\cdot,t) - u^2(\cdot,t)||_{L^1} \le ||u^1(\cdot,0) - u^2(\cdot,0)||_{L^1}\,. \tag{3.12}$$

Hence, since by (3.10) $u^1(x,t) \equiv u^2(x,t)$, the proof of Theorem 3.3 is complete. ∎

Remark. To prove (3.12) we used the technique established by Quinn [7]. In the same way the L^1-contraction of the solution operator can be proved for the regularized equation, (1.4).

Finally, we return to the question of uniqueness of the weak solution of (1.1),(1.3).

THEOREM 3.4 *Assume that condition* (1.2) *holds. Then for all* $W^1(L^\infty)$-*initial data* (1.3) *that satisfy* (3.5)*, the weak solution of* (1.1) *is unique.*

PROOF: We have defined the weak solution of (1.1),(1.3) as a limit of $u^{\delta_n}(x,t)$ as $\delta_n \downarrow 0$, where $u^{\delta_n}(x,t)$ are the solutions to the problem (1.4),(1.3) with $\delta = \delta_n$.

Let us consider a smoothed initial datum $u(x,0) = u_0^\varepsilon(x)$ where

$$u_0^\varepsilon(x) \equiv u_0 * \phi_\varepsilon(x)\,. \tag{3.13}$$

Here $\phi_\varepsilon(x)$ is a standard mollifier satisfying the following conditions:

$$\phi \in C_0^\infty\,; \quad \phi(x) > 0; \quad \int_{-1}^{1} \phi(x)dx = 1\,;$$
$$\operatorname{supp}\phi \subset [-1,1]\,; \quad \phi_\varepsilon(x) \equiv \frac{1}{\varepsilon}\phi\left(\frac{x}{\varepsilon}\right).$$

Note that the L^∞-norms of both $u_0(x)$ and $u_0'(x)$ do not increase after smoothing. Therefore, by Theorem 3.3 there exists a unique classical solution of (1.4) with the C^∞ initial datum (3.13). We denote this solution by u^{ε,δ_n}. Obviously, $u^{\varepsilon,\delta_n} \to u^\varepsilon$ pointwise as $\delta_n \downarrow 0$. Here u^ε is the solution of (1.1),(3.13).

Therefore, to conclude our proof it suffices to show that

$$||u^{\delta_n} - u^\varepsilon||_{L^1} \to 0 \quad \text{as } \varepsilon \downarrow 0,\ \delta_n \downarrow 0\,. \tag{3.14}$$

We have

$$\begin{aligned} \|u^{\delta_n} - u^\varepsilon\|_{L^1} &= \|u^{\delta_n} - u^{\varepsilon,\delta_n} + u^{\varepsilon,\delta_n} - u^\varepsilon\|_{L^1} \\ &\le \|u^{\delta_n} - u^{\varepsilon,\delta_n}\|_{L^1} + \|u^{\varepsilon,\delta_n} - u^\varepsilon\|_{L^1}\,. \end{aligned} \tag{3.15}$$

The second term in the RHS of (3.15) converges to zero as $\delta_n \downarrow 0$, while the first term, due to the L^1-contraction of the solution operator of equation (1.4), can be bounded as follows:

$$\|u^{\delta_n} - u^{\varepsilon,\delta_n}\|_{L^1} \le \|u_0 - u_0^\varepsilon\|_{L^1} = O(\varepsilon)\,. \tag{3.16}$$

Hence we have shown that all converging sequences $u^{\delta_n}(x,t)$ tend to the same limit, $u(x,t)$, which is the unique weak solution of the problem (1.1),(1.3). ∎

4 Convergence as $\nu \downarrow 0$

As we mentioned earlier, the solution operator of equation (1.5) is an L^1-contraction and is thus monotone [1, lemma 3.2] and BV bounded; that is,

$$\|u^\nu(\cdot,t)\|_{BV} \le \|u^\nu(\cdot,0)\|_{BV}\,. \tag{4.1}$$

THEOREM 4.1 *Let condition* (1.2) *hold and let* u^ν *be a solution of* (1.5) *subject to* L^∞*-bounded initial conditions* $u^\nu(x,0) \equiv u_0(x)$. *Then* u^ν *converges to the unique entropy solution of* (1.6) *as* $\nu \downarrow 0$, *and the following error estimates hold for all* $t \ge 0$:

$$\|u^\nu(\cdot,t) - u(\cdot,t)\|_{W^{-1}(L^\infty)} \le \mathrm{const}_t \cdot \nu\,, \tag{4.2}$$

$$\|u^\nu(\cdot,t) - u(\cdot,t)\|_{L^p} \le \mathrm{const}_t \cdot \nu^{1/p}\,, \quad \le p \le \infty\,, \tag{4.3}$$

$$\|u^\nu(\cdot,t) - u(\cdot,t)\|_{L^1} \le \mathrm{const}_t \cdot \sqrt{\nu}\,. \tag{4.4}$$

PROOF: Let $U(x,t) \equiv \int^x u(x,t)dx$ and $U^\nu(x,t) \equiv \int^x u^\nu(x,t)dx$ denote the primitives of solutions of (1.6) and (1.5), respectively. Then they satisfy the following two equations:

$$U_t^\nu + f(U_x^\nu) = \nu Q(U_{xx}^\nu)\,, \tag{4.5}$$

$$U_t + f(U_x) = 0\,. \tag{4.6}$$

Subtracting (4.6) from (4.5) and denoting by $E(x,t)$ the error $U^\nu(x,t) - U(x,t)$, we obtain

$$E_t + f'\Big(\xi(x,t)\Big) \cdot E_x = \nu Q(U_{xx}^\nu)\,, \tag{4.7}$$

where $\xi(x,t)$ is between $U^\nu(x,t)$ and $U(x,t)$. It follows from (4.7) and (1.2) that

$$-\nu \le E_t + f'(\xi)E_x \le \nu\,,$$

and hence

$$\|E(\cdot,t)\|_{L^\infty} \le \|E(\cdot,0)\|_{L^\infty} + \mathrm{const}_t \cdot \nu\,.$$

Since $u^\nu(x,0) \equiv u_0(x)$, the last inequality implies (4.2).

Finally, interpolating between the $W^{-1}(L^\infty)$-error estimate, (4.2), and the BV-boundedness of the error (which follows from (4.1) and the well-known BV-boundedness of the entropy solution of (1.6)), we are able to convert the weak error estimate (4.2) into a strong one, (4.3) (e.g., [2, theorem 9.3]). Since this estimate does not hold for $p = 1$, the L^1-estimate (4.4) has to be deduced using an interpolation between $W^{-1}(L^1)$- and BV-spaces (in fact, all error estimates of (1.7) were derived in [5, 12] in this manner). This concludes the proof of Theorem 4.1. ∎

Remarks.

1. For equation (1.7) the following error estimates were shown by Nessyahu and Tadmor [5, 12]:

$$\|e\|_{W^{-1}(L^\infty)} \leq C_t \cdot \sqrt{\varepsilon}\,, \quad \|e\|_{W^{-1}(L^1)} \leq C_t \cdot \varepsilon\,, \quad \|e\|_{L^p} \leq C_t \cdot \varepsilon^{1/2p}\,,$$

   where $e(x,t)$ denotes the difference between the solutions of (1.7) and (1.6). Our differential approximation (1.5) enables us to derive better estimates in the $W^{-1}(L^\infty)$- and L^p-norms ($p > 1$). However, insofar as the $W^{-1}(L^1)$- , L^1-, and L^∞-norms are concerned, our estimates are the same as the ones derived for the vanishing viscosity approximation (1.7).

2. Unlike the standard vanishing viscosity model [5, 12], all our error estimates apply to any smooth $f(u)$, *not necessarily convex.*

5 Existence of the Classical Solution

In this section we demonstrate the existence of the classical solution of the problem (1.1),(1.3) and thus complete the proof of Theorem 3.3. Again consider the regularized viscosity initial value problem, (1.4),(1.3), that admits a unique global classical solution. To derive a weak solution of the nonviscous problem, (1.1),(1.3), defined as a L^1-limit of the sequence u^{δ_n}, we assume adequately small and smooth initial data (3.5). We then demonstrate that this solution is smooth and satisfies (1.1) in the classical sense.

First, we note that equation (1.4) is parabolic and hence its solution satisfies the maximum principle (e.g., [4]), that is,

$$|u^\delta(x,t)| \leq \|u_0\|_{L^\infty}\,, \quad t \geq 0\,. \tag{5.1}$$

We now turn to the major part of the proof and show a uniform boundedness of $|u_x^\delta|$ for small δ's. To this end we rewrite (1.4) as

$$u_t^\delta = z_x\,, \tag{5.2}$$

where

$$z := \nu Q(u_x^\delta) + \delta u_x^\delta - f(u^\delta)\,. \tag{5.3}$$

Differentiating (5.3) with respect to t and using (5.2), we obtain the following parabolic equation:

$$z_t = \nu Q'(u_x^\delta) z_{xx} + \delta z_{xx} - f'(u^\delta) z_x\,. \tag{5.4}$$

Its solution, $z(x,t)$, satisfies a maximum principle:

$$|\nu Q(u_x^\delta(x,t)) + \delta u_x^\delta(x,t) - f(u^\delta(x,t))| \le \|\nu Q(u_0') + \delta u_0' - f(u_0)\|_{L^\infty}, \quad t \ge 0. \tag{5.5}$$

Due to the boundedness of u^δ, (5.1), and the smoothness of f and u_0, one deduces that for δ sufficiently small so that

$$\alpha + \delta\|u_0\|_{L^\infty} \le \beta < \nu \cdot \min(-a,b),$$

the inequalities (5.5) and (3.5) imply

$$|\nu Q(u_x^\delta(x,t)) + \delta u_x^\delta(x,t)| \le \beta, \quad t \ge 0. \tag{5.6}$$

Using the monotonicity of Q, we apply its inverse to (5.6) and conclude that

$$|u_x^\delta(x,t)| \le \mathrm{const}, \quad t \ge 0, \tag{5.7}$$

where $\mathrm{const} = Q^{-1}(\beta)$ does not depend on δ. This is the desired estimate of $|u_x^\delta|$.

Remark. A similar estimate holds for the Riemann problem and can be used toward a partial explanation of the sub- and supercritical solutions.

Equipped with the estimate (5.7) we now turn to proving the uniform boundedness of u_t^δ and u_{xx}^δ. We differentiate (1.4) with respect to t and then denote $w := u_t^\delta$ to obtain

$$w_t + f''(u^\delta)u_x^\delta w + f'(u^\delta)w_x = \nu Q''(u_x^\delta)u_{xx}^\delta w_x + \nu Q'(u_x^\delta)w_{xx} + \delta w_{xx}. \tag{5.8}$$

This is a parabolic equation. As we have shown earlier, the coefficient of w, $f''(u^\delta)u_x^\delta$, is uniformly bounded. Therefore, the maximum principle for (5.8) gives the following estimate:

$$|u_t^\delta(x,t)| \le \mathrm{const}_T, \quad 0 \le t \le T, \tag{5.9}$$

where $\mathrm{const}_T = e^{CT}\|u_t^\delta(\cdot,0)\|_{L^\infty}$ and C is a constant that depends on $\|u_0\|_{L^\infty}$ and $\|u_0'\|_{L^\infty}$ but not on δ. Note that $\|u_t^\delta(\cdot,0)\|_{L^\infty}$ is bounded since the initial condition is assumed to be smooth and since

$$u_t^\delta(x,0) = \nu Q'(u_0'(x))u_0''(x) + \delta u_0''(x) - f'(u_0(x))u_0'(x).$$

Next, the estimate (5.7) implies that $Q'(u_x^\delta)$ is bounded away from zero, that is, $Q'(u_x^\delta) \geq K > 0$, where K is independent of δ. Consequently, from (1.4) we obtain the following estimate:

$$|u_{xx}^\delta| \leq \frac{|u_t^\delta| + |f'(u^\delta)u_x^\delta|}{K}\,, \tag{5.10}$$

which implies a uniform boundedness of $|u_{xx}^\delta|$.

We now recall that in Theorem 3.2 we showed the existence of a sequence u^{δ_n} such that

$$u^{\delta_n}(x,t) \xrightarrow{L^1} u(x,t)\,. \tag{5.11}$$

Due to the uniform boundedness of u^δ, u_x^δ, and u_t^δ, the L^1-convergence of u^{δ_n} implies a pointwise convergence. In addition, there exists a subsequence also denoted by u^{δ_n} that converges to u uniformly. Moreover, u, u_x, u_t, and u_{xx} are also bounded, and therefore in order to conclude that $u(x,t) \in C^{2,1}(x,t)$, it suffices to show that $u_{xx} \in W^1(L^2(x))$.

To this end we differentiate (1.4) three times with respect to x, then multiply by u_{xxx}^δ and integrate over the x-domain. Integrating by parts and taking into account the estimates (5.7), (5.9), and (5.10), we obtain:

$$\begin{aligned} \frac{d}{dt}\|u_{xxx}^\delta\|_{L^2(x)}^2 &\leq K_1\|u_{xxx}^\delta\|_{L^2(x)}^2 + K_2 - 2\delta\|u_{xxxx}^\delta\|_{L^2(x)}^2 \\ &- 2\int\limits_x Q'(u_x^\delta)(u_{xxxx}^\delta)^2 dx + 18\int\limits_x \left|Q''(u_x^\delta)u_{xx}^\delta u_{xxx}^\delta u_{xxxx}^\delta\right| dx\,, \end{aligned} \tag{5.12}$$

where K_1 and K_2 are constants that depend only on the initial data and T. The last term in the RHS of (5.12) can be estimated as follows:

$$\begin{aligned} &18\int\limits_x \left|Q''(u_x^\delta)u_{xx}^\delta u_{xxx}^\delta u_{xxxx}^\delta\right| dx \\ &\leq 9K_3 \cdot \left(\frac{1}{\varepsilon^2}\|u_{xxx}^\delta\|_{L^2(x)}^2 + \varepsilon^2\|u_{xxxx}^\delta\|_{L^2(x)}^2\right). \end{aligned} \tag{5.13}$$

Here $K_3 = \|Q''(u_x^\delta)u_{xx}^\delta\|_{L^\infty}$ and ε is an arbitrary number. Consequently, taking ε such that

$$9K_3\varepsilon^2 \leq 2\|Q'(u_x^\delta)\|_{L^\infty}\,,$$

we can estimate the RHS of (5.12) and obtain a differential inequality,

$$\frac{d}{dt}\|u_{xxx}^\delta\|_{L^2(x)}^2 \leq K_4\|u_{xxx}^\delta\|_{L^2(x)}^2 + K_2\,, \tag{5.14}$$

where K_4 still depends on the initial data and T but does not depend on δ. Thus, (5.14) implies the uniform boundedness of $\|u^\delta_{xxx}\|_{L^2(x)}$, which in turn yields the desired $W^1(L^2(x))$-boundedness of u_{xx}.

To prove that $u(x,t)$ is a classical solution of (1.1), we multiply the “viscous" equation (1.4) by a smooth, compactly supported test function $\varphi(x,t) \in C_0^{2,1}(x,t)$ and integrate it with respect to x and t. Integrating by parts, we obtain

$$\int\limits_{t=0}^{T}\int\limits_{x}\Big\{u^{\delta_n}\varphi_t + f(u^{\delta_n})\varphi_x\Big\}\,dx\,dt = \nu\int\limits_{t=0}^{T}\int\limits_{x} Q(u_x^{\delta_n})\varphi_x\,dx\,dt - \delta\int\limits_{t=0}^{T}\int\limits_{x} u^{\delta_n}\varphi_{xx}\,dx\,dt\,. \tag{5.15}$$

We now pass to the limit in (5.15) as $\delta_n \downarrow 0$. The uniform convergence of u^{δ_n} to u implies that

$$\int\limits_{t=0}^{T}\int\limits_{x}\Big\{u^{\delta_n}\varphi_t + f(u^{\delta_n})\varphi_x\Big\}dx\,dt \xrightarrow{\delta_n\downarrow 0} \int\limits_{t=0}^{T}\int\limits_{x}\Big\{u\varphi_t + f(u)\varphi_x\Big\}dx\,dt\,, \tag{5.16}$$

and

$$\delta\int\limits_{t=0}^{T}\int\limits_{x} u^{\delta_n}\varphi_{xx}\,dx\,dt \xrightarrow{\delta_n\downarrow 0} 0\,. \tag{5.17}$$

It remains to find the limit of the first term in the RHS of (5.15). To this end, since we have already shown the L^∞-boundedness of u^δ_{xx}, (5.10), it is enough to estimate u^δ_{xt}. We differentiate (1.4) with respect to x and t, multiply by $\operatorname{sgn}(u^\delta_{xt})$, and integrate over the x-domain. We obtain

$$\frac{d}{dt}\|u^\delta_{xt}\|_{L^1(x)} + \int\limits_x f(u^\delta)_{xxt}\operatorname{sgn}(u^\delta_{xt})dx = \nu\int\limits_x Q(u^\delta_x)_{xxt}\operatorname{sgn}(u^\delta_{xt})dx + \delta\int\limits_x u^\delta_{xxxt}\operatorname{sgn}(u^\delta_{xt})dx\,. \tag{5.18}$$

For the second term in the LHS of (5.18) we have,

$$\int_x \left(f''(u^\delta)u_x^\delta u_t^\delta + f'(u^\delta)u_{xt}^\delta\right)_x \operatorname{sgn}(u_{xt}^\delta)dx$$
$$= \int_x \left[f'''(u^\delta)(u_x^\delta)^2 u_t^\delta + f''(u^\delta)u_{xx}^\delta u_t^\delta + f''(u^\delta)u_x^\delta u_{xt}^\delta\right] \operatorname{sgn}(u_{xt}^\delta)dx\,.$$

Hence, due to the L^∞-boundedness of u^δ, u_t^δ, u_x^δ, and u_{xx}^δ, it can be estimated as

$$\left|\int_x f(u^\delta)_{xxt} \operatorname{sgn}(u_{xt}^\delta)dx\right| \le C_1\|u_{xt}^\delta\|_{L^1(x)} + C_2\,, \tag{5.19}$$

where the constants C_1 and C_2 depend on the initial data and T. The second term in the RHS of (5.18) is clearly nonpositive. Finally, the first term in the RHS of (5.18) is equal to

$$\int_x \left(Q''(u_x^\delta)u_{xx}^\delta u_{xt}^\delta + Q'(u^\delta)u_{xxt}^\delta\right)_x \operatorname{sgn}(u_{xt}^\delta)dx$$
$$= \int_x \left(Q'(u^\delta)u_{xxt}^\delta\right)_x \operatorname{sgn}(u_{xt}^\delta)dx\,,$$

and, consequently, it is also nonpositive because of our assumption (1.2).

Thus, from (5.18) we obtain the following differential inequality:

$$\frac{d}{dt}\|u_{xt}^\delta\|_{L^1(x)} \le C_1\|u_{xt}^\delta\|_{L^1(x)} + C_2\,. \tag{5.20}$$

Its solution integrated over $(0,T)$ yields the desired L^1-estimate,

$$\|u_{xt}^\delta\|_{L^1(x,t)} \le \mathrm{const}_T\,, \quad 0 \le t \le T\,, \tag{5.21}$$

where const_T depends on $\|u_0\|_{L^\infty}$, $\|u_0'\|_{L^\infty}$, $\|u_0''\|_{L^\infty}$, T, $\|u_{xt}^\delta(\cdot,0)\|_{L^1}$, and the measure of the x-domain, which is assumed to be finite. Note that the boundedness of $\|u_{xt}^\delta(\cdot,0)\|_{L^1}$ follows from the assumption that the initial data is in C^3 and from equation (1.4), which after differentiation with respect to x yields

$$u_{xt}^\delta = -f''(u^\delta)(u_x^\delta)^2 - f'(u^\delta)u_{xx}^\delta$$
$$+ \nu\left[Q''(u_x^\delta)(u_{xx}^\delta)^2 + Q'(u_x^\delta)u_{xxx}^\delta\right] + \delta u_{xxx}^\delta\,.$$

In conclusion, the estimates (5.10) and (5.21) imply that

$$u_x^\delta \in W^1(L^1(x,t))\,,$$

which is compactly imbedded in $L^1(x,t)$. Consequently, there exists a subsequence of u^{δ_n} (also denoted by u^{δ_n}) such that

$$u_x^{\delta_n} \xrightarrow{L^1} u_x\,. \tag{5.22}$$

Therefore, due to the uniform boundedness of u_x^δ and smoothness of Q, we may pass to the limit in the first term in the RHS of (5.15), obtaining

$$\nu \int\limits_{t=0}^{T} \int\limits_{x} Q(u_x^{\delta_n})\varphi_x\,dx\,dt \xrightarrow{\delta_n \downarrow 0} \nu \int\limits_{t=0}^{T} \int\limits_{x} Q(u_x)\varphi_x\,dx\,dt\,. \tag{5.23}$$

Finally, combining (5.16), (5.17), and (5.23) we conclude that, for any test function $\varphi(x,t)$, the limit function $u(x,t)$ satisfies the nonviscous equation (1.1) in the integral sense:

$$\int\limits_{t=0}^{T} \int\limits_{x} \{u\varphi_t + f(u)\varphi_x\}\,dx\,dt = \nu \int\limits_{t=0}^{T} \int\limits_{x} Q(u_x)\varphi_x\,dx\,dt\,. \tag{5.24}$$

But as we have shown earlier, $u(x,t) \in C^{2,1}(x,t)$. Combined with (5.24) this means that $u(x,t)$ is a classical solution of the problem (1.1),(1.3). The proof of Theorem 3.3 is thus completed.

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