(ii) The collision operator must have a numerically efficient "fast" solver; one should be able to exploit the structure in a parallel computer environment. Stiffness problems must be reduced by an efficient handling of collision parameters which is to be implemented in an adaptive scheme.

(iii) There must be a strategy to pass over gradually from rarefied regimes to the fluid dynamic limit. This may be obtained by a hierarchical collision model with small scale and large scale structures.

3. Design of kinetic models. The first feature of our artificial discrete systems is based on the observation that special collision systems with parameters which are defined according to a partition of a rectangular grid give rise to a hierarchical system of kinetic equations. Properly transformated, this system decouples a small part of nonlinear equations from a large linear one. The study of the linearized equations demonstrates that the linear operator may be decomposed in a way which reduces the number of required operations impressively (very much like the decomposition of the discrete Fourier transformation leads to the Fast Fourier Transformation). This decomposition rule can be carried over to the nonlinear equations. This model has first been proposed in [3] and is also described in [5, 6]. It is a preliminary model which due to the restriction to "axiparallel" collision laws has artificial collision invariants which have to be suppressed by appropriate projection methods. This feature can be readily removed by using rotating grids (in velocity space). On the other hand, the projections can be handled such that the fluid dynamic limit results in the correct Euler equations.

Based on the model above, a system of more and more refining multilevel structures in velocity space can be constructed (resembling a Haar wavelet decomposition of the velocity space). This system keeps all the desirable properties of the one-level scheme described above. One way to pass to the fluid dynamic limit is to speed up the small scale collision procedure up to a certain level, thus reducing the number of flow variables. This reduction can be done adaptively in a fluid domain thus establishing a "smooth" transition from gas kinetics to fluid dynamics. Such a scheme has been proposed in [4] (see also [5]). A more sophisticated and flexible model with the correct number of collision invariants is presently to be constructed [1].

4. Conclusions. We are going to give impulses for the design of discrete kinetic models which combine a large amount of complexity as given in the models of Rarefied Gas Dynamics with efficient solvability and which allow for a hierarchical structure which makes it possible to study in detail questions concerning the passage to the fluid dynamic limit.

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L^1 stability estimate of hyperbolic semilinear systems with quadratic nonlinear source terms

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In this talk, we consider hyperbolic semilinear systems with quadratic nonlinear source terms:

$$\partial_t f_i + \partial_x (v_i(x,t) f_i) = \sum_{j,k} B_i^{jk} f_j f_k.$$

This system is assumed to be strictly hyperbolic in the sense that all characteristic velocities are different and it includes the general one-dimensional discrete Boltzmann models such as 1D-Broadwell equations. Under various assumptions on B_i^{jk} and initial data, we construct nonlinear functionals $\mathcal{H}(t)$ which is equivalent to the L^1 distance and non-increasing in time t. Using this nonlinear functional, we prove the L^1 stability of mild solutions:

$$||f(\cdot,t) - \bar{f}(\cdot,t)||_{L^1(\mathbb{R})} \leq G||f_0(\cdot) - \bar{f}_0(\cdot)||_{L^1(\mathbb{R})},$$

where f and $\bar{f}$ are mild solutions corresponding to two initial data f_0 and $\bar{f}_0$ and G is a positive constant independent of time t.

4.22 Tuesday, Session 5 (afternoon): Entropic schemes

Finite Volume Schemes for Van der Waals Fluids and Fully Discrete Cell-Entropy Inequalities

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The present paper is devoted to the numerical approximation of the solutions of mixed (hyperbolic-elliptic) systems of conservation laws modeling the dynamics of compressible with phase change. Here the non isothermal framework is addressed when considering the capillarity coecient as a general (positive) function of the specifc volume. Starting from a given capillarity law, we first introduce an extended version of the system still in conservation form and which by construction encode all the smooth solutions of the original PDEs. A given solution is recovered provided that the initial data of the extended version is suitably prescribed. Taking advantage of such a result, the structure of the extended system allows us to derive a natural numerical procedure where the approximation of the interstitial energy plays a central role. The numerical procedure we propose results in fully discrete Finite Volume Methods that obey cell-entropy inequalities under a natural CFL condition.

To the theory of generalized entropy solutions of the Cauchy problem for a first order quasilinear equation in the class of locally integrable functions

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Consider the Cauchy problem for a first order quasilinear equation

$$u_t + \mathrm{div}_x \varphi(u) = g(t,x,u), \tag{1}$$

$\varphi = (\varphi_1, \dots, \varphi_n)$, $u = u(t,x)$, $(t,x) \in \Pi_T = (0,T) \times \mathbb{R}^n$, $T > 0$, with the initial condition

$$u(0,x) = u_0(x) \in L^1_{loc}(\mathbb{R}^n). \tag{2}$$

We assume that the flux vector $\varphi(u)$ is only continuous and satisfies the growth restriction

$$|\varphi(u)| \le C_0(1 + |u|),\ C_0 = \mathrm{const}. \tag{3}$$

Suppose that the source function $g = g(t,x,u) \in L^1_{loc}([0,T) \times \mathbb{R}^n)$ for any fixed $u \in \mathbb{R}$, is continuous with respect to the variable u for all $(t,x) \in \Pi_T$ and for some constant $L = L(g) \ge 0$ $g(t,x,u) - g(t,x,v) \le L(u-v)\ \forall u, v \in \mathbb{R}$, $u > v$.

Definition 1 (S.N.Kruzhkov, [4]). A locally integrable function $u = u(t,x) \in L^1_{loc}(\Pi_T)$ is called a generalized entropy solution (briefly - g.e.s.) of the Cauchy problem (1), (2) if $g = g(t,x,u(t,x)) \in L^1_{loc}(\Pi_T)$ and
a) $\forall k \in \mathbb{R}\ |u-k|_t + \mathrm{div}_x[\mathrm{sign}\,(u-k)(\varphi(u) - \varphi(k))] \le \mathrm{sign}\,(u-k)g$ in the sense of distributions on Π_T (in $\mathcal{D}'(\Pi_T)$); b) $\operatorname{ess}\lim_{t \to 0+} u(t,\cdot) = u_0$ in $L^1_{loc}(\mathbb{R}^n)$.

Formally one could introduce a g.e.s. $u \in L^p_{loc}(\Pi_T)$ for $p > 1$ replacing (3) to the requirement $|\varphi(u)| \le C_0(1 + |u|^p)$ (without any restrictions for $p = \infty$). But, in this case, the Cauchy problem seems to be ill-posed. Even if $n = 1$, $g \equiv 0$ none of the positive results (established below for $p = 1$) remains valid in the general case (see, for instance, [3]).

In the case under consideration when the property of finite speed of propagation for initial perturbation can be violated uniqueness of g.e.s. to the problem (1), (2) is no longer valid even in the class of bounded g.e.s. (see examples in [5,6]). In this

connection the following result about existence of the maximal and the minimal solution plays important role. In the class of bounded g.e.s. similar results were established under various additional restrictions in the papers [1,2,8].

Theorem 1. *There exist the maximal and the minimal g.e.s. of the problem* (1), (2).

Some sufficient conditions for uniqueness of g.e.s. in the class L^∞ were found in [5,6]. After some corrections these conditions seem to be suitable for the general case as well.

Theorem 2. *Suppose that there exist nondecreasing subadditive functions* $\omega_i(r)$ *on* $\mathbb{R}_+ = (0, +\infty)$, $\omega_i(r) > 0$ *for* $r > 0$ *such that*

$$|\varphi_i(u) - \varphi_i(v)| \le \omega_i(|u - v|)\ \forall u, v \in \mathbb{R},\ i = 1, \dots, n,$$

$$\liminf_{r \to 0+} r^{1-n} \prod_{i=1}^{n} \omega_i(r) < \infty.$$

Then a g.e.s. of the problem (1), (2) *is unique.*

We also establish the following estimates of L^p-norms of g.e.s. (cf. [2,7]).

Theorem 3. *Suppose that for* $1 \le p \le \infty$ $u_0(x) \in L^p(\mathbb{R}^n)$, $g_0(t,x) = g(t,x,0) \in L^1((0,T), L^p(\mathbb{R}^n))$; $u = u(t,x)$ *is a g.e.s. of the problem* (1), (2). *Then* $u(\tau, \cdot) \in L^p(\mathbb{R}^n)$ *for a.e.* $\tau \in (0,T)$ *and*

$$\|u(\tau,\cdot)\|_p \le e^{L\tau} \left(\|u_0\|_p + \int_0^\tau \|g_0(t,\cdot)\|_p dt \right), \quad L = L(g).$$

Let $\xi = \{\xi_1, \dots, \xi_n\}$ be a basis on $\mathbb{R}^n$. Functions $u(x) \in L^1_{loc}(\mathbb{R}^n)$, $u(t,x) \in L^1_{loc}(\Pi_T)$ are called ξ-periodic if $u(x + \xi_i) = u(x)$ a.e. on $\mathbb{R}^n$ and respectively $u(t, x+\xi_i) = u(t,x)$ a.e. on Π_T for all $i = 1, \dots, n$. In the same way we define ξ-periodicity of functions $g(t,x,u)$.

Theorem 4. *Suppose that input data* $u_0(x)$, $g(t,x,u)$ *are* ξ*-periodic. Then a g.e.s. of the problem* (1), (2) *is also* ξ*-periodic and unique.*

The proof is based on the simple observation that the maximal and the minimal g.e.s. are ξ-periodic functions that easy implies their coincidence.

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The Upwind Interface Source method for hyperbolic conservation laws with a source term

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We shall report on joint works with B. Perthame (refer to [6] and [7]), which deal with typical questions arising in the analysis of numerical approximations for hyperbolic conservation laws with a source term. The presence of source terms deeply influences the numerical approach to the problem, as it was pointed out by several authors (refer to [3] and [5]), because of the difficulty to

investigate approximations preserving the steady state solutions. We focus our attention on semi-discrete finite volume schemes, in the general case of a nonuniform spatial mesh, in a one-dimensional framework. To define appropriate discretizations of the source term, we introduce the formalism peculiar to the Upwind Interface Source method (originally formulated by Roe [8]) and we establish conditions on the numerical functions so that the discrete solver preserves the steady states. Then we formulate a rigorous definition of consistency, adapted to the class of well-balanced schemes, for which we are able to prove a Lax-Wendroff type convergence theorem. Some examples of numerical solvers are discussed (see [1], [2], [4] for instance), in order to validate the arguments we propose.

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A novel theory of implicit methods for scalar conservation laws

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The aim of the talk is to present a novel theory of implicit methods for conservation laws which does not rely on classical structures in numerical analysis e.g. like employing the notion of total variation stability. Thereby, we improve the range of applicability of classical numerical methods considerably and we give rigorous theoretical results concerning the applicability of implicit schemes. The presented results are fundamental for useful extensions to higher spatial dimensions and conservation laws with source terms. The usefulness of the presented theory is reflected by the applicability of the developed implicit notions with respect to various applications, see e.g. [2] for a typical example of non-classical applications.
We consider the following Cauchy problem for nonlinear hyperbolic scalar conservation laws:

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial\,(u)}{\partial x} &= 0, \text{ with} t \in (0,T], \\ u\,(x,0) &= u_0\,(x)\,, \end{aligned}$$

with the flux function $f(u)$ in $C(\mathbf{R})$ and the initial data u_0 in $\left(L_1^{loc}\,(\mathbf{R}) \cap L_\infty\,(\mathbf{R})\right)$. As it is well-known, this problem in general does not admit smooth solutions, so that the notion of weak solutions is normally used together with an entropy condition as a criterion for the uniqueness of the weak solution.

The stated assumptions are rather weak since we do not assume the classical Lipschitz-continuity or differentiability of the flux function (or convexity or concavity) and we also do not assume that $u_0 \in BV(\mathbf{R})$.

The most general entropy condition valid for the problem under consideration is the entropy condition due to Kruzhkov, see e.g. [1]. As it can be shown directly by the help of the example described in [1], the stated weak assumption on the flux function – namely merely continuity – has the effect that initial data with compact support in $\mathbf{R}$ can be distributed on the whole real axis after an arbitrarily small timestep. One may consider this phenomenon as a parabolical degeneracy of the solution since it is evident by heuristic arguments, that only rarefaction waves can extend to infinity which gives the solution a parabolic flavour. However, the physical process modeled by the considered problem is nevertheless a transport process, i.e. data can be transported with infinite speed without showing some diffusive habits, and so one cannot treat such problems in the same manner as parabolic problems. In the parabolical case one could define a numerical method with explicit time integration which gives correct solutions in the sense of a truncation error since the diffusive parabolical influence means that the influence of the data decreases exponentially to zero, and exactly such estimates for the influence of data cannot be given.

Concerning the construction of numerical methods for such cases, this also means that nonlinear stability of a method cannot be guaranteed by employing the notion of solutions with bounded variation because these rely on the compact support of the data over a period of time $[0, T]$. Because of this, the concept of TVD-schemes is useless in this context. Also, explicit methods cannot be used since the CFL number which would have to be employed is effectively zero. Another possible notable behaviour of merely continuous fluxes is of course non-convexity (and non-concavity), i.e. solutions can be composed of mixed waves. Since we intend to construct methods without relying on the classical notions for explicit methods, we have to verify that this case is included by our considerations in order to show the validity of the developed implicit notions. The entropy condition due to Kruzhkov which is the natural condition of choice within this work incorporates an integral inequality where already the weak formulation of the original problem comes in conjuncture with the entropy functional and entropy flux of Kruzhkov, i.e. technically we do not need both the notion of weak solutions of the original problem and a further entropy condition since this is the same in Kruzhkovs sense. However, in the course of the technical reasoning we employ in the construction of our implicit theory, we copy the standard approach to show first the convergence to a weak solution and we then employ this knowledge to show the convergence to the enropy solution.

When employing implicit schemes, trying to prove monotonicity or convergence to the entropy solution, etc., one realizes quickly that rigorous criteria for properties of implicit schemes are not available. It is generally assumed that implicit methods inherit the basic properties of corresponding explicit schemes without doing a rigorous analysis. In the analysis of numerical methods for the defined initial value problem, linearization techniques cannot be applied since in the case of merely continuous fluxes even a generalized derivative can take on infinite values and is of no use; these infinite values for the flux derivative indeed occur in the case of infinite information speed. This in turn also means that Taylor expansions as well as criteria on derivatives of f (like in classical monotonicity criteria) are not available as mechanisms in the devicing of numerical procedures for this case. Also, it is suggested that implicit schemes can capture all information since the numerical characteristics of implicit schemes can capture information from the whole computational domain since the numerical characteristics cover it totally while the properties of the scheme remain invariant. We present a counterexample, namely the implicit version of the classical Lax-Friedrichs-scheme which loses some of its properties – in fact its most valuable property, namely the monotonicity – when the transport of information is too fast.

We present a novel general theory for implicit methods for scalar conservation laws which incorporates simple criteria for implicit monotonicity, general assertions about consistency of implicit schemes with the entropy condition of Kruzhkov and L_∞-stability and an innovative convergence proof which does not rely on the TV-notion and which is to our knowledge new. The mentioned implicit notions are proven to be sensible extensions of their explicit counterparts without using any derivatives or other related techniques which would restrict the range of our approach, i.e. we use totally different techniques then it is the case in the standard theory.

Three implicit variants of classical explicit schemes are discussed by using the theory showing the validity and usefulness of the implicit notions. The theoretical results of these investigations are validated by numerical results for two test cases showing exactly the difficulties one can encounter in the case of conservation laws with continuous fluxes: Propagation of information with infinite speed and mixed-wave-solutions. As an example for the first class of problems we use an example given by Kruzhkov and Panov in [1]. An example of a solution featuring mixed waves is discussed by the help of a Riemann-problem using the famous Buckley-Leverett-equation. In both cases, we compare our numerical results with the exact solution, showing both the robustness of the tested methods for distinguished problems in real life applications and the reliable significant improvement achieved concerning the range of applicability of implicit methods for conservation laws.

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4.23 Tuesday, Session 6 (morning): Multiphase I

Stochastic Modeling of Immiscible Flow with Moment Equations

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We study a model of two-phase oil-water flow in a heterogeneous reservoir, and present a direct method of obtaining statistical moments. The method is developed as an approach either to scale-up, or to uncertainty propagation, for a general class of nonlinear hyperbolic equations. Second-order moment differential equations are derived using a perturbation expansion in the standard deviation of an underlying random process, which in this application is log permeability. The perturbation approach is taken because test results do not support the use of a multivariate Gaussian assumption to close the system. Moments may depend on location; the common assumption of statistical homogeneity is not necessary.

Classification of the resulting coupled system of nonlinear equations will be discussed. In one space dimension, the system is hyperbolic, and the analytical solution exhibits a bimodal character. The theory does not extend to 2D, but qualitative numerical results are similar. These will be compared to the results of Monte Carlo simulations, which are smoother and shock-free. Moment equations can yield approximate statistical information considerably more efficiently. This is joint work with Kenneth D. Jarman of Pacific Northwest National Laboratory.

Existence of Oxidation Fronts in a Two-Phase Flow in Porous Media

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In this work we discuss the traveling wave solutions [5] for a 2x2 parabolic system with source terms which governs a two-phase flow consisting of oxygen and oil moving linearly in a porous media, taking into account an oxidation process, (see [1]). These solutions represent oxidation fronts propagating with a certain speed, leading to an oxidized state behind and a non oxidized state ahead. The resulting system is described in [2], and it may be writen as

$$s_t + f_x = (hs_x)_x + \chi q \tag{1}$$

$$\theta_t = (\gamma\theta_x)_x + \eta q\,, \tag{2}$$

where the state variables depending on (x,t) are the oil saturation denoted by s ($1-s$ is the gas saturation) and the temperature by θ. We have the following functions with variables s and θ: fractional flow function of oil, denoted by f; a positive function relating the gas-oil capillary pressure difference, denoted by h; and the reaction rate function denoted by q, considered as an Arrhenius type law stating that the total amount of oxygen mass consumed per unit time is proportional to the oxygen available and to a temperature dependent factor

$$q(s,\theta) = A_r s(1-s)e^{-\frac{E}{R(\theta-\theta_0)}} \quad \text{if} \quad \theta > \theta_0 \quad \text{or} \quad q(s,\theta) = 0 \quad \text{if} \quad 0 \le \theta \le \theta_0\,, \tag{3}$$

where A_r is the Arrhenius´s rate coefficient, E is the activation energy, and θ_0 is the temperature where the reaction starts. Finally, χ, η, and γ are positive constants, defined in terms of physical properties of the fluid and rock.
To analyze the oxidation fronts, it is necessary to write the source terms in equations (1), (2) as derivatives. So, we introduce an artificial variable defined by

$$Q = \int_x^\infty q(s,\theta)\, dx' \,. \tag{4}$$

With this new variable the system (1)-(2) has to be increased by one equation and may be rewritten as

$$s_t + f_x = (hs_x)_x - \chi Q_x \tag{5}$$

$$\theta_t = (\gamma\theta_x)_x - \eta Q_x \tag{6}$$

$$Q_x = -\,q\,. \tag{7}$$

From Equation (4)

$$\lim_{x\to\infty} Q(x) = Q^+ = 0, \text{ and we assume that } \lim_{x\to-\infty} Q(x) = Q^- < \infty.$$

A traveling wave solution with propagation speed σ connecting a non oxidized state $U^+ = (s^+, \theta^+, Q^+ = 0)$ to an oxidized state $U^- = (s^-, \theta^-, Q^-)$ is a function of the type $U(\xi) = (s(\xi), \theta(\xi), Q(\xi))$, depending on the single variable $\xi = x - \sigma t$ with a profile that satisfies the ordinary differential system

$$\frac{ds}{d\xi} = \sigma(s^+ - s) - (f^+ - f) + \chi Q \tag{8}$$

$$\frac{d\theta}{d\xi} = \frac{h}{\gamma}\{\sigma(\theta^+ - \theta) + \eta Q\} \tag{9}$$

$$\frac{dQ}{d\xi} = -q\,, \tag{10}$$

and the asymptotic limits

$$\lim_{\xi\to-\infty} U(\xi) = (s^-, \theta^-, Q^-)\,, \quad \lim_{\xi\to+\infty} U(\xi) = (s^+, \theta^+, Q^+ = 0)\,, \tag{11}$$

$$\lim_{\xi\to-\infty} \frac{dU}{d\xi} = (0,0,0)\,, \quad \text{and} \quad \lim_{\xi\to+\infty} \frac{dU}{d\xi} = (0,0,0)\,. \tag{12}$$

The speed σ and the states U^+ and U^- are related by the following Rankine-Hugoniot condition, obtained by taking the limit as $\xi \to -\infty$ in equations (8)-(10),

$$\sigma(s^+ - s^-) - (f(s^+, \theta^+) - f(s^-, \theta^-)) + \chi Q^- = 0 \tag{13}$$

$$\sigma(\theta^+ - \theta^-) + \eta Q^- = 0 \tag{14}$$

$$q(s^-, \theta^-) = 0\,. \tag{15}$$

Consequently the non oxidized state U^+ and the oxidized state U^- are equilibrium points of the System (8)-(10). By physical reasons we can consider $\theta^- > \theta_0$, and from equations (3) and (15) it follows that $s^- = 1$ or $s^- = 0$. If we consider that all oxygen is consumed by the chemical reaction and all produced CO_2 is immediately dissolved in the oil phase, the case of interest is $s^- = 1$.
Our main result is that if the speed σ is large enough, $\sigma > f_s(s^+, \theta^+)$, then there exists a unique heteroclinic orbit of the System (8)-(10) connecting the oxidized state U^- to the non oxidized state U^+. In this case the temperature of the oxidized state is larger than the temperature of the non oxidized state, as physically expected. This heteroclinic orbit is the traveling wave solution we are looking for. If $0 \le \sigma < f_s(s^+, \theta^+)$, there is no heteroclinic connection, at least generically. If $\sigma < 0$ there is a heteroclinic connection, but in this case the temperature of the oxidized state is less than the temperature of the non oxidized state, and this is physically not expected.
The proofs is based on arguments of continuity and on the invariant manifold theory [4, 3].

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4.24 Tuesday, Session 6 (afternoon): Multiphase II

Fundamental Waves of the Riemann Problem for a Simplified Oxidation Model in Porous Media

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A procedure which is sometimes used to increase oil production in petroleum reservoirs consists in pumping air into injection wells in order to maintain pressure and to expel oil to producing wells. This procedure usually leads to low temperature oxidation [1, 2] and it is necessary to guarantee that all oxygen is consumed before reaching the production wells, because of safety concerns.

We present a simplified model in which we consider a two-phase flow consisting of gas and oil moving linearly in porous media. We assume that the injected gas is pure oxygen and that it forms CO_2 in an exothermic chemical reaction which is totally dissolved in the oil phase as soon as it is generated. The oxidation rate is approximated by an Arrhenius type law which states that the total amount of oxygen mass consumed per unit time is proportional to the oxygen available and to a temperature dependent factor. For simplicity, in the current work we assume that the densities are constant and we neglect the gravity effects and heat loss to the rock formation, but we take into account longitudinal heat conductivity and capillary forces. The viscosity of the oil phase depends on the temperature.

The derivation of the mathematical model follows from the mass balance of the phases, the balance of energy and from Darcy's law of force applied to each phase [3]. Denoting the oil saturation by s (consequently the gas saturation is $1-s$), the temperature by θ and the total seepage velocity of the fluid by v, under the above simplified assumptions the mathematical model may be written as

$$\frac{\partial s}{\partial t} + \frac{\partial}{\partial x}(vf) = \frac{\partial}{\partial x}\left(h\,\frac{\partial s}{\partial x}\right) + \chi q \tag{1}$$

$$\frac{\partial}{\partial t}((\alpha+s)\theta) + \frac{\partial}{\partial x}(v(\beta+f)) = \frac{\partial}{\partial x}\left(\theta h\,\frac{\partial s}{\partial x} + \gamma\frac{\partial \theta}{\partial x}\right) + \eta q \tag{2}$$

$$\frac{\partial v}{\partial x} = \delta q\,. \tag{3}$$

In system (1)-(3), the function $f = f(s,\theta)$ is the fractional flow function of the oil phase, which depends on the relative mobilities of both phases. The function $h = h(s,\theta)$ is a nonnegative function depending on the gas relative mobility and on the gas-oil capillary pressure difference. The function $q(s,\theta)$ represents the reaction term. Finally, the Greek letters α, β, γ, δ, η and χ are constants defined in terms of physical properties of the fluid and of the rock. In particular, the constant δ in system (1)-(3) is proportional to the difference between the densities of the oil and gas phases. Thus, if we consider the unrealistic assumption that the phase densities are equal, the system may be reduced to the equation (3).

This mathematical model allows us to develop some theoretical analysis and its numerical solution allows us to predict temperature and saturations in space and time. Since we neglect heat loss to the rock formation and include the longitudinal conductivity, the oxidation gives rise to a temperature front with small amplitude. We assume that at the right of the oxidation front there is a non-oxidized mixture of gas (oxygen) and oil phases, at a certain temperature. At the left of the front, there is only an oleic phase, with dissolved CO_2, at a higher temperature resulting from oxidation.

Our main result states that the Riemann solution of system (1)-(3) consists of sequences of three types of fundamental waves. One fundamental wave is the oxidation wave, which is analyzed in [4] as a traveling wave solution. In the non-oxidized region either the temperature is constant and the system is reduced to equation (1) corresponding to the Buckley-Leverett's two phase flow equation, or the temperature possesses a finite jump and a contact wave is added to the solution. In the oxidized region, since all oxygen is supposed to be consumed by the chemical reaction and all produced CO_2 is assumed to be dissolved in the oil phase, the system of differential equations is reduced to equation (2) corresponding to a linear advection-diffusion equation in temperature.

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A Simple Method for Compressible Multiphase Mixtures and Interfaces

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We consider the numerical resolution of a compressible multiphase flow model, first proposed by Baer and Nunziato [1] for detonation waves in granular explosives, and modified in Saurel and Abgrall [4] for the resolution of multiphase mixtures and interface problems between pure compressible materials. The governing equations for the one-dimensional compressible

two-phase flow are [4]

$$\begin{aligned}
\frac{\partial \alpha_g}{\partial t} + V_{\text{int}}\frac{\partial \alpha_g}{\partial x} &= \mu(P_g - P_l) \\
\frac{\partial \alpha_g \rho_g}{\partial t} + \frac{\partial \alpha_g \rho_g u_g}{\partial x} &= 0 \\
\frac{\partial \alpha_g \rho_g u_g}{\partial t} + \frac{\partial \alpha_g \rho_g u_g^2 + \alpha_g P_g}{\partial x} &= P_{\text{int}}\frac{\partial \alpha_g}{\partial x} + \lambda(u_l - u_g) \\
\frac{\partial \alpha_g \rho_g E_g}{\partial t} + \frac{\partial \alpha_g u_g(\rho_g E_g + P_g)}{\partial x} &= P_{\text{int}}V_{\text{int}}\frac{\partial \alpha_g}{\partial x} + \lambda(u_l - u_g)V_{\text{int}} \\
&\quad -\mu P_{\text{int}}(P_g - P_l) \\
\frac{\partial \alpha_l \rho_l}{\partial t} + \frac{\partial \alpha_l \rho_l u_l}{\partial x} &= 0 \\
\frac{\partial \alpha_l \rho_l u_l}{\partial t} + \frac{\partial \alpha_l \rho_l u_l^2 + \alpha_l P_l}{\partial x} &= -P_{\text{int}}\frac{\partial \alpha_g}{\partial x} - \lambda(u_l - u_g) \\
\frac{\partial \alpha_l \rho_l E_l}{\partial t} + \frac{\partial \alpha_l u_l(\rho_l E_l + P_l)}{\partial x} &= -P_{\text{int}}V_{\text{int}}\frac{\partial \alpha_g}{\partial x} - \lambda(u_l - u_g)V_{\text{int}} \\
&\quad +\mu P_{\text{int}}(P_g - P_l).
\end{aligned} \tag{1}$$

Here, the gas and liquid phases are denoted with the subscripts g and l, respectively, and the interface parameters with the subscript "int". Let α_k be the volume fractions, ρ_k the material densities, P_k the pressures, and $E_k = e_k + u_k^2/2$ the specific total energies for $k = g, l$. The parameters λ and μ determine the relaxation rates of velocities and pressures of the phases, see [4]. In [4], the following definitions are proposed for the interface pressure P_{int} and velocity V_{int},

$$\begin{aligned}
P_{\text{int}} &= \alpha_g P_g + \alpha_l P_l, \\
V_{\text{int}} &= (\alpha_g \rho_g u_g + \alpha_l \rho_l u_l)/(\alpha_g \rho_g + \alpha_l \rho_l).
\end{aligned}$$

Other choices are possible according to the physical situation of interest. The equations (1) are closed by two equations of state (EOS), here we use the *stiffened gas EOS*, and the saturation constraint for the volume fractions,

$$\begin{aligned}
P_k &= (\gamma_k - 1)\rho_k e_k - \gamma_k \pi_k, \\
\alpha_g + \alpha_l &= 1,
\end{aligned} \tag{2}$$

where γ_k and π_k are constants, specific for each phase. For gas we take $\gamma_g = 1.4$, $\pi_g = 0$, for liquid $\gamma_l = 4.4$, $\pi_l = 6 \cdot 10^8$ Pa. An important feature of the system (1) is the presence of so-called non-conservative terms. These are the terms $P_{\text{int}}\frac{\partial \alpha_g}{\partial x}$, $P_{\text{int}}V_{\text{int}}\frac{\partial \alpha_g}{\partial x}$ at the right hand side of (1) and the transport equation of the gas volume fraction. Due to these terms, one cannot write the system (1) in a divergence form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0.$$

Consequently, the usual notions of the weak solution and the Rankine-Hugoniot conditions are not applicable for the system (1). The characteristic analysis for the system (1) shows that it is hyperbolic, although not strictly hyperbolic. Indeed, situations are possible, when some of its eigenvalues

$$\begin{aligned}
&\lambda_1 = V_{\text{int}}, \\
&\lambda_2 = u_g - c_g, \quad \lambda_3 = u_g, \quad \lambda_4 = u_g + c_g, \\
&\lambda_5 = u_l - c_l, \quad \lambda_6 = u_l, \quad \lambda_7 = u_l + c_l
\end{aligned} \tag{3}$$

can coincide with each other. Moreover, it is easy to see that the corresponding eigenvectors can become linearly dependent. For the 2–7 characteristic fields, we can find the corresponding Riemann invariants. Firstly, the gas volume fraction

$$\alpha_g = \text{const}$$

across λ_2–λ_7. Then, the liquid parameters do not change across the gas waves, i.e.

$$\rho_l = \text{const}, \; u_l = \text{const}, \; P_l = \text{const} \;\; \text{across} \;\; \lambda_2, \lambda_3, \lambda_4,$$

and the gas parameters are constant across the liquid waves, i.e.

$$\rho_g = \text{const},\ u_g = \text{const},\ P_g = \text{const}\ \text{ across }\ \lambda_5, \lambda_6, \lambda_7.$$

One can also see that the 2, 4, 5, and 7 characteristic fields are genuinely nonlinear, and 3 and 6 linearly degenerate. The corresponding Riemann invariants are

$$\begin{aligned}
&\frac{P_g + \pi_g}{\rho_g^{\gamma_g}} = \text{const},\ u_g + \frac{2c_g}{\gamma_g - 1} = \text{const across } \lambda_2\text{-rarefaction}\\
&P_g = \text{const},\ u_g = \text{const}\ \text{ across }\ \lambda_3\\
&\frac{P_g + \pi_g}{\rho_g^{\gamma_g}} = \text{const},\ u_g - \frac{2c_g}{\gamma_g - 1} = \text{const}\ \text{ across }\ \lambda_4\text{-rarefaction.}
\end{aligned}$$

Analogously,

$$\begin{aligned}
&\frac{P_l + \pi_l}{\rho_l^{\gamma_l}} = \text{const},\ u_l + \frac{2c_l}{\gamma_l - 1} = \text{const}\ \text{ across }\ \lambda_5\text{-rarefaction}\\
&P_l = \text{const},\ u_l = \text{const}\ \text{ across }\ \lambda_6\\
&\frac{P_l + \pi_l}{\rho_l^{\gamma_l}} = \text{const},\ u_l - \frac{2c_l}{\gamma_l - 1} = \text{const}\ \text{ across }\ \lambda_7\text{-rarefaction}
\end{aligned}$$

for the liquid. We see that the Riemann invariants for the phases $k = g, l$ coincide with those for the Euler equations.
The characteristic analysis of the first characteristic field, which corresponds to the material interface, appears to be very extensive and complicated.
In this work, we are interested in the numerical solution of the system (1). The methodology which we follow was introduced in Saurel and Abgrall [4]. We use the Strang splitting technique for the numerical solution of (1):

$$\mathbf{V}_i^{n+1} = L_s^{\Delta t/2} L_h^{\Delta t} L_s^{\Delta t/2} \mathbf{V}_i^n,$$

where $\mathbf{V}_i^n$ is the vector of state variables on a mesh cell i and time t_n, i.e.

$$\mathbf{V}_i^n = (\alpha_g, \alpha_g\rho_g, \alpha_g\rho_g u_g, \alpha_g\rho_g E_g, \alpha_l\rho_l, \alpha_l\rho_l u_l, \alpha_l\rho_l E_l)^T,$$

L_h is the operator of numerical solution of the hyperbolic part of the system (1), and L_s is the operator of integration of the source and relaxation terms. Here we focus on the hyperbolic operator; the details on relaxation procedures may be found in Lallemand and Saurel [3].
For the numerical solution of the hyperbolic part of (1), we employ a Godunov-type scheme

$$\mathbf{U}_i^{n+1} \quad = \quad \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\left[\mathbf{f}(\mathbf{U}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)) - \mathbf{f}(\mathbf{U}^*(\mathbf{U}_{i-1}^n, \mathbf{U}_i^n))\right] + \Delta t\, \mathbf{H}_i^n \Delta, \tag{4}$$

Here, $\mathbf{U}$ is the state vector, $\mathbf{f}$ is the conservative flux,

$$\mathbf{U} = \begin{bmatrix} \alpha_g\rho_g \\ \alpha_g\rho_g u_g \\ \alpha_g\rho_g E_g \\ \alpha_l\rho_l \\ \alpha_l\rho_l u_l \\ \alpha_l\rho_l E_l \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} \alpha_g\rho_g u_g \\ \alpha_g\rho_g u_g^2 + \alpha_g P_g \\ \alpha_g u_g(\rho_g E_g + P_g) \\ \alpha_l\rho_l u_l \\ \alpha_l\rho_l u_l^2 + \alpha_l P_l \\ \alpha_l u_l(\rho_l E_l + P_l) \end{bmatrix}, \quad \mathbf{H} = \begin{bmatrix} 0 \\ P_{\text{int}} \\ P_{\text{int}} V_{\text{int}} \\ 0 \\ -P_{\text{int}} \\ -P_{\text{int}} V_{\text{int}} \end{bmatrix},$$

the state $\mathbf{U}^*(\mathbf{U}_i^n, \mathbf{U}_{i+1}^n)$ is the intermediate state in the solution of the Riemann problem (1) with the states $\mathbf{U}_i^n$, $\mathbf{U}_{i+1}^n$, and Δ is some discretization of the non-conservative terms at the right hand side of (1). Assume for the moment that the state $\mathbf{U}^*$ has been determined. Then, we determine Δ, as well as the discretization for the transport equation for the gas volume fraction in such a way, that the numerical approximation to the system would preserve a moving contact discontinuity. Some computation gives us the following discretizations

$$\alpha_i^{n+1} \quad = \quad \alpha_i^n - (V_{\text{int}})_i^n \frac{\Delta t}{\Delta x}(\alpha_{i+1/2}^* - \alpha_{i-1/2}^*), \tag{5}$$

$$\Delta \quad = \quad \frac{1}{\Delta x}(\alpha_{i+1/2}^* - \alpha_{i-1/2}^*). \tag{6}$$

Following the idea of [2], we find $\mathbf{U}^*$ as follows. Consider the Riemann problem for the hyperbolic part of the system (1), written in the primitive variable formulation,

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}\frac{\partial \mathbf{W}}{\partial x} = 0, \tag{7}$$

$$\mathbf{W}(x,0) = \begin{cases} \mathbf{W}_l, & x \le 0 \\ \mathbf{W}_r, & x > 0, \end{cases} \tag{8}$$

where

$$\mathbf{W} = \begin{bmatrix} \alpha_g \\ \rho_g \\ u_g \\ P_g \\ \rho_l \\ u_l \\ P_l \end{bmatrix}, \mathbf{A} = \begin{pmatrix} V_{\text{int}} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\rho_g}{\alpha_g}(u_g - V_{\text{int}}) & u_g & \rho_g & 0 & 0 & 0 & 0 \\ \frac{P_g - P_{\text{int}}}{\alpha_g \rho_g} & 0 & u_g & 1/\rho_g & 0 & 0 & 0 \\ \frac{\rho_g c^2_{\text{int},g}}{\alpha_g}(u_g - V_{\text{int}}) & 0 & \rho_g c_g^2 & u_g & 0 & 0 & 0 \\ -\frac{\rho_l}{\alpha_l}(u_l - V_{\text{int}}) & 0 & 0 & 0 & u_l & \rho_l & 0 \\ -\frac{P_l - P_{\text{int}}}{\alpha_l \rho_l} & 0 & 0 & 0 & 0 & u_l & 1/\rho_l \\ -\frac{\rho_l c^2_{\text{int},l}}{\alpha_l}(u_l - V_{\text{int}}) & 0 & 0 & 0 & 0 & \rho_l c_l^2 & u_l \end{pmatrix}. \tag{9}$$

Following [2], we calculate the Jacobian matrix $\mathbf{A}(\bar{\mathbf{W}})$ in the average state

$$\bar{\mathbf{W}} = \frac{\mathbf{W}_l + \mathbf{W}_r}{2}.$$

The intermediate state in the solution of the Riemann problem (7)-(8) is

$$\mathbf{W}^* = \mathbf{W}_l + \sum_{\lambda_i < 0} a_i \mathbf{r}_i,$$

where λ_i are the eigenvalues and $\mathbf{r}_i$ the corresponding eigenvectors of the matrix $\mathbf{A}(\bar{\mathbf{W}})$.
Recalculating $\mathbf{W}^*$ into the conservative vector $\mathbf{U}^*$, we fully determine the Godunov-type scheme (4)-(5) for the system (1).
We also extend the scheme (4)-(5) to the second order accuracy by the MUSCL approach.
The scheme (4)-(5), built as described above, preserves the moving contact discontiuity by construction. For conservative systems, it reduces to the VFRoe scheme of Gallouet et. al. [2]. It is written in conservative form, and the numerical flux function is obviously consistent. Note that the scheme will be in conservative form without fulfiling the Roe condition on the matrix $\mathbf{A}$. Then, by Lax–Wendroff theorem, the numerical solution, if convergent, converges to a weak solution.
To prove the efficiency of the proposed scheme, we have chosen essentially the same test cases as in [4]. These are the water-air shock tube, the water faucet problem, and the shock wave propagation in the mixture of solids. We compare the numerical solution with the exact one, and with the original method of [4]. The numerical results of the proposed scheme clearly show better accuracy in comparison with the original method of [4].

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Two-phase flow with hysteresis in porous media

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We discuss the Riemann problem for a scalar hyperbolic conservation law with hysteresis modelling two-phase flow in porous media. The ”flux” depends not only on density, it also depends on the density history. The envelope curves describe loading and unloading regimes. Reversible ”scanning” curves fill the region within the envelope.
The solution of the Riemann problem consists of rarefaction waves and shocks. The loading, unloading and scanning rarefactions are described. The definitions of shock admissibility of Lax and Oleinik are applied, resulting in multiple solutions of Riemann problem. Nevertheless, the so called monotonic solution, where the flux is monotonic along self-similar path in x,t space, is unique and stable with respect to small perturbations of initial data.
The solution of a ”smoothed” Riemann problem is obtained by solving for the interaction of the hyperbolic waves. A solution with smoothed data where the flux function is monotonic tends to a monotonic solution of the Riemann problem, i.e. the solution of the Riemann problem is independent of the smoothing. A solution with smoothed data where the flux function is non-monotonic tends to the solution of the Riemann problem containing a shock from the left state to an intermediate scanning curve, i.e. the solution of the Riemann problem depends on the type of smoothing. Nevertheless, it is possible to propose a simple criterion to predict the self-similar limit of the initial problem depending on the type of smoothing.
It is interesting to mention that the solutions of equations with scanning curves for a specific type of perturbations tend to solutions of the Riemann problem for an equation without the scanning curves.
The Riemann problem for a specific class of 2x2 hyperbolic system of conservation laws with hysteresis is discussed. For the systems under consideration, the rarefactions of the second family degenerate into contact shocks. The solutions with monotonic flux function along self-similar path are considered. The monotonic solution always exists and is unique. Nevertheless, other non-monotonic solutions are presented. The example showing dependence of the Riemann problem solution on small perturbation of the initial data is presented.
The solutions presented are realized in different flows in porous media. One hyperbolic equation with hysteresis describes secondary migration of hydrocarbons and formation of petroleum reservoirs, exploitation of an in situ gas storage, flow of polymers in porous media. Displacement of oil by polymer solutions, water-alternate gas injection, cyclic injection and production of polymers are described by 2x2 hyperbolic system. All the above mentioned models contain hysteresis due to irreversibility of adsorption, additional entrapment of oil during the imbibition after the primary drainage, hysteresis in relative permeability and contact angle. Comparison with hysteresis-free solutions shows that even weak hysteretic behavior can completely change the flow structure and the behavior at large times.

Dynamics of multilayer shear flows

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The multi-layer shear flows are one of the most graphic and experimentally accessible examples of conservation laws system. The flow is interesting due to its conceptual simplicity, rich dynamical phenomenology, and technological relevance. The simplest system is multilayer thin film flowing down an inclined plane - the process used in manufacturing of color photographic films and papers. Conservation of mass in each layer leads to a system of coupled PDEs. Its hyperbolic part corresponds to a kinematic part of the motion. Investigation of its structure reveals a very nontrivial types of behaviour that might appear in such

system. Namely, in addition to hyperbolic waves, the system may also exhibit elliptic behaviour. The influence of dissipative effects results in a simple and remarkable manifestation of the result obtained by Majda and Pego (1985): that the dissipative terms may in fact destabilize the hyperbolic system, and that the solutions of viscosity-smoothed system do not converge to the solutions of the corresponding hyperbolic system.

4.25 Thursday, Session 1 (morning): Euler equations

On the classes of globally smooth solutions to the Euler equations in several dimensions

Olga S. Rozanova

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The aim of the work is an investigation of solutions to the system of Euler equations with the right-hand sides describing the diverse external forces:

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{V}) = 0, \tag{1}$$

$$\partial_t(\rho \mathbf{V}) + (\rho \mathbf{V}, \nabla)\,\mathbf{V} + \nabla p = \rho \mathbf{f}(\mathbf{x}, t, \mathbf{V}, \rho, S), \tag{2}$$

$$\partial_t S + (\mathbf{V}, \nabla S) = 0 \tag{3}$$

with the state equation $p = e^S \rho^\gamma$, $\gamma = const > 1$, we suppose $\mathbf{f}$ to be a smooth function of all arguments. Here $\rho(t, \mathbf{x})$, $\mathbf{V}(t, \mathbf{x})$, $S(t, \mathbf{x})$ are the components of solution, given in $\mathbb{R} \times \mathbb{R}^n$, $n \geq 1$ (density, velocity and entropy, correspondingly).
Put the Cauchy problem for (1–3):

$$\rho(0, \mathbf{x}) = \rho_0(\mathbf{x}) \geq 0,\ \mathbf{V}(0, \mathbf{x}) = \mathbf{V}_0(\mathbf{x}),\ S(0, \mathbf{x}) = S_0(\mathbf{x}). \tag{4}$$

We are especially interested in the classical solutions to (1–3) with the component of density sufficiently quickly decreasing as $|\mathbf{x}| \to \infty$ to fulfil the convergency of the integral $\int\limits_{\mathbb{R}^n} \rho |\mathbf{x}|^2 d\mathbf{x}$ (so called solutions with a finite moment).

As well known, the solution to Cauchy problem for system (1–3) may lose the initial smoothness during a finite time. Sometime there is a possibility to estimate the time of the singularity formation from above (see, f.e., [1] and references therein). Moreover, in the case $\mathbf{f} = 0$ the singularity appears in any solution with initially compactly supported data (f.e., [2]).
At the same time it is interesting that there exist some nontrivial classes of globally smooth solutions.

DEFINITION. We shall call the globally in time smooth solution $(\bar\rho(t, \mathbf{x}),\ \bar{\mathbf{V}}(t, \mathbf{x}),\ \bar S(t, \mathbf{x}))$ to system (1–3) *the interior solution* if any solution $(\rho(t, \mathbf{x}),\ \mathbf{V}(t, \mathbf{x}),\ S(t, \mathbf{x}))$ to Cauchy problem (1–4) with the sufficiently small norm

$$\|(\rho_0^{(\gamma-1)/2}(\mathbf{x}) - \bar\rho^{(\gamma-1)/2}(0, \mathbf{x}),\ \mathbf{V}_0(\mathbf{x}) - \bar{\mathbf{V}}(0, \mathbf{x}),\ S_0(\mathbf{x}) - \bar S(0, \mathbf{x})\|_{H^m(\mathbb{R}^n)}$$

is globally smooth as well and

$$(\rho^{(\gamma-1)/2} - \bar\rho^{(\gamma-1)/2},\ \mathbf{V} - \bar{\mathbf{V}},\ S - \bar S) \in \cap_{j=0}^1 C^j([0, \infty); H^{m-j}(\mathbb{R}^n)),\ m > 1 + n/2.$$

Note that the trivial solution is not interior at least for $\mathbf{f} = \mathbf{0}$. But the set of the interior solutions is not empty. In the paper [4] (a generalization of [3]) for $\mathbf{f} = \mathbf{0}$ it was shown that the solution $(0,\ \bar{\mathbf{V}}(t, \mathbf{x}),\ const)$ to (1–4) is interior, where $\bar{\mathbf{V}}(t, \mathbf{x})$ is a globally smooth solution to the equation $\partial_t \mathbf{V} + (\mathbf{V}, \nabla)\,\mathbf{V} = 0$ such that the spectrum of its Jacobian initially is separated from the real negative semi-axis, $D\bar{\mathbf{V}}(0, \mathbf{x}) \in L^\infty(\mathbb{R}^n)$, $D^2\bar{\mathbf{V}}(0, \mathbf{x}) \in H^{m-1}(\mathbb{R}^n)$ (we denote by D^k the vector of all spatial derivatives of order k).
Note that if $\mathbf{f} = 0$ the velocity components of the solutions do not belong to the Sobolev class, but the other components may be even compactly supported.
The result is clear from the physical point of view: the velocity field with a positive divergency "spreads" the initially concentrated small mass, that prevents the singularity formation.
The question arises: can one construct the interior solution with the density not close to zero? It occurs that the answer is positive.
Below we *suppose* firstly the existence of globally smooth in time solution with linear profile of velocity having certain properties and show that the solution is interior. Then we construct some of the solutions for $n = 2$.

What global solution with linear profile of velocity is interior?

Denote $C_b^r(\mathbb{R}^n)$ the space of continuous functions having continuous and bounded in $\mathbb{R}^n$ derivatives up to order $r = 0, 1, \ldots$. Further, denote also $\mathbf{f_V}$ the matrix $\|\frac{\partial f_i}{\partial V_j}\|$.

Theorem 1 *Let the function* $\mathbf{f}_1(\mathbf{x}, t, \Pi, \mathbf{V}, S) = \mathbf{f}(\mathbf{x}, t, e^{-\frac{S}{\gamma}}\Pi^{\frac{2}{\gamma-1}}, \mathbf{V}, S)$ *have the derivatives with respect to all variables up to order* $m+1$, *that are continuous and bounded for bounded* $(\Pi, \mathbf{V}, S)$. *Suppose the system (1–3) has the globally in time smooth solution* $\bar{U} = (\bar{\rho}, \bar{\mathbf{V}}, \bar{S})$ *with linear profile of velocity* $\bar{\mathbf{V}} = A(t)\mathbf{r}$ *such that*
a) $\bar{\rho}^{\frac{\gamma-1}{2}}(0, \mathbf{x}) \in \cap_{j=0}^{m+1} C_b^j(\mathbb{R}^n)$; $D\,\bar{S}(0, \mathbf{x}) \in \cap_{j=0}^{m} C_b^j(\mathbb{R}^n)$;
b) $\xi(t) = \det A(t) > 0$ *for* $t \geq t_0 > 0$;
and there exist a smooth real-valued function $\lambda(t)$, *a constant* q *and a matrix with the real-valued coefficients* $U_\phi(t)$ *such that* $(\mathbf{w}, U_\phi(t)\mathbf{w}) = 0$ *for any vector* $\mathbf{w} \in \mathbb{R}^n$ *with the following properties:*
c) $\int_{t_0}^{+\infty} \lambda(\tau) d\tau < \infty$,
d) $\int_{t_0}^{+\infty} \lambda^q(\tau)\xi^{1/n}(\tau)\, d\tau < \infty$,
e) functions

$$Q_1(t)R(t),$$

$$(Q_2(t) - \lambda^{-1}(A(t)\mathbf{f_V}A^{-1}(t) - U_\phi(t)),$$

$$(\ln R(t))'$$

are bounded on $\mathbb{R}_+ \times \mathbb{R}^n$.
Then the solution $\bar{U}$ *is interior.*

REMARK. The solutions with linear profile of velocity satisfying the conditions of Theorem 1 exist. In the case investigated in [3] $f = 0, \gamma \leq 1 + \frac{2}{n}$, $A(t) = (E + tA(0))^{-1}A(0)$, $\mathrm{Sp}A(0) \notin \mathbb{R}_-$, $\mathrm{tr}A(t) \sim \frac{n}{t}, \det A(t) \sim \frac{1}{t^n}$, $Q_1 = 0, Q_2 \sim 2\left(\frac{1}{n}((\mathrm{tr}A^{-1})E - A^{-1}\right)$, as $t \to \infty$. Here $\lambda = (1+t)^{-2}, q = \frac{n(\gamma-1)}{4}$.
We shall get two corollaries from Theorem 1, basing on which we can assert that the solution will be constructed in the next section are interiors.

Corollary 1 *Let* $\mathbf{f} = \mathbf{0}$. *If system (1–3) has the globally smooth in time solution with linear profile of velocity, described in the statement of Theorem 1 and* $A(t) \sim \frac{\delta}{t}E$, $t \to \infty$, *where* δ *is a positive constant, then the solution is interior.*

Corollary 2 *Let* $\mathbf{f} = L\mathbf{V}$, *where* L *is a matrix with the smooth coefficients such that* $A_0 L A_0^{-1} - U_1(t) = -\mu E$, *where* A_0 *is a matrix with the positive determinant,* $A_0 = \delta E + U_2$, *the matrices* U_i *are such that* $(\mathbf{w}, U_i(t)\mathbf{w}) = 0$, $i = 1, 2$, *for any vector* $\mathbf{w} \in \mathbb{R}^n$, δ *is a constant,* μ *is a positive constant. If system (1–3) has the globally smooth in time solution with linear profile of velocity, described in the statement of Theorem 1 and* $A(t) \sim \frac{1}{t}A_0$, $t \to \infty$, *then the solution is interior.*

Construction of some interior solutions

Now we construct some interior solutions for the important case $n = 2$ and $\mathbf{f} = L\mathbf{v}$, with matrix $L = \begin{pmatrix} -\mu & -l \\ l & -\mu \end{pmatrix}$, $\mu = const > 0$, $l = const$. So we describe in the simplest way the Coriolis force and the Rayleigh friction in the meteorological model neglecting the vertical processes.
If we succeed to find the velocity field, then from linear with respect to ρ and S equations (1), (3) we can find the solution to Cauchy problem (4), moreover, if the velocity field is smooth, then the other components of solution will be smooth as well.
It needs to note that the values ρ and p can became unbounded in some point of the particle trajectory during a finite time T only if $\int_0^T div\mathbf{V}dt = -\infty$ in the point.
For smooth solutions to system (1–3), so quickly decreasing at infinity to fulfil the convergency of all integrals involved, the total mass $m = \int\limits_{\mathcal{R}^2} \rho\, d\mathbf{x}$ is conserved. At $\mu = 0$ the total energy $E = \int\limits_{\mathcal{R}^2} \left(\frac{\rho|\mathbf{V}|^2)}{2} + \frac{p}{\gamma-1}\right) d\mathbf{x} = E_k(t) + E_p(t)$ and the momentum $J = \int\limits_{\mathcal{R}^2} \left(\rho(\mathbf{V}_\perp, \mathbf{r}) + \frac{l}{2}|\mathbf{x}|^2\right) d\mathbf{x}$ are conserved as well. The conservation laws hold for the solutions to system

(1–3) with the density (and pressure) vanishing at the space infinity rather quickly, whereas the velocity components may be even increasing.
Consider another integral functionals characterizing the average properties of solutions:

$$G(t) = \frac{1}{2}\int\limits_{\mathcal{R}^2} \rho|\mathbf{r}|^2\, d\mathbf{x}, \quad F_i(t) = \int\limits_{\mathcal{R}^2} (\mathbf{V}, \mathbf{X}_i)\rho\, d\mathbf{x}, \quad i = 1, 2,$$

where $\mathbf{X}_1 = \mathbf{r} = (x, y)$, $\mathbf{X}_2 = \mathbf{r}_\perp = (y, -x)$.
Note that $G(t) > 0$, $lG(t) - F_2(t) = J = const$.
For the *smooth* solutions to (1 – 3) the following relations hold [5]:

$$G'(t) = F_1(t), \qquad F_2'(t) = lF_1(t) - \mu F_2(t),$$

$$F_1'(t) = 2(\gamma - 1)E_p(t) + 2E_k(t) - lF_2(t) - \mu F_1(t), \qquad E'(t) = -2\mu E_k(t). \tag{5}$$

In a general case the functions $E_p(t)$ and $E_k(t)$ cannot be expressed through $G(t), F_1(t), F_2(t)$. But if we choose the velocity field with a linear profile

$$\mathbf{V} = A(t)\mathbf{r}, \tag{6}$$

with a (2×2) matrix $A(t)$, we obtain the *closed* system of ODE to find the coefficients of the matrix. In general it is rather complicated and can be solved only numerically.
The simplest result can be obtained if

$$\mathbf{V} = \alpha(t)\mathbf{r} + \beta(t)\mathbf{r}_\perp,$$

It is easy to see that in the case

$$F_1(t) = 2\alpha(t)G(t), \quad F_2(t) = 2\beta(t)G(t),$$
$$E_k(t) = (\alpha^2(t) + \beta^2(t))G(t).$$

¿From the equation

$$\partial_t p + (\mathbf{V}, \nabla p) + \gamma\, p\, div\, \mathbf{V} = 0$$

following from (1–3) and the state equation, we can obtain

$$E_p(t) = E_p(0)G^{\gamma-1}(0)\frac{1}{G^{\gamma-1}(t)}.$$

In that way, all function involved in system (5) are expressed through $G(t), \alpha(t), \beta(t)$. For the convenience denote $G_1(t) = 1/G(t)$ and obtain the system

$$G_1'(t) = -2\alpha(t)G_1(t),$$
$$\beta'(t) = \alpha(t)(l - 2\beta(t)) - \mu\beta(t),$$
$$\alpha'(t) = -\alpha^2(t) + \beta^2(t) - l\beta(t) - \mu\alpha(t) + (\gamma - 1)E_p(0)G_1^{1-\gamma}(0)G_1^\gamma(t). \tag{7}$$

Note that for $\gamma > 1$ the functions $\alpha(t)$ and $\beta(t)$ are bounded, it follows from the expression for the total energy:

$$E(t) = (\alpha(t)^2 + \beta(t)^2)G(t) + E_p(0)G^{\gamma-1}(0)\frac{1}{G^{\gamma-1}(t)} \le E(0),$$

thus

$$\alpha^2(t) + \beta^2(t) \le E(0)G_1(t) - E_p(0)G_1^{1-\gamma}(0)G_1^\gamma(t) < +\infty,$$

$|\mathrm{div}\, \mathbf{V}| < \infty$ and, consequently, the values of density and pressure will be bounded for all solutions of the class we want to construct.
If $\mu = 0$ the system (7) can be integrated:

$$\alpha(G_1) = \pm\sqrt{KG_1^\gamma - C^2G_1^2 + (E - lC)G_1 - l^2/4},$$

$$\beta(t) = CG_1(t) + l/2,$$

$$-\int_{G_1(0)}^{G_1(t)} \frac{dG_1}{2G_1\alpha(G_1)} = t,$$

with constants $C = \frac{2\beta(0)-l}{2G_1(0)}$, $K = (\alpha^2(0) + C^2G_1^2(0) - (E - lC)G_1(0) + l^2/4)/G_1^\gamma(0)$.
In the cases $\mu > 0$ and $\mu = l = 0$ there is a unique stable equilibrium in the origin. We have the following asymptotics of the solution components as $t \to \infty$:

$$\text{if } \mu > 0,\ l = 0, \text{ then } \alpha(t) \sim \frac{1}{2\gamma}t^{-1},\ \beta(t) \sim C\left(\frac{\mu}{2K\gamma}\right)^{1/\gamma} t^{-1/\gamma}\exp\{-\mu t\},$$

$$G_1(t) \sim \left(\frac{\mu}{2K\gamma}\right)^{1/\gamma} t^{-1/\gamma};$$

$$\text{if } \mu > 0, l \neq 0, \text{ then } \alpha(t) \sim \frac{1}{2\gamma}t^{-1},\ \beta(t) \sim \frac{\mu}{2l\gamma}t^{-1},\ G_1(t) \sim \left(\frac{l^2+\mu^2}{2K\mu\gamma}\right)^{1/\gamma} t^{-1/\gamma};$$

$$\text{if } \mu = l = 0, \text{ then } \alpha \sim \left(\sqrt{G_1(0)/E} + t\right)^{-1},\ G_1 \sim \alpha^2,\ \beta = CG_1,$$

with $K = (\gamma - 1)E_p(0)G_1^{1-\gamma}(0)$.
Thus, as follows from Corollaries 1 and 2, in the cases $\mu = l = 0$ and $\mu > 0$ we have constructed the velocity field for the interior solution.
As we know $\alpha(t)$ and $\beta(t)$, we can elementary find components of density and entropy (or pressure):

$$\rho(t, |r|, \phi) = \exp(-2\int_0^t \alpha(\tau)d\tau)\rho_0(|r|\exp(-\int_0^t \alpha(\tau)d\tau), \phi + \int_0^t \beta(\tau)d\tau),$$

$$S(t, |r|, \phi) = S_0(|r|\exp(-\int_0^t \alpha(\tau)d\tau), \phi + \int_0^t \beta(\tau)d\tau),$$

$$p(t, |r|, \phi) = \exp(-2\gamma\int_0^t \alpha(\tau)d\tau)p_0(|r|\exp(-\int_0^t \alpha(\tau)d\tau), \phi + \int_0^t \beta(\tau)d\tau).$$

¿From (2) and (7) we obtain that for the smooth solution

$$\nabla p = -(\gamma - 1)G_1^{1-\gamma}(0)E_p(0)G_1^\gamma(t)\rho\mathbf{r}.$$

It follows that the function ρ_0 and $p_0 = e^{S_0}\rho_0^\gamma$ must be axisymmetric (do not depend on ϕ) and compatible, i.e.

$$\nabla p_0 = -(\gamma - 1)G_1(0)E_p(0)\rho_0\mathbf{r}.$$

Note also that for constructing the interior solution we must choose the initial data ρ_0 and S_0 to satisfy the condition 2a) of Theorem 1.

REMARK. One can show that if the velocity field has the general form (6) in the cases $\mu = l = 0$ and $\mu > 0,\ l = 0$ there is the asymptotics $A(t) \sim \frac{\delta}{t}E,\ t \to \infty$, that is according to Corollary 1 the corresponding solution with a linear profile of velocity is interior.

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A class of global non smooth axisymmetric solutions to the Euler equations of an isentropic perfect gas in 2 space dimensions

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In papers by Serre, Grassin-Serre, and Grassin, global existence results were obtained for solutions to the Euler equations of a perfect gas, under some smoothness and growth assumptions on the initial data, provided the initial velocity is dispersive and the initial sound speed is small. In the present work we consider axisymmetric flows (for the isentropic Euler equations of a perfect gas) in 2 space dimensions, with non smooth initial data. More precisely, we consider the case that initial gradients may jump across a given circle (centered at 0); the tangential component of the initial velocity may also jump there. In each component of the complement of this circle, we assume that the initial data are restrictions of global rotation invariant data which satisfy the assumptions of Grassin-Serre and Grassin. We prove the global existence of a solution with the given initial data. To achieve this, we rely on some previous work of Li-Yu and Li.

4.26 Thursday, Session 1 (afternoon): Riemann problems II

A Riemann solver applied to porous medium equations

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A Riemann solver has been applied to several types of numerical computations in the fluid dynamics. In this talk we propose a numerical scheme based on it to the following one-dimensional nonlinear diffusion equation:

$$v_t = (v^m)_{xx} - cv^p, \quad t > 0,\ x \in \boldsymbol{R} \tag{1}$$

with the initial condition

$$v(0,x) = v^0(x), \quad x \in \boldsymbol{R}, \tag{2}$$

where $m > 1$, $c > 0$ and $p > 0$ are constants, and the initial function v^0 is nonnegative and has a compact support.
The aim is to explain the construction of the numerical scheme with convergence (see Theorems 1–4), and to show some mathematical results concerned with the dynamical behavior of solution v of (1)–(2) by using the property of numerical approximations (Theorems 5 and 6). We also show some numerical simulations to the two-dimensional version of (1)–(2).
The equation (1) is known as a simple model describing the nonlinear fluid-transfer process with an absorption, where $v = v(t,x)$ is the density of fluid. Here $(v^m)_{xx}$ describes the nonlinear diffusion, and $-cv^p$ the absorption. We note that, when $c = 0$, the equation (1) is called the *porous medium equation*, which describes the flow of liquids in porous media. The most remarkable phenomenon in the solution v of (1)–(2) is the *finite propergation of the support*; that is, if the support of v^0 is compact, so is $\operatorname{supp} v(t,\cdot)$ for all $t \geq 0$. The behavior of $\operatorname{supp} v(t,\cdot)$ is qualitatively classified into the following three cases:
i) For $m \geq p$, $\operatorname{supp} v(t,\cdot)$ expands and $\operatorname{supp} v(t,\cdot) \to \boldsymbol{R}$ as $t \to \infty$;
ii) For $1 \leq p < m$, $\operatorname{supp} v(t,\cdot)$ also expands and there exists a bounded set $B \subset \boldsymbol{R}$ satisfying $\operatorname{supp} v(t,\cdot) \subset B$ for all $t \geq 0$;
iii) For $0 < p < 1$, $\operatorname{supp} v(t,\cdot)$ may expand and/or shrink and v vanishes in a finite time; that is, there exists a constant $T^* > 0$ satisfying $\operatorname{supp} v(t,\cdot) = \emptyset$ for $t \geq T^*$ and $\operatorname{supp} v(t,\cdot) \neq \emptyset$ for $t < T^*$. Such an absorption is said to be a strong absorption and T^* is called an extinction time.
In Cases i) and ii) it is obvious that $\operatorname{supp} v(t,\cdot)$ never becomes disconnected, even if the initial function $v^0(x)$ has zeros in the interval (α_1, α_2), where $\operatorname{supp} v^0 = [\alpha_1, \alpha_2]$. However, in Case iii) the support may shrink and become disconnected for such an initial function $v^0(x)$. The most striking phenomenon is the possibility of the support to split into several disjoint sets, even if $v^0(x)$ is positive on (α_1, α_2). This motivates us to investigate the problem "How does the support vary when t varies?" To answer this problem we construct the numerical scheme which realizes such a behavior of the support in the following specific case of iii) where

$$m + p = 2 \ \text{ and } \ 0 < p < 1. \tag{3}$$

In this case, Kersner (1978) constructed an explicit solution, which enables us to verify the accuracy of numerical approximations.
Now let us state our numerical scheme. To analyze the behavior of $\operatorname{supp} v(t,\cdot)$ we construct numerical approximations to the left interface $\ell(t)$ and right one $r(t)$ defined by

$$\ell(t) = \sup\{\xi \in \boldsymbol{R};\ v(t,x) = 0 \text{ on } x < \xi\},$$

$$r(t) = \inf\{\xi \in \boldsymbol{R};\ v(t,x) = 0 \text{ on } x > \xi\}.$$

We rewrite (1)–(2) as the following equations by putting $u = v^{m-1}$:

$$u_t = Pu + Hu + Du, \quad t > 0,\ x \in \boldsymbol{R}, \tag{4}$$

$$u(0,x) = u^0(x) \equiv (v^0(x))^{m-1}, \quad x \in \boldsymbol{R}, \tag{5}$$

where

$$Pu = muu_{xx}, \quad Hu = a(u_x)^2, \quad Du = -c', \quad a = \frac{m}{m-1}, \quad c' = c(m-1).$$

Our numerical scheme approximates (4)–(5) instead of (1)–(2). Let k and h be the time step and the space mesh width, respectively. Then the parabolic term Pu and the dynamics term Du are approximated by standard explicit schemes $P_{k,h}u_h$ and $D_{k,h}u_h$, respectively, where u_h the numerical solution. For the hyperbolic term Hu a Riemann solver is applied to the construction of the scheme $H_{k,h}u_h$. Moreover, the Rankine-Hugoniot jump equation is used to approximate the interfaces of the solution of $u_t = Hu$. We note that Pu does not change the position of interfaces but Du changes it, which can be easily approximated. The key point to obtain good numerical interfaces is the construction of numerical scheme $H_{k,h}u_h$ without the artificial viscosity. We state the details of a Riemann solver in this talk.

When $c = 0$, some numerical schemes are already known. The schemes by Baklanovskaya (1961) and Graveleau and Jamet (1971) give good numerical approximations to v, however, they does not consider the approximation of the interfaces. DiBenedetto and Hoff (1984) proposed the interface tracking algorithm. Though they proved the convergence of numerical interfaces by introducing an artificial viscosity, it takes smaller h to obtain the accurate numerical approximations. On the other hand, by using the Riemann solver, our scheme excludes such an artificial viscosity, so that we may expect more accurate numerical approximations in both cases where $c = 0$ and $c > 0$.

The proof of the convergence of numerical approximations can be obtained under the following condition imposed on the initial function:

Condition A. $u^0 \in C^0(\boldsymbol{R})$ is a nonnegative function with compact support and $u^0_x \in L^\infty(\boldsymbol{R}) \cap BV(\boldsymbol{R})$.

Let $\ell_h(t)$ and $r_h(t)$ be the numerical left and right interfaces given by our scheme, respectively; that is,

$$\begin{aligned} \ell_h(t) &= \sup\{\xi \in \boldsymbol{R};\ u_h(t,x) = 0 \text{ on } x < \xi\}, \\ r_h(t) &= \inf\{\xi \in \boldsymbol{R};\ u_h(t,x) = 0 \text{ on } x > \xi\}. \end{aligned}$$

Then we have the following four theorems:

Theorem 1 (Convergence of the numerical solution u_h). *Assume Condition A. Let $\{h\}$ be an arbitrary sequence which tends to zero. Then there exist a subsequence $\{h'\}$ of $\{h\}$ and a function u with the following properties:*

i) $u \in C^0(\mathcal{H}) \cap L^\infty(\mathcal{H}), \quad u_x \in L^\infty(\mathcal{H})$;

ii) As $h' \to 0$,

$$\begin{aligned} &\|u_{h'} - u\|_{L^\infty(\mathcal{H})} \to 0, \\ &\|(u_{h'})_x - u_x\|_{L^p(\mathcal{H})} \to 0 \quad (1 \le p < +\infty); \end{aligned}$$

iii) u is a weak solution of (4)–(5);

iv) $u_{xx} \in \mathcal{E}'$ *and* $u_t \in \mathcal{E}'$,

where $\mathcal{H} = [0,\infty) \times \boldsymbol{R}$, and $\mathcal{E}'$ is the dual of the space $\mathcal{E}$ consisting all continuous functions with compact support in $(0,\infty) \times \boldsymbol{R}$.

Let $v_h(t,x) = (u_h(t,x))^{1/(m-1)}$. Then by the above theorem and the uniqueness of solution of (1)–(2) (Herrero and Vázquez (1987)), we can prove

Theorem 2 (Convergence of the numerical solution v_h). *Let the same assumptions as stated in Theorem 1 be satisfied. Then v_h converges uniformly on $\mathcal{H}$ to the unique weak solution v of (1)–(2) as h tends to zero.*

The convergence of the numerical interface is given as follows:

Theorem 3 (Convergence of the left numerical interface). *Let the same assumptions as stated in Theorem 1 be satisfied and assume*

(i) $(u^0)_x(x)$ *is absolutely continuous on* $\boldsymbol{I} \equiv \operatorname{supp} u^0$;

(ii) $\operatorname{ess.inf}_{x \in \boldsymbol{I}} (u^0)_{xx}(x)$ *is finite.*

Suppose M and ε are positive constants satisfying

$$u^0_x(x) > M \qquad \text{for } x \in [\ell(u^0),\ \ell(u^0) + \varepsilon). \tag{6}$$

Then the left numerical interface $\ell_h(t)$ converges uniformly to the exact one on $[0,\ \tilde{T}]$, where

$$\tilde{T} = \max_{0 \le \eta \le M} \frac{(M-\eta)\eta}{(2a+m)C_1C_2\eta + 3c'C_2}, \quad C_1 = \|u^0_x\|_\infty \text{ and } C_2 = -\operatorname{ess.inf}_{x \in \boldsymbol{I}} u^0_{xx}.$$

Remark 1. The convergence of the right numerical interface $r_h(t)$ can be similarly shown.

Theorem 4 (Convergence of the numerical extinction time). *Let the same assumptions as stated in Theorem 1 be satisfied. Then*

$$T_h^* \equiv \inf\{T > 0;\ \operatorname{supp} v_h(t,0) = \emptyset \text{ on } t > T\} \to T^* \quad \text{as } h \to 0.$$

We state the support splitting phenomenon, which is justified by the proof of the existence of two constants T_1 and T_2 ($0 < T_1 < T_2$) such that $\operatorname{supp} v(t,\cdot)$ is connected for $0 < t < T_1$ and is not connected for $T_1 < t < T_2$. This means that the support splits into at least two disjoint sets at $t = T_1$. Such a phenomenon is numerically shown by Rosenau and Kamin (1983). But the theoretical justification is not discussed. We prove that this phenomenon occurs under some condition imposed on the initial function.

Theorem 5. *Let the same assumptions as stated in Theorem 3 be satisfied, and assume*

$$\frac{u^0(\beta_j)}{c' + mC_0C_2} > \frac{\|u^0\|_{L^1[\gamma_1,\gamma_2]}}{c'(\gamma_2-\gamma_1) - (m+a)C_0TV(u_x^0)} > 0 \quad (j = 1,2) \tag{7}$$

for some β_j and γ_j ($j = 1,2$), where $C_0 = \|u^0\|_\infty$ and $\beta_1 < \gamma_1 < \gamma_2 < \beta_2$. Then there exist $T_1 > 0$ and $\tilde{x} \in [\gamma_1,\ \gamma_2]$ such that

$$v(T_1,\tilde{x}) = 0 \quad \text{and} \quad v(T_1,\beta_j) > 0\ (j = 1,2).$$

Also, if $u^0(x)$ satisfies

$$a\|u_x^0\|_\infty^2 < c', \tag{8}$$

then there exists a positive constant T_2 such that $\operatorname{supp} v(t,\cdot)$ *is disconnected for each $t \in (T_1,\ T_2)$.*

We note that the initial function which satisfies (7) can be easily constructed.

Finally, we state the derivation and justification of the equation which the interface satisfies. Our numerical scheme suggests that the left interface satisfies

$$\frac{d}{dt}\ell(t) = -\frac{m}{m-1}(v^{m-1})_x(t,\ell(t)+0) + \frac{c'}{(v^{m-1})_x(t,\ell(t)+0)}. \tag{9}$$

The first term of the right hand side of (9) is caused by the nonlinear diffusion $(v^m)_{xx}$, and the second one the absorption term $-cv^p$. By taking the properties of our numerical scheme into consideration and by using the comparison theorem, we can obtain

Theorem 6. *Let the same assumptions as stated in Theorem 3 be satisfied. Then the left interface equation (9) holds a.e. in $[0,\tilde{T})$, where $\tilde{T} > 0$ is the constant in Theorem 3.*

Geometry of Hugoniot curves in 2×2 systems of hyperbolic conservation laws with quadratic flux functions

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This talk analyzes a simple discontinuous solution to non strictly hyperbolic 2×2 systems of conservation laws having quadratic flux functions and an isolated *umbilic point* where the characteristic speeds are equal. We study the Hugoniot curves especially in Schaeffer-Shearer's case I & II which are relevant to the three-phase Buckley-Leverett model for oil reservoir flow. The compressive and overcompressive parts are determined. The wave curves through the umbilic point are discussed and their compressive and overcompressive parts are also determined.

Let us consider a 2×2 system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{1}$$

where $U = {}^t(u,v) \in \Omega$ for some connected region $\Omega \subset \mathbf{R}^2$ and $F : \Omega \to \mathbf{R}^2$ is a smooth map. We say that this system of equations is *hyperbolic*, when the Jacobian matrix $F'(U)$ has *real* eigenvalues $\lambda_1(U), \lambda_2(U)$ for any $U \in \Omega$. If, in particular, these eigenvalues are *distinct*: $\lambda_1(U) < \lambda_2(U)$, the system is called *strictly hyperbolic* at U. A state $U^* \in \Omega$ is called an *umbilic* point, if $\lambda_1(U) = \lambda_2(U)$ and $F'(U)$ is diagonal at $U = U^*$. In a strictly hyperbolic region, we have a pair of *characteristic fields* $R_1(U), R_2(U)$ which are right eigenvectors corresponding to $\lambda_1(U), \lambda_2(U)$, respectively. We choose left eigenvectors $L^1(U), L^2(U)$ such that

$$L^1(U)R_1(U) = L^2(U)R_2(U) = 1, \quad L^2(U)R_1(U) = L^1(U)R_2(U) = 0.$$

Suppose that $U = U^*$ is an *isolated* umbilic point. We have the Taylor expansion of $F(U)$ near $U = U^*$:

$$F(U) = F(U^*) + \lambda^*(U - U^*) + Q(U - U^*) + O(1)|U - U^*|^3$$

where $\lambda^* = \lambda_1(U^*) = \lambda_2(U^*)$ and $Q : \mathbf{R}^2 \to \mathbf{R}^2$ is a homogeneous quadratic mapping. After the Galilean change of variables: $x \to x - \lambda^* t$ and $U \to U + U^*$, we observe that the system of equations (1) is reduced to

$$U_t + Q(U)_x = 0, \quad (x,t) \in \mathbf{R} \times \mathbf{R}_+ \tag{2}$$

modulo higher order terms. Now by a change of unknown functions $V = S^{-1}U$ with a regular constant matrix S, we have a new system of equations $V_t + P(V)_x = 0$ where $P(V) = S^{-1}Q(SV)$. Thus we come to

Definition 1 *Two quadratic mappings $Q_1(U)$ and $Q_2(U)$ are said to be* equivalent, *if there is a constant matrix $S \in GL_2(\mathbf{R})$ such that*

$$Q_2(U) = S^{-1}Q_1(SU) \quad \text{for all} \quad U \in \mathbf{R}^2. \tag{3}$$

A general quadratic mapping $Q(U)$ has six coefficients and $GL_2(\mathbf{R})$ is a four dimensional group. Thus by the above equivalence transformations, we can eliminate four parameters. These procedures are successfully carried out by Schaeffer-Shearer (CPAM 1987) and they obtained the following *normal forms*.

Let $Q(U)$ be a hyperbolic quadratic mapping with an isolated umbilic point $U = 0$, then there exist two real parameters a and b with $a \neq 1 + b^2$ such that $Q(U)$ is equivalent to $\frac{1}{2}\nabla C$ where $\nabla = {}^t(\partial_u, \partial_v)$ and

$$C(U) = \tfrac{1}{3}au^3 + bu^2v + uv^2. \tag{4}$$

Moreover, if $(a,b) \neq (a',b')$, then the corresponding quadratic mappings: $\frac{1}{2}\nabla C$ and $\frac{1}{2}\nabla C'$ are not equivalent.

In the following argument, we shall confine ourselves to the quadratic mapping:

$$Q(U) = \frac{1}{2}\nabla C(U) = \frac{1}{2}\begin{pmatrix} au^2 + 2buv + v^2 \\ bu^2 + 2uv \end{pmatrix}. \tag{5}$$

Geometric properties of the mapping $Q(U)$, for example the integral curves of characteristic vector fields, change as (a,b) varies in the ab-plane. Schaeffer-Shearer's classification (CPAM 1987) is the following: Case I is $a < \frac{3}{4}b^2$; Case II is $\frac{3}{4}b^2 < a < 1 + b^2$; for $a > 1 + b^2$, the boundary between Case III and Case IV is $4\{4b^2 - 3(a-2)\}^3 - \{16b^3 + 9(1-2a)b\}^2 = 0$. The drastic change across $a = 1 + b^2$ was recognized by Darboux (1896) even in the 19th century. We notice that these 2×2 system of hyperbolic conservation laws with an isolated umbilic point is a generalization of a three phase Buckley-Leverett model for oil reservoir flow where the flux functions are represented by a quotient of polynomials of degree two. In Appendix of Schaeffer-Shearer's paper (CPAM 1987): in collaboration with Marchesin and Paes-Leme, they show that the quadratic approximation of the flux functions is either Case I or Case II.
The Riemann problem for (1) is the Cauchy problem with initial data of the form

$$U(x,0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ U_R & \text{for} \quad x > 0 \end{cases} \tag{6}$$

where U_L, U_R are constant states in Ω. A jump discontinuity defined by

$$U(x,t) = \begin{cases} U_L & \text{for} \quad x < st, \\ U_R & \text{for} \quad x > st \end{cases} \tag{7}$$

is a piecewise constant weak solution to the Riemann problem, provided these quantities satisfy the *Rankine-Hugoniot condition*:

$$s(U_R - U_L) = F(U_R) - F(U_L). \tag{8}$$

We say that the above discontinuity is a *j-compressive* shock wave $(j = 1, 2)$ if it satisfies the *Lax entropy conditions* :

$$\lambda_j(U_R) < s < \lambda_j(U_L), \quad \lambda_{j-1}(U_L) < s < \lambda_{j+1}(U_R) \tag{9}$$

(Lax (CPAM 1957, 1971)). Here we adopt the convention $\lambda_0 = -\infty$ and $\lambda_3 = \infty$. In Case II, we shall also face with the *overcompressive* shock wave: a jump discontinuity satisfying

$$\lambda_1(U_R) < s < \lambda_1(U_L), \quad \lambda_2(U_R) < s < \lambda_2(U_L). \tag{10}$$

The *Hugoniot loci* over U_0 are the set of (U, s) satisfying

$$H_{U_0}(U, s) = s(U - U_0) - \{F(U) - F(U_0)\} = 0. \tag{11}$$

Their projections on to the U-plane are called the *Hugoniot curves* through U_0. If U_0 is not an umbilic point, Lax (CPAM 1957) shows that there exist over U_0 two Hugoniot loci $\{(Z_j(\mu), s_j(\mu))\}$ $(j = 1, 2)$ for small $|\mu|$ satisfying

$$Z_j(0) = U_0, \quad s_j(0) = \lambda_j(U_0) \quad (j = 1, 2). \tag{12}$$

Their projections $\{Z_j(\mu)\}$ $(j = 1, 2)$ are called the *j-Hugoniot curves* through U_0.

In this note, we shall confine ourselves to Case I and II of the representative quadratic mapping $F(U) = Q(U)$ defined by (5). Our aim is to determine rigorously compressive parts of the Hugoniot curves. Although we have an extensive bibliography: Gomes (Adv. Appl. Math. 1989), Isaacson-Marchesin-Plohr-Temple (SIAM 1988, SIAM 1988, SIAM 1988, SIAM 1990), Isaacson-Marchesin-Palmeira-Plohr (CMP 1992), Schaeffer-Shearer (CPAM 1987, Trans. AMS 1987), Shearer (JDE 1989), Schaeffer-Shearer-Marchesin-Paes-Leme (Arch. Rational Mech. Anal. 1987), etc., study of Hugoniot curves has been carried out mainly through numerical computations so far and rigorous mathematical study will be appreciated. Chen-Kan (Arch. Rational Mech. Anal. 1995) is mainly concerned with Case IV, obtaining global in time solutions via compensated compactness method. In their argument, studies on the singular entropy equation and construction of regular entropy functions are applicable also to Case I and II. On the other hand, Gomes (Adv. Appl. Math. 1989) reports that there exist, on a detached branch of Hugoniot curves, compressive shock waves that do not have viscous profiles. Čanić-Plohr (JDE 1995) treats systems of conservation laws with general quadratic flux functions admitting a compact elliptic region. They adopt the viscosity admissibility criterion: the discontinuous solution (7) has a viscous profile. The boundary of the region of admissible shock waves are shown to consist of portion of loci corresponding to the heteroclinic bifurcations, limit cycles, homoclinic orbits, Bogdanov-Takens and Hopf bifurcations; explicit formulas for certain parts of the boundary are presented.

The Hugoniot loci are represented as an intersection of two quadratic surfaces and the Hugoniot curves are plane curves of the third degree. Incidentally, these curves are rational curves, which is already pointed out by Schaeffer-Shearer (Trans. AMS 1987). Our study is based on these facts and our main tools are Wendroff's lemma, first proved Wendroff (J. Math. Anal. Appl. 1972). First, we obtain parametrizations of these curves by rational functions. We also review Wendroff's lemma and its consequences. Second, we determine compressive and overcompressive parts of the Hugoniot curve. Then the wave curves through the umbilic point are discussed and their compressive and overcompressive parts are determined. We shall discuss the relation between our argument and Lemma 4.2, 4.3 in Schaeffer-Shearer's paper (CPAM 1987).

The Riemann Problem for a Phase Transition Problem

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Introduction

We consider the structure of the solutions to the Riemann problem for a system of conservation laws related to a phase transition problem. The system is nonisentropic and given by

$$\begin{aligned} v_t - u_x &= 0, \\ u_t - f_x &= 0, \\ \mathcal{E}_t - (uf)_x &= 0, \end{aligned} \tag{1}$$

where v, u, $\mathcal{E}$, and f are strain, velocity, total energy, and stress, respectively. The total energy is given by $\mathcal{E} = e + \frac{1}{2}u^2$, where e is the internal energy. We take strain and entropy s as state variables. Therefore, the stress and internal energy are expressed as $f = f(v,s)$ and $e = e(v,s)$, respectively. We assume that e is a smooth function of v and s, $e_s > 0$, and $e_{vs} \neq 0$. Specifically we assume that $e_{vs} < 0$. The thermodynamic relation is given by

$$de = fdv + \theta ds, \tag{2}$$

where θ is temperature. This implies that $\frac{\partial e}{\partial v} = f$, $\frac{\partial e}{\partial s} = \theta$, and $f_s < 0$. We also assume that there exit constants c for which the level curves $f(v,s) = c$ are not monotone in vs-plane as depicted in Figure 1.1. The curve $f_v = 0$ is also sketched. For $s < s_c$ the horizontal line $s = \text{const}$ intersects with $f_v = 0$ twice. If the value of v is in the region bounded by $s = s_c$ and $f_v = 0$ on the left and the right of C, v is in the α-phase and β-phase, respectively. In the region $s < s_c$, we assume that $f_{vv} < 0$ in the α-phase and $f_{vv} > 0$ in the β-phase. Normally we use v and θ as the state variables because often the temperature is the control variable for the phase transition problems. Nevertheless, there are a few reasons for choosing the entropy as the state variable. First, this makes the entropy condition easier to apply. Second, the relation between the thermodynamic relation and the Rankine-Hugoniot condition become clear; see Lemma 4.1 and Remark 4.2.

The Riemann problem is a special initial value problem in which the initial data are given by

$$U(x,0) = (v,u,s)(x,0) = \begin{cases} U_l = (v_l, u_l, s_l) & x < 0, \\ U_r = (v_r, u_r, s_r) & x > 0, \end{cases} \tag{3}$$

where (v_l, u_l, s_l) and (v_r, u_r, s_r) are two different constant states. As shown above we may use the vector notation and set $U = (v,u,s)$. We assume that the solution is self-similar and consists of constant states separated by the backward wave, the phase boundaries, the contact discontinuity, and the forward wave. The phase boundaries are jump discontinuities across which the phase changes. We discuss the case where the speeds of phase boundaries are much smaller than those of the forward and backward waves.

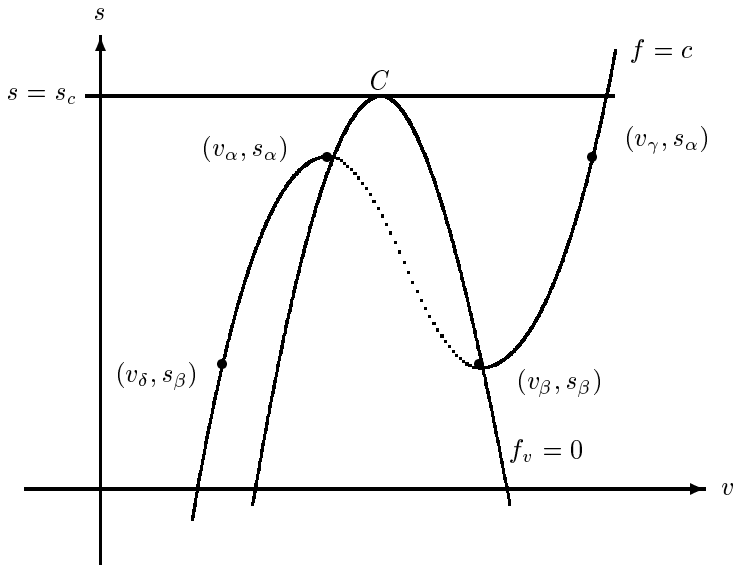

Figure 1.1

The goal of this paper is to examine the role of admissibility criteria in the context of the Riemann problem when the latent heat is taken into account. Namely, $e_{vs} \neq 0$. The solutions to the Riemann problem for (1) are not unique and the admissibility criteria are used to select a physically relevant solution. One criterion we use is the entropy condition. We use the "negative physical entropy" as the entropy. This criterion imposes that the entropy decreases across jump discontinuities. The rate of decay of the entropy is given by

$$E(v_-, s_-, v_+, s_+) = \sigma(v_-, v_+)(s_+ - s_-),$$

where $\sigma(v_-, v_+) = \pm\sqrt{\frac{f_+ - f_-}{v_+ - v_-}}$ is the speed of the jump discontinuity and the subscripts $-$ and $+$ denote the states to the left and right of discontinuity, respectively. The entropy condition requires that $E(v_-, s_-, v_+, s_+) \leq 0$ holds across each discontinuity. In Section 3, we examine the possible number of phase boundaries consistent with the entropy condition in the solution of the Riemann problem.

It turns out that the entropy condition itself is not enough to single out an admissible solution. Therefore, we need to impose further conditions. One such criterion is the kinetic relation proposed by Abeyaratne and Knowles [1, 2]. This condition postulates that there exists a nondecreasing function $\phi(g)$ of the driving traction g satisfying $\phi(0) = 0$ such that the speed of discontinuity is given by

$$\sigma = \phi(g).$$

They propose that

$$g = -(s_+ - s_-) \quad \text{or} \quad g = -(s_+ - s_-)(\theta_+ + \theta_-)$$

is a possible choice as the driving traction. In order that this relation is consistent with the entropy condition, we require that $\phi' > 0$ so that $\sigma g > 0$ holds. In this paper we choose $g = -(s_+ - s_-)$ for the driving traction. Another criterion is the entropy rate admissibility criterion which is proposed by Dafermos [4, 5]. This criterion roughly says that the rate of entropy decay is the fastest for the admissible solution. The entropy rate admissibility criterion postulates that the solution is admissible if it solves (1) and minimizes

$$\sum_{\text{jump discontinuities}} \sigma(s_+ - s_-). \tag{4}$$

In Section 4 we study the consequence of the kinetic relation and entropy rate admissibility criteria. They are separately applied in addition to the entropy condition We observe that if v_l and v_r are specified in the same phase, these conditions generally

select a unique solution. On the other hand, if v_l and v_r are specified in the different phases, there are initial data for which the kinetic relation may choose three different solutions.
Concerning the thermoelastic case, Hattori [6] considered the Riemann problem of the above system using the entropy rate admissibility criterion. He considered only the single phase boundary problem. Abeyaratne and Knowles [2] discussed the trilinear case where the Helmholtz free energy $\psi(v,\theta)$ satisfies $\psi_{v\theta} = 0$. Note that $\psi_{v\theta}(v,\theta) = e_{vs}(v,s)s_\theta$. Therefore, the condition $\psi_{v\theta} = 0$ is equivalent to $e_{vs} = 0$ provided that $s_\theta \neq 0$. Asakura [3] studied the stability of Maxwell states. Slemrod [8] discussed the effects of viscosity and capillarity. Ngan and Truskinovsky [7] studied the effect of latent heat.
This paper consists of four sections. In Section 2, we summarize the properties of the phase boundary curves. In Section 3, we examine when and how many phase boundaries are consistent with the entropy condition in the solution of the Riemann problems. We discuss two cases where the v_l and v_r are specified in the same phase and in the different phases. Finally in Section 4, we study the consequence of the kinetic relation and entropy rate admissibility criterion.

Phase boundary curves

A phase boundary is a line of discontinuity in the xt-plane across which the phase changes. It satisfies the Rankine-Hugoniot condition. The phase boundary curve $P(U_o)$ is the set of U connected to U_o by a phase boundary. One important difference between a phase boundary curve and a shock wave curve is that v_o and v are in the different phases. Therefore, unlike shock wave curves, U_o is not on the phase boundary curve $P(U_o)$. Let U_o^* be the state at which the phase boundary curve for U_o emanates. Denote by (v_α, s_α) and (v_β, s_β) the states in the vs-plane at which $f_v = 0$ on a level curve $f = f_o$; see Figure 1.1. In the following we assume that U_o and U_o^* are in the α-phase and β-phase, respectively.

Lemma 4.1 *If $e_\beta - e_o \le f_o(v_\beta - v_o)$, then in the vs-plane the phase boundary curve for a given point (v_o, s_o) emanates from the point (v_o^*, s_o^*) with v_o^* in the other phase, satisfying the following two conditions*

$$f(v_o^*, s_o^*) = f_o, \quad e_o^* - e_o = f_o(v_o^* - v_o). \tag{5}$$

If $e_\beta - e_o > f_o(v_\beta - v_o)$, the phase boundary curve emanates from

$$f_v(v_o^*, s_o^*) = 0, \quad e_o^* - e_o = \frac{1}{2}(f_o^* + f_o)(v - v_o), \tag{6}$$

provided that (v, s) satisfying (6) exists.

Remark 4.2 *From (5) we have*

$$e_o - f_o v_o = e_o^* - f_o^* v_o^*.$$

This shows that the enthalpy $e - fv$ is equal if the two sides of the stationary phase boundary is connected by the phase boundary curve only. Since in general a part of stationary phase boundary is a contact discontinuity, the above relation does not hold for an arbitrary stationary phase boundary.

Lemma 4.3 *If the level curve $f = f_o$ is not monotone, there exists a unique state (v_o^m, s_o^m) on $f = f_o$ with v_o^m in the α-phase such that the phase boundary emanates from (v_1^m, s_o^m) on $f = f_o$ with v_1^m in the β-phase. Since $e_{vs} < 0$, this also implies that if $s_o > s_o^m$, $s_o < s_o^*$ and if $s_o < s_o^m$, $s_o > s_o^*$.*

Riemann problem with entropy condition

First, we state the lemma limiting the number of phase boundaries allowed in the solution of the Riemann problem.

Lemma 4.4 *Suppose two phase boundaries move in the same direction. Then, the entropy condition is not satisfied across one of the phase boundaries.*

First we consider the case where v_l and v_r are specified in the same phase. Let U_1, U_2, U_3, and U_4 be the middle constant states from left to right with the backward phase boundary, the contact discontinuity, and the forward phase boundary in between. We have a two-parameter family of solutions and the speeds of phase boundaries σ_{p_1} and σ_{p_2} $(\sigma_{p_1} \le 0 \le \sigma_{p_2})$ can be chosen as the parameters. We study the case where the hyperbolic Riemann problem with the same initial data is constructed. This solution is the same as the double phase boundary solution with $\sigma_{p_1} = \sigma_{p_2} = 0$.

Theorem 4.5 *There are four possible solution configurations near $\sigma_{p_1} = \sigma_{p_2} = 0$ depending on the values of s_1 and s_4 at $\sigma_{p_1} = \sigma_{p_2} = 0$.*
(1) If $s_1 \le s_1^m$ and $s_4 \le s_4^m$ at $\sigma_{p_1} = \sigma_{p_2} = 0$, the hyperbolic Riemann solution is the only admissible solution.
(2) If $s_1 > s_1^m$ and $s_4 \le s_4^m$ at $\sigma_{p_1} = \sigma_{p_2} = 0$, the speed of the backward phase boundary can be negative and the forward phase boundary has the zero speed.
(3) If $s_1 \le s_1^m$ and $s_4 > s_4^m$ at $\sigma_{p_1} = \sigma_{p_2} = 0$, the backward phase boundary has the zero speed and the speed of the backward phase boundary can be negative.
(4) If $s_1 > s_1^m$ and $s_4 > s_4^m$ at $\sigma_{p_1} = \sigma_{p_2} = 0$, the backward phase boundary with the negative speed and the forward wave with the positive speed are possible.

Next, we study the case where v_l and v_r are specified in the different phases. We consider the case where the solution with the stationary phase boundary constructed. Let U_1 and U_4 be the middle constant states on the right of the backward and on the left of the forward waves, respectively. If there is only one phase boundary, we have one parameter family of solutions and we take its speed σ_p as a parameter.

Theorem 4.6 *If v_l and v_r are specified in the different phases, there are four different solution configurations near $\sigma_p = 0$ depending on the values of s_1 and s_3 at $\sigma_p = 0$.*
(1) If $s_1 \le s_1^m$ and $s_4 \ge s_4^m$ at $\sigma_p = 0$, then the solution with the stationary phase boundary is the only one solution satisfying the entropy condition.
(2) If $s_1 > s_1^m$ and $s_4 \ge s_4^m$ at $\sigma_p = 0$, there is a one-parameter family of solutions with the backward phase boundary.
(3) If $s_1 \le s_1^m$ and $s_4 < s_4^m$ at $\sigma_p = 0$, there is a one-parameter family of solutions with the forward phase boundary.
(4) If $s_1 > s_1^m$ and $s_4 < s_4^m$ at $\sigma_p = 0$, there is a one-parameter family of solutions with the backward phase boundary or the forward phase boundary. Also, it is possible to have the solution with three phase boundaries where the left phase boundary moves backward, the middle one is stationary, and the right phase boundary moves forward. In this case we have a two-parameter family of solutions.

The other admissibility criteria

The entropy condition itself is not enough to select a unique solution. In this section we examine the consequence of two admissibility criteria, the kinetic relation and the entropy rate admissibility criterion. We start from the kinetic relation.
The kinetic relation select a unique solution if v_l and v_r are specified in the same phase. On the other hand the kinetic relation may not select a unique solution if v_l and v_r are specified in the different phases. If $s_1 > s_1^m$ and $s_4 < s_4^m$, there are four solution configurations, the solution with a backward phase boundary, with a forward phase boundary, and with three phase boundaries.

Theorem 4.7 *If v_l and v_r are specified in the different phases and both $s_1 > s_1^m$ and $s_4 < s_4^m$ hold at $\sigma_p = 0$ (or $\sigma_{p_1} = \sigma_{p_2} = 0$), then there exists a positive ε such that, for every ϕ with $0 \le \phi' \le \varepsilon$, all three types solutions are admissible according to the kinetic relation of the form*

$$\sigma = \phi(s_- - s_+).$$

Concerning the entropy rate admissibility criterion, an interesting question in the Riemann problem is whether the criterion should be applied to each phase boundary separately or to all jump discontinuities. The reason for this distinction is that they produce different solutions. The former means that the entropy rate admissibility criterion is applied to each phase boundary so that each phase boundary is stable in the sense that if the constant states adjacent to the phase boundary are taken as the initial data for the Riemann problem, the solution consists of these initial data and the phase boundary with the same speed. In other words, no new waves are generated for this Riemann problem. This is justified if there is only one solution configuration. On the other hand, if there are more than one solution configuration, one way is to apply the criterion to all jump discontinuities for each configuration and choose the configuration minimizing the entropy rate. Then, it is applied to each phase boundary separately in the chosen configuration.

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4.27 Thursday, Session 2 (morning): Relaxation I

Relaxation models and finite element schemes for the shallow water equations

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We consider classical and novel relaxation models for hyperbolic balance laws and in particular for the shallow water system. We then present a class of fully discrete finite element relaxation schemes based on these models. The resulting schemes use a combination of appropriate finite elements and Runge-Kutta time stepping mechanisms and have a regularization mechanism with finite speed of propagation. The methods considered are designed for computations with arbitrary meshes and require mesh refinement close to the shocks. Numerical results are presented for several test problems.

High–resolution, Riemann–solver–free methods for conservation laws

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This talk describes how to construct Riemann–solver–free high–resolution methods for conservation laws. Two classes of methods shall be explained and compared in numerical experiments.
The first class is based on the relaxation approximation [4], where the flux is treated as an additional unknown

$$\begin{array}{rcrcl} u_t & + & v_x & = & 0 \\ v_t & + & a u_x & = & \dfrac{1}{\varepsilon}(v - f(u)) \ . \end{array} \tag{1}$$

Solving linear Riemann problems for the homogeneous advection equation on the left hand side in (1) results in an explicit formula for the additional unknowns v on the cell boundary. According to the relaxation concept, that part of the linear Riemann problem solution $v_{j+1/2} \approx f(u)_{j+1/2}$ is used as a numerical flux approximation in the semi discrete conservation law

$$u_t + \frac{1}{\Delta x}\left(v_{j+1/2} - v_{j-1/2}\right) = 0 \ , \tag{2}$$

which is integrated by appropriate time stepping methods. These schemes can be extended to higher order accuracy, applying non–oscillatory reconstructions to the characteristic variables $v \pm \sqrt{a}u$ of the linear advection system. The semi discrete system (2) is then integrated by SSP–RK methods [3]. Following these lines, relaxed ENO and relaxed PH (piecewise hyperbolic) [5] schemes shall be presented and compared. In the case of two shock waves colliding at opposite speed, the ENO reconstruction leads to grid dependent oscillations after the interaction has occurred. The more local piecewise hyperbolic reconstruction avoids this artifact and is computationally more efficient than the ENO based method.
These methods, based on relaxation, are closely related to central schemes [2]. In the first order form, without applying reconstructions, both approaches arrive at the Lax–Friedrichs or the Rusanov (local LxF) scheme. The difference is how the reconstructions are applied.

The second class of methods is based on discrete kinetic approximations [1], where the vector of unknowns is decomposed into N parts $u^\varepsilon = \sum_{n=1}^N u_n^\varepsilon$, which are governed by a weakly coupled system of advection equations An particular realization of a discrete kinetic approximation is the flux decomposition method (FDM) $F^{(d)}(u) = F_+^{(d)}(u) - F_-^{(d)}(u)$, where the Jacobians of the partial fluxes have positive eigenvalues only. For simplicity of presentation, we state the FDM in one space dimension here. Then the state vector is decomposed into three parts $u^\varepsilon = u_+^\varepsilon + u_0^\varepsilon + u_-^\varepsilon$ which are governed by

$$\begin{aligned}
\partial_t u_+^\varepsilon + \lambda \partial_x u_+^\varepsilon &= \frac{1}{\varepsilon}\left(\frac{1}{\lambda}F_+(u^\varepsilon) - u_+^\varepsilon\right) \\
\partial_t u_0^\varepsilon &= \frac{1}{\varepsilon}\left(u_+^\varepsilon + u_-^\varepsilon - \frac{1}{\lambda}(F_+(u^\varepsilon) + F_-(u^\varepsilon))\right) \\
\partial_t u_-^\varepsilon - \lambda \partial_x u_-^\varepsilon &= \frac{1}{\varepsilon}\left(\frac{1}{\lambda}F_-(u^\varepsilon) - u_-^\varepsilon\right) \quad .
\end{aligned}$$

Summing up these three PDEs it follows that $\partial_t u^\varepsilon + \lambda \partial_x (u_+^\varepsilon - u_-^\varepsilon) = 0$. Using the convergence $u^\varepsilon \to u$ and $\lambda u_\pm^\varepsilon \to F_\pm(u)$ as $\varepsilon \to 0$, the original conservation law is recovered.
In a similar way as for the relaxation approximation, numerical flux approximations are obtained from the homogenous advection system $\partial_t u_\pm \pm \lambda \partial_x u_\pm = 0$. These flux approximations $F_\pm(u)_{j+1/2} \approx \lambda(u_\pm)_{j+1/2}$ are used in the semi discrete formulation

$$u_t + \frac{\lambda}{\Delta x}\left((u_+ - u_-)_{j+1/2} - (u_+ - u_-)_{j-1/2}\right) = 0 \quad .$$

With explicit, first order time integration this leads to the Enquist–Osher scheme in the scalar case. For Euler equations, applying the van Leer flux splitting, the classical van Leer upwind scheme [6] is obtained. We shall present a high order extension applying PH reconstructions to the kinetic variables $u_\pm^\varepsilon$.
Test examples include the shallow water system and compressible gas dynamics in one and two space dimensions. Our numerical results show an increase in resolution for the upwind methods derived via the discrete kinetic formalism compared to the relaxed high resolution schemes. However, the applicability of these methods is limited, as appropriate flux decompositions are not easily available for any complex system of conservation laws.
Figure 1 shows density contours of solutions of the 2D Euler–equations with different Riemann initial data. The approximate solutions are computed by the relaxed PH scheme on a 400×400 grid.

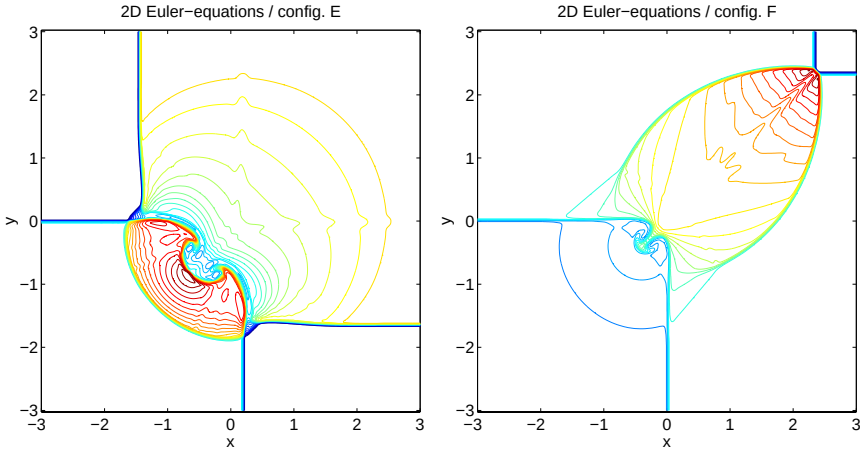

Figure 1: 2D Riemann problems

Figure 2 shows the Osher–Shu accoustic shock interaction computed by the relaxed PH scheme (left) and the flux decomposition PH method (right). The resolution is 400 points in the interval $[0, 10]$.

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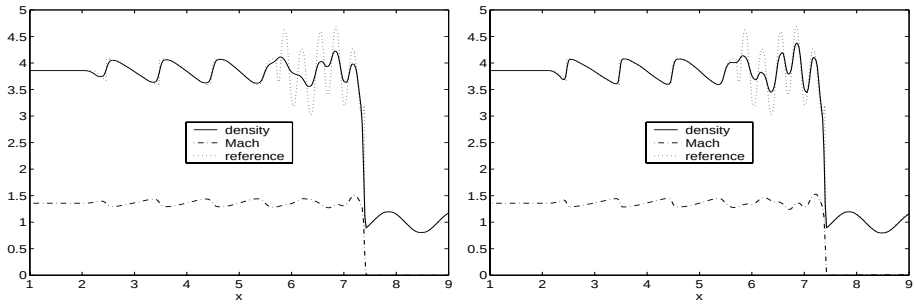

Figure 2: Osher–Shu accoustic shock interaction

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Numerical Study of Dynamic Phase Transitions in 2-D with a Relaxed Scheme

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The model of van der Waals fluid was proposed one century ago to describe dynamic phase transitions. It is a system of mixed-type partial differential equations. When lying in the unstable phase, the system is elliptic, and an instability of Hadamard's type occurs, prohibiting stable numerical simulation. This fatal difficulty had made the model abandoned for a long time. After the eighties, new attacks were made by either regularizing the system with high-order dissipations, such as viscosity-capillarity, or specifying generalized entropy conditions to single out solution within the hyperbolic domain. It was found that dynamic phase transitions are not completely described by the structural relations, but also further conditions supplied by the materials, i.e. kinetic relations and nucleation criteria.

In a previous study, we have discovered that low-order dissipations is capable of balancing with the instability in van der Waals fluid. Moreover, a category of discrete BGK models may be constructed systematically to supply a big variety of kinetic relations, and nucleation criteria that may well be relevant to different materials.

In two dimensional case, there are only a few results, and the understanding is much limited. To our knowledge, there exists a linear stability theory, as well as some numerical simulations on a mass-viscosity model. On the other hand, most complicate phenomena in physics appear in multi-dimensions. So, there is a big gap to fill, both theoretically and numerically.

The dynamic phase transition in 2-D is governed by a system in Eulerian coordinates

$$\begin{array}{lcllcll} \rho_t & + & (\rho u)_x & + & (\rho v)_y & = & 0, \\ (\rho u)_t & + & (\rho u^2 + p(\rho,T))_x & + & (\rho uv)_y & = & 0, \\ (\rho v)_t & + & (\rho uv)_x & + & (\rho v^2 + p(\rho,T))_y & = & 0. \end{array} \tag{1}$$

Here the pressure takes the form of

$$p(\rho, T) = \frac{8\rho T}{3-\rho} - 3\rho^2. \tag{2}$$

It is easy to check that at a subcritical temperature $T < 1$, the pressure is non-monotone, yielding instability.
For each equation in the form of

$$f_t + A(f)_x + B(f)_y = 0, \tag{3}$$

we may apply a relaxation approximation by

$$\begin{aligned}
g_t^+ + \frac{\lambda}{2} g_x^+ &= \frac{M^+(f) - g^+)}{\epsilon}, & M^+(f) &= \frac{f}{4} + \frac{A(f)}{\lambda}, \\
g_t^- - \frac{\lambda}{2} g_x^- &= \frac{M^-(f) - g^-}{\epsilon}, & M^-(f) &= \frac{f}{4} - \frac{A(f)}{\lambda}, \\
h_t^+ + \frac{\omega}{2} h_x^+ &= \frac{N^+(f) - h^+}{\epsilon}, & N^+(f) &= \frac{f}{4} + \frac{B(f)}{\omega}, \\
h_t^- + \frac{\omega}{2} h_x^- &= \frac{N^-(f) - h^-}{\epsilon}, & N^-(f) &= \frac{f}{4} - \frac{B(f)}{\omega}.
\end{aligned} \tag{4}$$

We remark that each of the above equations is only one-dimensional. We use a splitting technique to resolve the source term and the advection term subsequenly for one time-step marching. In one-dimensional case, numerical experiences suggest that a better result (in the sense of less smearing in resolving a shock) may be achieved with a relaxed scheme. This means, we solve the ODE with $\epsilon \to 0$, i.e. project the entity to its Maxwellian function. In the advection step, a standard second order MUSCL scheme with minmod limiter is used. Further more, to reach a second order accuracy in time, we perform a limit $\triangle t \to 0$, to obtain a semi-continuous form that reads

$$\begin{aligned}
&\frac{df_{ij}}{dt} = \frac{A_{i-1,j} - A_{i+1,j}}{2\triangle x} + \frac{\lambda}{\triangle x}\left(\frac{f_{i-1,j} + f_{i+1,j}}{2} - u_{ij}\right) - \frac{\lambda}{2\triangle x}\cdot \\
&\Big([\text{minmod}(M^+_{i+1,j} - M^+_{ij}, M^+_{ij} - M^+_{i-1,j}) - \text{ minmod}(M^+_{ij} - M^+_{i-1,j}, M^+_{i-1,j} - M^+_{i-2,j})] \\
&+[\text{minmod}(M^-_{i+2,j} - M^-_{i+1,j}, M^-_{i+1,j} - M^-_{ij}) - \text{minmod}(M^-_{i+1,j} - M^1_{ij}, M^-_{ij} - M^-_{i-1,j})]\Big) \\
&+\text{y-direction terms}.
\end{aligned} \tag{5}$$

We note that the only parameters coming from the relaxation approximation are the discrete velocities λ and ω. In the numerical studies presented here, we take $\lambda = \omega$ to attain symmetry in x and y directions. It is worth mentioning that the nonlinear interaction between ellipticity and the diffusion mechanisms is quite complicated here, and demands an intensive theoretical study. Different schemes and parameters leads to different diffusion effects, and therefore, possibly different evolution results. This may well correspond to the high sensitivity in physical situations of phase transition.
Through numerical studies with this relaxed scheme, we find that the nonlinear interaction between dissipation and instability generates patterns for different initial profiles. In particular, we shall present the formation of an octagon pattern, as well as a roll-like pattern after the disspearance of a fascinating double spiral. In these two special cases, we recognize clearly the developing of phase transition boundaries, as well as the wave interactions.

4.28 Thursday, Session 2 (afternoon): Viscous conservation laws II

On Initial-Boundary Value Problems for Viscous Conservation Laws

Konstantina Trivisa

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This research project deals with the well-posedness, regularity, asymptotic behavior of solutions to various initial-boundary value problems of viscous conservation laws. The main focus of the talk will be given to models governing compressible, reacting fluids formulated as the Navier-Stokes Equations. Special emphasis will be given to the precise description of the qualitative behavior of the solutions with large discontinuous initial data as well as to issues of stability in the case of models which describe dynamic combustion.
(*Joint work with G.Q. Chen and D. Hoff.*)

Asymptotic stability of the planar waves for the multidimensional viscous conservation laws in half space

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The purpose of the present talk is to show the asymptotic stability of one dimensional planar waves to the multidimensional viscous conservation laws in the half space:

$$u_t + f(u)_x + \sum_{i=1}^{n-1} g(u)_{y_i} = u_{xx} + \Delta_y u, \quad x > 0, \tag{1}$$

where $(x, y) = (x, y_1, y_2, \cdots, y_{n-1}) \in \mathbf{R}^n$ ($n \geq 2$). Here, the flux function f is assumed to be uniformly convex. It is assumed that $u_b < u_+$, where $u(0, y, t) = u_b$ and $\lim_{x\to\infty} u(x, y, 0) = u_+$.
The one dimensional planar waves are the solution $\phi(x, t)$ to (1) independent of y: $\phi_t + f(\phi)_x = \phi_{xx}$, with $\phi(0, t) = u_b$ and $\lim_{x\to\infty} \phi(x, 0) = u_+$. The asymptotic states of $\phi(x, t)$ in the one dimensional half space are classified into three cases: the asymptotic states are rarefaction waves if $u_* \leq u_b < u_+$, stationary waves if $u_b < u_+ \leq u_*$ and the superposition of rarefaction and stationary waves if $u_b < u_* < u_+$, where u_* is the zero of f'. We show that this classification also holds for the multidimensional model (1). Precisely, we prove that the planar waves $\phi(x, t)$ are asymptotically stable if the initial perturbation $u(x, y, 0) - \phi(x, 0)$ is located in $L^1 \cap L^\infty$. Moreover, the decay estimate:

$$\sup_{x>0,\ y\in\mathbf{R}^{n-1}} |u(x, y, t) - \phi(x, t)| \leq C(1 + t)^{-n/2+\varepsilon} \tag{2}$$

is obtained, where ε is an arbitrarily small positive constant.

4.29 Thursday, Session 3 (morning): Discontinuous fluxes

Conservation laws with space dependent flux: theory and numerics

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Introduction

We shall discuss a conservation law of the form

$$u_t + f(x,u)_x = 0 \tag{1}$$

where the coefficient may be a discontinuous function of x.

This can model continuous sedimentation. Here the settling of solid particles in a liquid under the influence of gravity is described.

In the study of this problem numerical approximations and theoretical proofs are closely interwoven. We shall bring two numerical schemes to bear: the front tracking approximation and a relaxation approximation. Whereas the former is a great tool both for accurate and fast numerical simulations and to obtain new theoretical results, the latter is easier to implement while still providing good numerical results.

Front tracking approximation

This method is based on the Riemann solution which gets applied to a conservation law approximating (1) by making a piecewise linear approximation to the flux. A unique solution to the Riemann problem is selected by the vanishing viscosity selection criterion. With the help of a Temple functional compactness of the front tracking approximation is shown which gives existence of weak solutions, [KR]. We can also give a stability estimate for the solution to the two phase polymer flow model. Numerical examples will be shown including the case of sedimantation, [BK].

Relaxation

Here the solution to (1), (2) is approximated by a relaxation system

$$u_t + v_x = 0 \tag{2}$$

$$v_t + cu_x = \frac{f(x,u)-v}{\epsilon} \quad . \tag{3}$$

Under suitable stabilty condition the convergence of the solutions to (3),(4) for $\epsilon \to 0$ is proven. Numerical examples show that a second order numerical approximation of (3), (4) is easy to implement and gives good numerical approximations to (1), (2), [KKR].

References

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On conservation laws modeling continuous sedimentation in vessels with discontinuously varying cross-sectional area

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We study the initial value problem for a scalar conservation law modeling the settling of solid particles in a viscous fluid under the influence of gravity in a so-called clarifier-thickener vessel. We start by discussing the mathematical model that yields this conservation law. The model incorporates a feed mechanism where the feed suspension enters the vessel, an overflow outlet at the top of the vessel, and a discharge outlet at the bottom of the vessel. The resulting initial value problem is

$$u_t + g(x,u)_x = 0, \quad x \in \mathbb{R},\ t > 0, \tag{1}$$

$$g(x,u) = \begin{cases} Q_{\mathrm{L}}u + (Q_{\mathrm{L}} - Q_{\mathrm{R}})u_{\mathrm{F}} & \text{for } x < -1, \\ Q_{\mathrm{L}}u + S(x)h(u) + (Q_{\mathrm{L}} - Q_{\mathrm{R}})u_{\mathrm{F}} & \text{for } -1 < x < 0, \\ Q_{\mathrm{R}}u + S(x)h(u) & \text{for } 0 < x < 1, \\ Q_{\mathrm{R}}u & \text{for } x > 1, \end{cases} \tag{2}$$

together with the initial condition

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}; \quad u_0(x) \in [0, u_{\max}]. \tag{3}$$

The spatially varying flux $g(x,u)$ has discontinuities at the feed level $x = 0$, at the discharge level $x = 1$, and at the overflow level $x = -1$. In addition, the spatially dependent cross sectional area $S(x)$ is allowed to have jump discontinuities. The specific properties of the suspension under study are described by the batch flux density function $h(u)$, which vanishes outside $(0, u_{\max})$, is nonnegative, and generally has at least one inflection point between 0 and $u_{\max}$, such that the flux is nonconvex. We find that it is possible view the flux as depending on a pair of spatially varying parameters, i.e., $g(x,u) = f(\gamma(x),u)$, with $\gamma(x) = (\gamma_1(x), \gamma_2(x))$. Once we have adopted this point of view, it is not hard to apply a slight extension of the Engquist-Osher scheme

$$U_j^{n+1} = U_j^n - \lambda \Delta_- f^{\mathrm{EO}}\left(\gamma_{j+\frac{1}{2}}, U_{j+1}^n, U_j^n\right), \tag{4}$$

with the parameterized version of the Engquist-Osher flux defined by

$$f^{\mathrm{EO}}(\gamma, v, u) = \frac{1}{2}\left(f(\gamma,u) + f(\gamma,v) - \int_u^v |f_u(\gamma,w)|\, dw\right). \tag{5}$$

Here we have discretized the spatially varying flux parameters on a mesh that is staggered with respect to that of the conserved variable. This simple idea is what allows us to use a standard *scalar* numerical flux, the advantage being we can avoid solving the full 2×2 Riemann problems that arise when the meshes are aligned. Note that the problem discussed here has a number of different types of 2×2 Riemann problems, one for each of the various types of jumps in the flux. The mesh staggering approach also simplifies the task of analyzing the scheme with respect to compactness of the approximations.

Our main result is convergence of a subsequence of the approximations toward a weak solution of the conservation law. The key ingredient in the proof is the use of a so-called singular mapping, defined by

$$\Psi(\gamma, u) = \int_0^u |f_u(\gamma, w)| dw.$$

For these problems, a bound on the total variation of the conserved quantity is very difficult, if not impossible, which makes it necessary to use the singular mapping approach. Letting $u^{\mathcal{D}}$ and $\gamma^{\mathcal{D}}$ denote our numerical approximations and the discretized version of γ, with $\mathcal{D}$ denoting the mesh refinement, we transform the numerical approximations according to $z^{\mathcal{D}} = \Psi(\gamma^{\mathcal{D}}, u^{\mathcal{D}})$, and prove compactness for the the transformed sequence $z^{\mathcal{D}}$. Since Ψ is strictly increasing in its second argument, it is possible to recover the limit of the conserved variable u by inverting Ψ.
To put this effort in the context of recent work in this area, convergence of a front tracking algorithm has been proven in [1] for the constant cross section problem. Also for the constant cross section problem, in [2] we introduced the scheme discussed here, and proved compactness in [3]. Only a minor modification is required to pass from the constant cross section scheme to the varying cross section scheme. However, in establishing compactness, we encounter some analytical difficulties that were not present in the constant cross section problem. Specifically, application of the singular mapping approach to the Engquist-Osher scheme requires a certain decomposition of the singular mapping Ψ. In previously considered problems, this decomposition has been straightforward. The varying cross section requires a new approach in order to achieve this decomposition.
Aside from compactness issues, the scheme satisfies a set of cell entropy inequalities, from which we conclude that the limit of any convergent subsequence satisfies the corresponding set of Kruzkov-type entropy inequalities. For piecewise smooth weak solutions, it follows from the results in [4] that certain geometric entropy inequalities are satisfied, and that such solutions form an L^1 contraction semigroup.
Finally, we present the results of numerical experiments.

References

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4.30 Thursday, Session 3 (afternoon): Godunov and ENO schemes

The generalised Riemann problem: the basis of ADER schemes

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Advection-reaction type partial differential equations model a wide variety of phenomena in several disciplines in physics, chemistry, environmental sciences, geometry, financial mathematics and many others. Generally, these non-linear inhomogeneous equations must be solved in complicated multi-dimensional domains and thus analytical solutions are only available under very special circumstances. Situations in which exact solutions are available include Riemann problems. Conventionally, the Riemann problem for a system of conservation laws in two independent variables x and t is the initial value problem for the system with initial conditions consisting of two *constant* states separated by a discontinuity at the origin $x = 0$; for background see [3] and [8], for example. Ben-Artzi and Falcovitz [1] and others have generalised the concept of Riemann problem by admitting initial conditions that are *linear* functions in x, separated by a discontinuity at $x = 0$. We introduce the notation GRP_1 to denote this generalisation of the conventional Riemann problem, denoted by GRP_0.

Here we extend further the concept of generalised Riemann problem in two respects, the first concerns initial conditions and the second concerns the type of governing equations. As to initial conditions, we admit k-th order polynomial functions of x and denote the corresponding generalised Riemann problem by GRP_k. The most general case is that in which the initial conditions are two arbitrary but infinitely differentiable functions of x, with the corresponding generalised Riemann problem denoted by GRP_∞. Concerning the governing differential equations, in addition to pure non-linear advection, we include here reaction-like terms; these source terms are assumed to be arbitrary but sufficiently smooth algebraic functions of the unknowns.

We propose a semi-analytical method of solution of the generalised Riemann problem GRP_∞ for non-linear advection-reaction partial differential equations. The method gives the solution at $x = 0$ at a time τ, assumed to be sufficiently small, in terms of a time Taylor series expansion at $x = 0$ about $t = 0$. The leading term in this expansion is the exact solution of a conventional non-linear Riemann problem, GRP_0, with piece-wise constant initial conditions. All remaining terms in the expansion have coefficients that are time derivatives of the solution; these time derivatives are replaced by spatial derivatives by repeated use of the differential equations, a technique known as the Lax-Wendroff procedure [4]. It is then shown that all spatial derivatives in the expansion obey inhomogeneous advection equations. As derivative values are required at time $t = 0^+$, all source terms in these evolution equations for all-order spatial derivatives vanish and one only requires the solution of the conventional Riemann problems with piece-wise constant initial conditions for each of these advection equations. In summary, we reduce the solution of the generalised advection-reaction Riemann problem GRP_∞ to that of solving a sequence of conventional Riemann problems GRP_0 for homogeneous advection equations, the solutions of which are pieced together to determine the complete solution.

The approach can be extended further to solve the *general initial value problem* for advection-reaction partial differential equations with piece-wise smooth initial conditions for sufficiently small times. At any position x where the initial condition is smooth the GRP_∞ solution procedure applies trivially. In the case in which the initial condition includes more than one discontinuity, one places local origins at these positions to define local problems GRP_∞ and applies the solution method as described. In this manner, for a sufficiently short time, one obtains the solution everywhere at any point x that is sufficiently far away from points of initial discontinuities. This restriction is necessary to preserve time smoothness in the local Riemann problems. The solution technique presented here can be used for assessing the performance of numerical methods intended for solving complicated problems involving both non-linear advection and reaction. Given its local character, our solution procedure has also the potential for providing sub-cell resolution in numerical computation procedures for the general initial-boundary value problem. Finally, our solution procedure can be used to construct numerical methods of very high order of accuracy in space and time to solve the general initial-boundary value problem; results reported in [9], [11], [6], [7], [10], [5] look very promising. These methods extend existing numerical approaches such as those of Godunov [3] and Glimm [2] in which local solutions of conventional Riemann problems with piece-wise constant initial conditions are pieced together to advance the solution in time. Finally we make some remarks concerning the generalised Riemann problem for advection-diffusion equations.