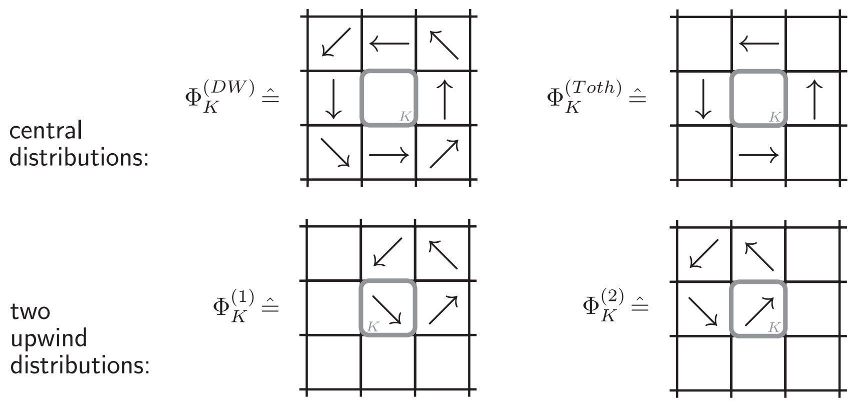

Figure 1: Divergence-free transport in two space dimensions. Examples of fieldline-fluxdistributions emerging from cell K. The updates by the fluxdistributions have to approximate closed curves to preserve the divergence locally in every timestep.

In this paper the central quantity is the so called *fluxdistribution* Φ_K, a grid function with $\operatorname{supp}\Phi_K = K \cup \mathcal{N}(K)$, where $\mathcal{N}(K)$ are the neighbouring cells of K. The value $\Phi_K(\hat{K})$ of the fluxdistribution gives the change of u at cell $\hat{K}$ caused by the cell K. Given a fluxdistribution then a numerical method is established by

$$u(t+\Delta t) = u(t) + \sum\nolimits_{\hat{K}} \Phi_{\hat{K}} + O(\Delta t^m) \tag{4}$$

where Δt is a timestep. To incorporate the constraint we require

$$\tilde{\mathcal{C}}_K \cdot \Phi_{\hat{K}} = 0 \quad \forall \hat{K}, K \tag{5}$$

to hold for the fluxdistribution. Thus it follows

$$\Rightarrow \quad \tilde{\mathcal{C}}_K \cdot u(t+\Delta t) = \tilde{\mathcal{C}}_K \cdot u(t) \tag{6}$$

i.e. the value of the discrete constraint is *preserved*. From Eqn (5) follows a linear system whose nontrivial solutions represent fluxdistributions that locally preserve the discrete constraint. The remaining parameters in the solutions of (5) are calculated by the consistency condition

$$\lim_{\Delta t, h \to 0} \frac{1}{\Delta t} \sum\nolimits_{\hat{K} \in \mathcal{N}(K)} \Phi_{\hat{K}}(K) \to \mathcal{L}(u)|_K \tag{7}$$

for the fluxdistribution with the transport equation in (1). The resulting method depends strongly on the used discretization $\tilde{\mathcal{C}}$ of the constraint and on the choice of the fluxdistribution out of the solutions of (5). In the following some fluxdistributions for a special case are discussed.

Divergence-free Advection

As a simple example for a constrained transport of type (B) we consider the divergence-free advection

$$\partial_t u + \operatorname{curl}(u \times v) = 0 \qquad \text{with} \ \operatorname{div} u = const \tag{8}$$

Such an equation arises in magnetohydrodynamics as well as in the vorticity equation for fluid flows.
In two dimensions the variable u has the form $\left(u^{(x)}(x,y), u^{(y)}(x,y), 0\right)$ and the given advection velocity is $v = \left(v^{(x)}(x,y), v^{(y)}(x,y), 0\right)$. Additionally we have $\Phi_K(\hat{K}) \in \mathbb{R}^2$. For simplicity of presentation the grid is chosen as uniform and rectangular with $K \hat{=} (i,j)$.
In two dimensions the set of all discrete divergence operator with second order accuracy and 3×3 stencil is described by a 3-parametric family of operator matrices in (3)

$$C(\alpha, \beta, \gamma) = C_0 + \alpha C_1 + \beta C_2 + \gamma C_3 \qquad \alpha, \beta, \gamma \in \mathbb{R}. \tag{9}$$

Any choice for α, β and γ produces a second order accurate approximation of the divergence.

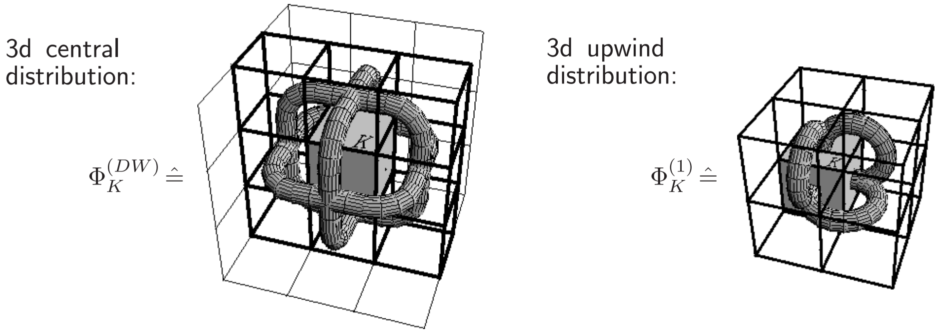

Figure 2: Examples of 3-dimensional fieldline-fluxdistributions that preserve the divergence locally in every timestep. Like in Fig.1 the updates by the fluxdistributions have to approximate closed curves which are overstated in the figure as tubes.

Classical Divergence Operator: If $\alpha = \beta = \gamma = 0$ is used in Eqn (9) the classical operator

$$\mathrm{div}\, u|_{i,j} = \frac{u^{(x)}_{i+1,j} - u^{(x)}_{i-1,j}}{2h} + \frac{u^{(y)}_{i,j+1} - u^{(y)}_{i,j-1}}{2h} \tag{10}$$

arises. Inserting this operator into condition (5) yields one nontrivial solution for the fluxdistribution

$$\Phi_{i,j} \in \mathrm{span}\left\{\Phi^{(Toth)}_{i,j}\right\} \tag{11}$$

The resulting scheme is the central scheme proposed by Toth in [6]. In the upper right corner of Fig.1 the corresponding fluxdistribution is displayed.
Extended Divergence Operator Another important discrete divergence operator is obtain out of (9) if $\alpha = \frac{1}{8}$, $\beta = \gamma = 0$ is chosen. This choice maximizes the number of nontrivial solutions of (5) while it minimizes the remaining constant in the rest term of (3). The fluxdistribution may now be formed out of four independent parts.

$$\Phi_{i,j} \in \mathrm{span}\left\{\Phi^{(1)}_{i,j}, \Phi^{(2)}_{i,j}, \Phi^{(3)}_{i,j}, \Phi^{(4)}_{i,j}\right\} \tag{12}$$

The staggered scheme of Dai/Woodward in [5] is obtained if one chooses

$$\Phi^{(DW)}_{i,j} \sim \Phi^{(1)}_{i,j} + \Phi^{(2)}_{i,j} + \Phi^{(3)}_{i,j} + \Phi^{(4)}_{i,j}. \tag{13}$$

In Fig.1 this fluxdistribution is shown in the upper left corner.
The DW scheme uses a symmetric distribution. An upwind method may be constructed by weighting the distributions $\Phi^{(i)}$ according to the direction of the wind (see Fig.1). In this paper the resulting scheme will be tested and compared to the schemes of Dai/Woodward and Toth.
Fluxdistributions in 3 Dimensions In three dimensions the variety of discrete divergence operators and thus of possible fluxdistributions is much higher. By use of an operator analoguous to the extended operator in 2d we obtain 28 independent fluxdistributions. From them a 3d-version of the DW-scheme as well as a upwind scheme may be build. Fig.2 shows the symmetric and a possible upwind distribution.

References

[1] K. W. Morton and P. L. Roe, *Vorticity-Preserving Lax–Wendroff-Type Schemes for the System Wave Equation*, SIAM J. Sci. Comp. **23**/1, (2001) p.170
[2] J. U. Brackhill and D. C. Barnes, *The effect of nonzero $\nabla \cdot B$ on the numerical solution of the magnetohydrodyanmic equations*, J. Comp. Phys. **35**, (1980) p.426
[3] C. R. Evans and J. F. Hawley, *Simulation of Magnetohydrodynamic Flows: A Constrained Transport Method*, Astrophys. J. **332**, (1988) p.659
[4] K. G. Powell, *An approximate Riemann solver for magnetohydrodynamics (that works in more than one dimension)*, ICASE Report No 94-24, (1994)

[5] W. Dai and P. R. Woodward, *On the divergence-free condition and conservation laws in numerical simulations for supersonic magnetohydrodynamic flows*, Astrophys. J. **494**, (1998) p.317
[6] G. Toth, *The $\nabla \cdot B$ Constraint in Shock-Capturing Magnetohydrodynamics Codes*, J. Comp. Phys. **161,** (2000) p.605
[7] H. DeSterck, *Multi-Dimensional Upwind Constrained Transport on Unstructured Grids for Shallow Water Magnetohydrodynamics*, AIAA Paper 2001-2623, (2001)
[8] R. Limacher, *Simulation der MHD-Gleichungen mit der Transportmethode*, diploma thesis, ETH Zurich (2000)

4.35 Thursday, Session 6 (morning): Boundary problems

A Numerical Method for Exact Boundary Controllability Problems for the Wave Equation

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The computational approximation of exact boundary controllability problems for the wave equation are studied. A numerical method is introduced which is based on the direct solution of optimization problems that are introduced in order to define unique solutions of the controllability problem. The uniqueness of the discrete solutions obtained in this manner is demonstrated and the convergence properties of the method are illustrated through computational experiments. Efficient implementation strategies for the method are also discussed. It is shown that for smooth, minimum L^2-norm Dirichlet controls, the method results in convergent approximations without the need to introduce regularization. Furthermore, for the generic case of non-smooth Dirichlet controls, convergence is also obtained with respect to L^2 norms. One of the strengths of the method is the flexibility it allows for treating other controls and other minimization criteria; such generalizations are discussed. (Joint work with L. Steven Hou and Lili Ju)

Oscillation and asymptotic behavior of the free boundary governed by a hyperbolic equation

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Introduction

Let us consider the following one-dimensional free boundary problem

$$(P)\quad \begin{cases} u_{xx} - u_{tt} = 0 & \text{in} \quad (0,\infty)\times\{t>0\}\cap\{u>0\}, \\ u_x^2 - u_t^2 = Q^2 & \text{on} \quad (0,\infty)\times\{t>0\}\cap\partial\{u>0\}, \end{cases}$$

with the initial conditions

$$(I)\quad \begin{cases} u(x,0) = e(x) & \text{in} \quad (-l_0, 0), \\ u_t(x,0) = g(x) & \text{in} \quad (-l_0, 0), \end{cases}$$

and the boundary condition

$$(B)\quad u(-l_0, t) = f(t) \qquad \text{for} \quad t \geq 0,$$

where $e(x)$, $g(x)$ and $f(t)$ are given functions, Q and l_0 are positive constants.
This problem arises from the following variational problem which is related to the physical model " A thin film is pasted on the plate. By lifting up the edge of the film in the vertical direction, the film is peeled from the plate." The shape of the film is described by the graph of a function $u : \Omega \to \mathbf{R}$. It can be assumed that the effect of the peeling front on the Lagrangian is constant Q^2. Then, this problem is to find a stationary point of the functional:

$$\begin{aligned} (J)\quad J(u) &= \int_0^{T^*}\int_{u>0}\left(\tau|\nabla u|^2 - \rho(D_t u)^2 + Q^2\right)dxdt \\ &= \int_0^{T^*}\int_{\Omega}\left(\tau|\nabla u|^2 - \rho(D_t u)^2 + Q^2\right)\chi_{u>0}\,dxdt \qquad u \in K, \end{aligned}$$

where Ω is a domain in $\mathbf{R}^n$, T^* is a positive constant, $\chi_{u>0}$ is a characteristic function of the set $\{(x,t) \in \Omega \times (0,T^*); u(x,t) > 0\}$ and K is a suitable function space. Here the constants τ and ρ are the tension, and the line density, respectively. To approach this problem, we assume that a stationary point is sufficiently smooth. Then we can derive (1) and (2) as the Euler-Lagrange equations from the functional (J) (cf. [1], [2] and [7]),

$$\tau \Delta u - \rho u_{tt} = 0 \qquad \text{in} \quad \Omega \times \{t > 0\} \cap \{u > 0\}, \tag{1}$$

$$\frac{\tau}{2}|\nabla u|^2 - \frac{\rho}{2}u_t^2 = \frac{Q^2}{2} \qquad \text{on} \quad \Omega \times \{t > 0\} \cap \partial\{u > 0\}. \tag{2}$$

In this talk, as a first step, the one-dimensional problem will be analyzed. By using a change of variable t, we can assume $\tau = 1$ and $\rho = 1$. Therefore, we will consider the problem (P).

The initial condition (I) implies that the thin film has been already peeled from the plate on the interval $(-l_0, 0)$ and the boundary condition (B) corresponds to the situation in which the height of the edge of the film is described by $f(t)$. Kikuchi and Omata [7] showed the existence of time-local solutions under the several conditions which were imposed on the functions $e(x)$, $g(x)$ and $f(t)$. On the global existence of solutions, however, they have not stated. In [5], numerical experiments were carried out. A sufficient condition for the global existence was given by Nakane [6] by using the iteration method. This method, however, can not be applied to the case of the peeling speed is zero.

In this talk, it is shown that if the peeling speed f' is non-negative then time global solutions for the problem (P), (I) and (B) are constructed (cf. Main Theorem). It is remarked here that this condition is improvement on the previous one in [6]. By admitting $f' < 0$, there exists an example whose solution can be constructed only locally in time with a pair of particular initial and boundary conditions.

On the other hand, in case the peeling speed f' is zero, we will mention the cause of the periodic behavior of the free boundary from the mathematical point of view.

To consider our problem, two variables ξ and η are introduced by

$$t = (\xi + \eta)/2,$$
$$x = (\xi - \eta)/2.$$

Since the initial values e and g are given on the line $\{(x,t); t = 0\}$, they are defined on the line $\{(\xi,\eta); \xi + \eta = 0\}$ in the new coordinate. We regard them as the functions of ξ and rewrite them e and g again. Similarly, the boundary value f is given on the line $\{(\xi,\eta); \xi - \eta + 2l_0 = 0\}$ and is a function of η. Therefore (P), (I) and (B) are transformed into

$$(P') \qquad \begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{u > 0\}, \\ -4u_\xi u_\eta = Q^2 & \text{on} \quad \partial\{u > 0\}, \end{cases}$$

$$(I') \qquad \begin{cases} u(\xi,-\xi) = e(\xi) & \text{in} \quad (-l_0, 0), \\ u_\eta(\xi,-\xi) + u_\xi(\xi,-\xi) = g(\xi) & \text{in} \quad (-l_0, 0), \end{cases}$$

$$(B') \qquad u(\eta - 2l_0, \eta) = f(\eta - l_0) \qquad \text{in} \quad [l_0, \infty).$$

By taking these equations into considerations, we state the following problem:

Problem. *Let T be a positive constant. Find a pair of functions $u \in C^0(\{(\xi,\eta); \xi \geq \eta - 2l_0, \xi \geq -\eta\})$ and $l \in C^0([0,T)) \cap C^1((0,T))$ which satisfies (P'), (I') and (B') for $\eta < T$ and*
(i) $l(0) = 0$,
(ii) $u \in C^2(\{(\xi,\eta); \eta - 2l_0 < \xi < l(\eta), \xi > -\eta\}) \cap C^1(\{(\xi,\eta); \eta - 2l_0 < \xi \leq l(\eta), \xi \geq -\eta\})$,
(iii) $u > 0$ *in* $\{(\xi,\eta); \eta - 2l_0 \leq \xi < l(\eta), \xi > -\eta\}$,
(iv) $u(\xi,\eta) = 0$ *in* $\{(\xi,\eta); \xi \geq l(\eta)\} \cup \{(\xi,\eta); \xi \geq -\eta, \eta < 0\}$.

Assumption. *Suppose that the functions $f(\eta - l_0) \in C^2([l_0,\infty))$, $e(\xi) \in C^2([-l_0, 0])$ and $g(\xi) \in C^1([-l_0, 0])$ satisfy the* *following conditions:*

$$\text{(A.0)} \quad \begin{cases} e(\xi) > 0 \quad \text{in} \quad (-l_0, 0), \\ g(\xi) > 0 \quad \text{in} \quad (-l_0, 0), \\ f(0) = e(-l_0) > e(0) = 0, \end{cases}$$

$$\text{(A.1)} \quad \begin{cases} f'(0) = g(-l_0), \\ f''(0) = e''(-l_0), \end{cases}$$

$$\text{(A.2)} \quad \begin{cases} e'(0)^2 - g(0)^2 = Q^2, \\ (e''(0) + g'(0))(e'(0) - g(0))^4 = Q^4(e''(0) - g'(0)), \end{cases}$$

$$\text{(A.3)} \quad \begin{cases} e'(\xi) < g(\xi) & \text{for} \quad (-l_0 \leq \xi \leq 0), \\ f'(\xi + l_0) - (e'(\xi) + g(\xi))/2 > 0 & \text{for} \quad (-l_0 \leq \xi \leq 0), \end{cases}$$

$$\text{(A.4)} \quad f'(\eta - l_0) \geq 0 \quad \text{for} \quad \eta \in [l_0, \infty).$$

Lemma. *Let c be a positive number and $I = [0, c)$ an interval on the η-axis. Let $\lambda(\eta) \in C^2(I)$ be a function which satisfies*
(i)$\lambda(0) = 0$,
(ii)$\lambda'(\eta) > 0$ for $\eta \in I$.
Then there exists a unique pair of functions $u(\xi, \eta) \in C^2(\{(\xi, \eta); \eta \in I, 0 < \xi < l(\eta)\}) \cap C^1(\{(\xi, \eta); \eta \in I, 0 \leq \xi \leq l(\eta)\})$ and $l(\eta) \in C^2(I)$ such that

$$(PL) \begin{cases} u_{\xi\eta} = 0 & \text{in} \quad \{(\xi, \eta); 0 < \xi < l(\eta), \eta \in I\}, \\ -4u_\xi u_\eta = Q^2 & \text{on} \quad \{(\xi, \eta); (l(\eta), \eta), \quad \eta \in I\}, \\ u(l(\eta), \eta) = 0 & \text{on} \quad \eta \in I, \\ u(0, \eta) = \lambda(\eta) & \text{on} \quad \eta \in I. \end{cases}$$

Main Theorem. *For any $T > 0$, there exists a unique solution to Problem.*
Remark. The solution can be constructed by pasting the local solutions, inductively. (A.1) and (A.2) give the regularity of the solution along the joint line. By (A.3), the condition (ii) of Lemma holds for each step. Moreover, from (A.4), we have that the pasting interval does not decrease. It implies that there exists a time global solution for the problem (P), (I) and (B).

References

[1] H. W. Alt and L. A. Caffarelli, *Existence and regularity for a minimum problem with free boundary,* J. Reine Angew. Math., **325** (1981), 105–144.

[2] H. W. Alt, L. A. Caffarelli and A. Friedman, *A free boundary problem for quasi-linear elliptic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **11** (1984), 1–44.

[3] E. DiBenedetto, “ Partial Differential Equations,” Birkhauser, 1995.

[4] R. L. Devaney, “ An introduction to Chaotic Dynamical Systems, ” Benjamin/Cummings, 1986.

[5] H. Imai, K. Kikuchi, K. Nakane, S. Omata and T. Tachikawa, *A Numerical Approach to the Asymptotic Behavior of Solutions of a One-Dimensional Free Boundary Problem of Hyperbolic Type,* J.J.I.A.M., **18-1** (2001), 43–58.

[6] K. Nakane, *Global existence of solutions to a one-dimensional free boundary problem of hyperbolic type,* Proceedings of Japan Academy, **Vol. 77, Ser. A, No. 7** (2001), 103–107.

[7] K. Kikuchi and S. Omata, *A Free boundary problem for one dimensional hyperbolic equation,* Adv. Math. Sci. Appl. **9**(2) (1999), 775–786.

Evaluation of the well-posed Perfectly Matched Layer for Advective Acoustics

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The well-posed PML for advective acoustics suggested by Abarbanel, Gottlieb and Hesthaven [1] is evaluated. For numerical simulations of the testproblem used in [1] we experience exponential growth inside the PML destroying the solution inside the computational domain. An analysis of the constant coefficient problem reveals the existence of eigenvalues connected to exponential growth. By numerical investigation of the eigenvalues and eigenfunctions of the spatially discretized testproblem we see the same characteristic growth, indicating that the same type of growing modes as for the constant coefficient problem are present. A simple fix to avoid the exponential growth is suggested. Numerical results are presented that show that the fix has small influence on the efficiency of the PML. More complex problems are simulated. Especially for problems with Mach-number varying perpendicular to the flow direction, as inside a jet, the PML works well. Simulations show that the efficiency of the PML is as good as for the constant Mach-number case.

References

[1] S. Abarbanel, D. Gottlieb, and J. S. Hesthaven. Well-posed perfectly matched layers for advective acoustics. *Journal of Computational Physics*, 154:266–283, 1999.

4.36 Thursday, Session 6 (afternoon): Applications III

Modelling of Traffic Flows

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Traffic flows with inhomogeneous road conditions are studied. The inhomogeneous road conditions we modelled include obstacles and others. Our model is a system of nonlinear hyperbolic equations with both relaxation and sources. The flux and the source terms depend on the space variable. The main difficulty is that the system of equations is inhomogeneous in the space variable x. The well-posedness of the model in L^1 topology is established and the unique zero relaxation limit is obtained.
Meanwhile, we study a traffic flow model with a nonconcave fundamental diagram. A fundamental diagram gives a correspondence of vehicle density to the flow rate in traffic. In general, a fundamental diagram is not concave as observed in real traffic flow. The model is a system of nonconcave hyperbolic conservation laws with relaxation. One of the characteristic fields of the system is neither linearly degenerate nor genuinely nonlinear. We establish the global existence and uniqueness of the solution to the Cauchy problem and zero relaxation limit of the solutions.

Conservation Laws and ODEs: a Traffic Problem

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The well known Lighthill-Whitham [4] and Richards [5] traffic flow model reads

$$\begin{cases} \partial_t \rho + \partial_x \left[\rho \cdot v(\rho)\right] = 0 \\ \rho(0,x) = \bar\rho(x) \end{cases} \tag{1}$$

where ρ is the car density, $v(\rho)$ is the speed of cars at traffic density ρ and $\bar\rho$ is a suitable initial datum. The speed $v(\rho)$ is assumed to fulfilled the following the standard assumptions

(H1) $v \colon [0,R] \mapsto [0,+\infty[$ is smooth, decreasing and $v(R) = 0$.

Assume that a driver travels along a road modeled by (1), being influenced by the traffic along the road but without influencing it significantly. This driver's position $p = p(t)$ then solves the Cauchy problem

$$\begin{cases} \dot p = w\left(\rho(t,p)\right) \\ p(0) = \bar p \end{cases} \tag{2}$$

for a suitable function w and an initial position $\bar p$.
We aim at studying problem (1)-(2) and some optimization problems related to it. Indeed, under assumptions reasonable from the point of view of the traffic flow models, (1)-(2) does not fit in the cases usually considered in the current literature.
We stress that this optimization is carried out in a *dynamic* situation. In fact, as it has to be expected, the optimal behavior of the driver depends on the traffic he will meet all along his way and this, in turn, may well be far from any equilibrium or stationary situation.
The first step in the program outlined above is the solvability of the Cauchy problem for (2). The main theorems on the well posedness of ordinary differential equations with discontinuous right hand side require either assumptions of Caratheodory type [3, §1] or the bounded directional variation of the vector field [1]. As noted in [2], equations of type (1)-(2) fall in neither of the two categories above. In fact, differently from the usual theory of o.d.e.s, here the solutions turn out to be a *H ölder* function of the initial data.

In [2], aiming at a result on the well posedness of a class of 2×2 conservation laws, it is proved an existence and uniqueness result for ordinary differential equations depending on the solution of a scalar conservation law. However, problem (1)-(2) does not fit in the result therein, for the assumption (A2) in [2] is unrealistic in the present framework. Here, this assumption would mean that the speed of the driver varies in an interval *disjoint* from that of the main traffic speed.
Let R stand for the maximal possible traffic density along the road. Denote by f the traffic flow, i.e. $f(\rho) = \rho \cdot v(\rho)$. The present construction is accomplished on the time interval $[0,T]$, for an arbitrary positive T. We look for a Filippov [3] solution to (2), i.e. an absolutely continuous function $p\colon [0,T] \mapsto \mathbf{R}$ such that

$$\begin{cases} \dot p(t) \in \overline{\mathrm{co}}\big\{ w(\rho) \ : \ \rho \in \mathcal{I}[\rho(t,p(t)-),\, \rho(t,p(t)+)] \big\}\,, \\ p(0) = \bar p\,. \end{cases} \tag{3}$$

Here $\overline{\mathrm{co}}$ stands for the closed convex hull, $\rho(t,x-)$, $\rho(t,x+)$ are respectively the left and right limits of $x \mapsto \rho(t,\cdot)$ at x, and $\mathcal{I}[a,\, b]$ is the smallest interval containing a and b. ¿From the physical point of view, our choice of Filippov solutions is motivated by the realistic situation of a driver being trapped in a queue. In fact, consider the case

$$\bar\rho(x) = \begin{cases} 0 & \text{if } x < 1 \\ R & \text{if } x > 1 \end{cases} \qquad \bar p = 0 \quad \text{and} \quad w = v \tag{4}$$

If the flux function f is strictly concave, the solution to (1) with initial data (4) is a single shock wave that travels with zero speed and represents the beginning of a queue. Independently of v, the driver reaches the queue but then can not pass through it (see figure 1). Hence a Caratheodory solution does not exist for all times. As it is well known, the choice of the wider class

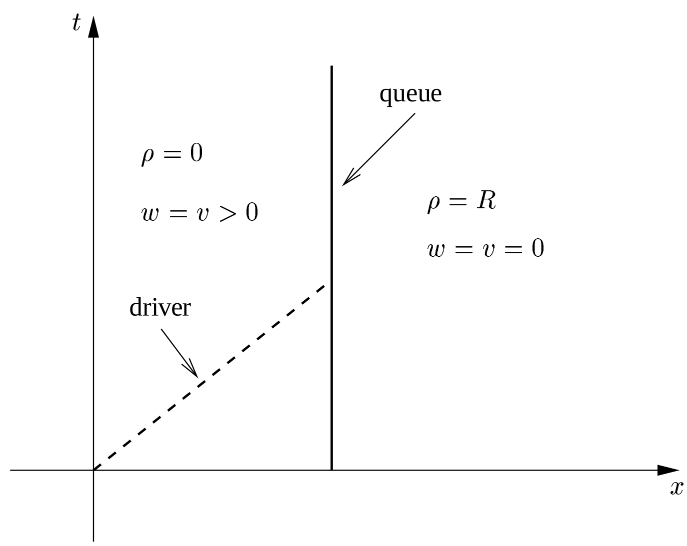

Figure 1: A Caratheodory solution may fail to exist for all times

of Filippov solutions easily guarantees the existence. In the present setting, in spite of this choice, uniqueness and continuous dependence continue to hold. This is stated in the following theorem.

Theorem 1 *Let v be as in* **(H1)** *and $w\colon [0,R] \mapsto [0,+\infty[$ be smooth. If*

$$w(\rho) > f'(\rho) \qquad \text{for all } \rho \in [0,R] \tag{5}$$

then for all initial data $\bar\rho \in \mathbf{BV}$, problem (1)–(2) admits a unique Filippov solution. Moreover, if $p_1(\cdot)$ and $p_2(\cdot)$ solve (2) with initial data respectively $\bar p_1$, $\bar p_2$, then we have the Hölder dependence

$$\left| p_1(t) - p_2(t) \right| \le C \cdot \left| \bar p_1 - \bar p_2 \right|^{\alpha}, \tag{6}$$

for some $\alpha \in \,]0,1[$ and $C > 0$.

If $w = w(\rho)$ attains values both lower and greater than $f'(\rho)$, then the continuous dependence of the solutions to (2) on the initial data may fail. Indeed, let $\bar\rho$ in (1) be defined by

$$\bar\rho(x) \doteq \begin{cases} \rho^\ell & \text{if } x < 0\,, \\ \rho^r & \text{if } x > 0\,, \end{cases} \tag{7}$$

with $\rho^\ell > \rho^r$, so that, whenever $f(\rho) = \rho \cdot v(\rho)$ is strictly concave, the solution of (1) with initial datum (7) is a centered rarefaction wave departing from $x = 0$. Assign w such that

$$w(\rho^\ell) < f'(\rho^\ell)\,, \qquad w(\rho^r) > f'(\rho^r)\,.$$

Then, for $\bar p < 0$, $p(t) = \bar p + w(\rho^\ell)t$, while for $\bar p > 0$, $p(t) = \bar p + w(\rho^r)t$ and we lose the continuous dependence (see figure 2).

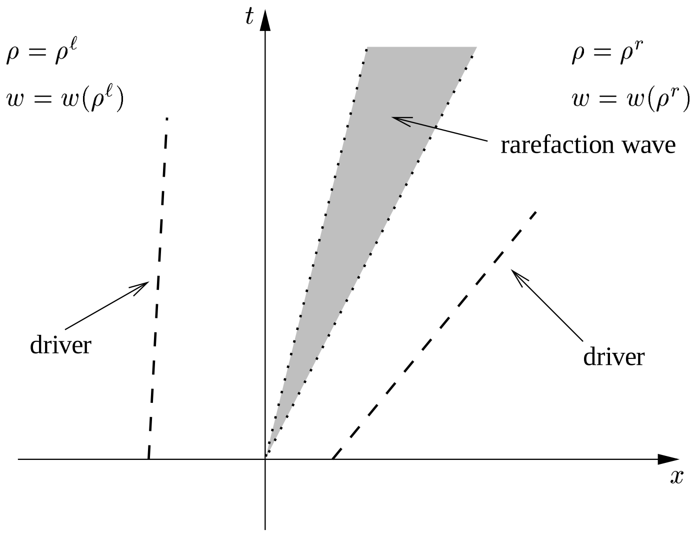

Figure 2: How continuous dependence can be lost

Remark that (5) is weaker than (A2) in [2], where the range of characteristic speeds of the conservation law and the range of the vector field of the o.d.e. are assumed disjoint.
Due to assumption **(H1)**, the condition (5) is fulfilled, for example, when

$$w(\rho) \geq v(\rho) \qquad \forall\, \rho \in]0, R] \qquad \textit{and} \qquad w(0) > v(0)\,. \tag{8}$$

This means that the driver travels faster than the main traffic. Condition (8) guarantees that Filippov solutions are also Caratheodory solutions since it implies that any discontinuity is crossed without the driver being trapped in it. In fact, by the Rankine-Hugoniot conditions, the speed Λ of a shock between the states ρ^ℓ and ρ^r is

$$\Lambda = v(\rho^r) + \frac{\rho^\ell}{\rho^r - \rho^\ell} \cdot \big(v(\rho^r) - v(\rho^\ell)\big) \leq v(\rho^r) \leq v(\rho^\ell)$$

and is lower than both $w(\rho^r)$ and $w(\rho^\ell)$.
The proof of Theorem 1 is obtained by taking a suitable piecewise constant approximation ρ^ε of the solution to (1), having discontinuities along a finite number of Lipschitz continuous polygonal lines. Then we look at the ODE

$$\dot p^\varepsilon(t) = w\big(\rho^\varepsilon(t, p(t))\big)\,, \tag{9}$$

and get an estimate of the Hölder continuous dependence of the Fillipov solutions to (9) on the initial data with constants independent on ε. Indeed, consider a reference trajectory $p^\varepsilon(\cdot)$ of(9) with initial datum $\bar p$. Following [2], we introduce the tangent vector

$$\xi(t) = \lim_{h\to 0} \frac{p_h^\varepsilon(t) - p^\varepsilon(t)}{h}\,,$$

where $p_h^\varepsilon(\cdot)$ is the solution to (9) with initial datum $\bar{p}+h$. Carrying out a careful analysis of how the tangent vector $\xi(\cdot)$ varies across the discontinuities of ρ^ε, we get that there exist $C>0$ and $\kappa\in]0,1[$ independent on ε such that

$$\frac{\xi(t)}{\xi(s)} \le C\left(\frac{t}{s}\right)^\kappa, \qquad 0<s<t, \tag{10}$$

holds. From (10) we obtain that, if $p_1^\varepsilon(\cdot)$ and $p_2^\varepsilon(\cdot)$ are solutions to (9) with initial data respectively $\bar{p}_1$ and $\bar{p}_2$, then there holds

$$\left|p_1^\varepsilon(t)-p_2^\varepsilon(t)\right| \le \begin{cases}(1+\sup w-\inf w)Ct^\kappa|\bar{p}_1-\bar{p}_2|^{1-\kappa} & \text{if} \quad |\bar{p}_1-\bar{p}_2|\le t,\\ (1+\sup w-\inf w)|\bar{p}_1-\bar{p}_2| & \text{otherwise,}\end{cases} \tag{11}$$

with all the constants independent on ε. Letting $\varepsilon\to 0$ and proving that all the solutions to (2) are obtained as limit of solutions to (9), we complete the proof of Theorem 1.
An entirely similar construction leads to an analogous result with the Cauchy problem for (1) substituted by the initial-boundary value problem.

Theorem 2 *Let v as in **(H1)**, $L>0$ and $T>0$. Fix an initial data $\bar\rho\in\mathbf{BV}([0,L])$ and a boundary data $\tilde\rho\in\mathbf{BV}([0,T])$ be given. Let $\rho\colon [0,+\infty[\times[0,L]\mapsto\mathbf{R}$ solve the initial - boundary value problem*

$$\begin{cases}\partial_t\rho+\partial_x(\rho\cdot v(\rho))=0\\ \rho(0,x)=\bar\rho(x)\\ \rho(\bar t,0)=\tilde\rho(t)\end{cases} \tag{12}$$

Assume the driver moves according to

$$\begin{cases}\dot p = w(\rho(t,p))\\ p(t_0))=0\end{cases} \tag{13}$$

with w as in (5). Then, (13) admits a unique solution p. If $p_1(\cdot)$ and $p_2(\cdot)$ solve (2), starting respectively at times $\bar t_1$, $\bar t_2$, then we have the Hölder dependence

$$\left|p_1(t)-p_2(t)\right| \le C\cdot\left|\bar t_1-\bar t_2\right|^\alpha, \tag{14}$$

for some $\alpha\in]0,1[$ and $C>0$.

Relying on Theorem 2 one can study some optimization problems related to (12)–(13). Consider, for instance, the following question: find at what time t_0 it is mostly convenient for the driver to leave in order to minimize the traveling time. Take the following two somewhat opposite situations:

$$\begin{cases}\bar\rho(x)=0\\ \tilde\rho(t)=\begin{cases}\rho_o & t\in[0,T_o]\\ 0 & \text{otherwise}\end{cases}\end{cases} \qquad \begin{cases}\bar\rho(x)=\rho_o\\ \tilde\rho(t)=\begin{cases}0 & t\in[T',T'']\\ \rho_o & \text{otherwise}\end{cases}\end{cases} \tag{15}$$

The former resembles a typical situation in which the single driver needs to travel along a road near to the rush hour. In the latter case, the driver has to find the best starting moment in the middle between two rush hours.
In the first case, the best strategy consists in leaving either before rush hour or after it. In the second case, there exists a best choice in the time interval $[T',T'']$..

References

[1] A. BRESSAN, *Unique solutions for a class of discontinuous differential equations*, Proc. Amer. Math. Soc., 104 (1988), pp. 772–778.

[2] A. BRESSAN AND W. SHEN, *Uniqueness for discontinuous ODE and conservation laws*, Nonlinear Anal., 34 (1998), pp. 637–652.

[3] A. F. FILIPPOV, *Differential equations with discontinuous righthand sides*, Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[4] M. LIGHTHILL AND G. WHITHAM, *On kinematic waves. II. A theory of traffic flow on long crowded roads*, Proc. Roy. Soc. London. Ser. A., 229 (1955), pp. 317–345.

[5] P. RICHARDS, *Shock waves on the highway*, Operations Res., 4 (1956), pp. 42–51.

Simulation-Based Risk Management of Data for Rapid Mass Flows

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Our research effort integrates mathematical modeling, high performance computing and database management, for hazard mitigation of rapid mass-flows at volcanoes. This program contains three main thrusts: (1) developing realistic simulations of geophysical mass flows; (2) integrating data from several sources, including simulation results, remote sensing data, and GIS data, and (3) extracting, and organizing information in a range of formats and fidelity, including audio, visual, and text, to scientists and decision-makers involved in risk management. In this talk, we focus on aspects of modeling and computing. Pioneering work of Savage and his colleagues formulated a model of debris flow and rock avalanches that is analogous to the shallow water system, the principal difference being the momentum source terms in the model. To date, numerical simulations of the governing equations have considered flow over a relatively simple topography. Here we present a parallel, adaptive grid simulation framework to solve model system. Our simulations incorporate topographical elevation data from specific volcanoes that we study. We also describe a data management scheme that facilitates processor communication and data retrieval.

Simulation of Cavitation in Thermodynamic Equilibrium

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The talk deals with the simulation of cavitation phenomena which plays an important role for the development of modern hydraulic tools and injection systems. The necessity to simulate cavitated flows arises from the demand to prevent damages and to understand the dynamic properties in the early development phase. Fast working valves, strongly time depending high and low pressure levels, and a large range of temperature can lead to evaporation of oil in hydraulic pipelines and valves. The evaporation leads to a significant reduction of the speed of sound in the mixture, reduces the mass flow, and therefore influences the working properties of the valves. Condensation of vapor bubbles can lead to noise and damages in the hydraulic components. In order to describe the complex time dependent physical processes and protect technical tools of cavitation erosion, numerical simulations prove to be very helpful.

Many papers deal with this topic, see for example [4,5,6,8]. Usually, a combination of continuous and discrete models is used. The continuous model is governed by the Euler equations or Navier-Stokes equations whereas the cavitation is modeled by an explicit description of the behaviour of the bubbles where oscillations of single bubbles or bubble clouds are considered, cf. [2,6,7,8]. These models include many unknown physical values and properties, such as the bubble diameter, the bubble number density, and the extreme complicated bubble interactions. Due to this reason, these models are difficult to handle for real industrial applications.

The consideration of a homogeneous mixture governed by conservation laws coupled with an additional source term for the phase transition was firstly introduced in [3] and [4]. In this case the evaporation and condensation process is described by the equation of state. This approach can lead to stability problems due to the large range of the mixture density and propagation speed. In [3] a combination of an explicit and locally implicit scheme is used successfully to avoid these stability problems. As a next step one will split the continuity equation for the gas and the liquid phase and couple it by a source term as done in [9] for isotropic water flow, i.e. the equation describing the conservation of energy was neglected.
Within the presentation this model will be extended by introducing an additional energy equation. Thereby, we have restricted our attention so far to a mixture of water and steam with a constant fraction of unsolved air in a one dimensional cavitating pipe flow. Consequently, the system of equations for homogeneous two phase flow with phase transition has the form

$$\begin{aligned} \tfrac{\partial}{\partial t}(\varepsilon\rho_{\mathrm{G}}) + \tfrac{\partial}{\partial z}(\varepsilon\rho_{\mathrm{G}} v) &= \Gamma, \\ \tfrac{\partial}{\partial t}\left((1-\varepsilon)\rho_{\mathrm{L}}\right) + \tfrac{\partial}{\partial z}\left((1-\varepsilon)\rho_{\mathrm{L}} v\right) &= -\Gamma, \\ \tfrac{\partial}{\partial t}(\rho v) + \tfrac{\partial}{\partial z}(\rho v^2 + p) &= 0, \\ \tfrac{\partial}{\partial t}\left(\rho(e + v^2/2)\right) + \tfrac{\partial}{\partial z}\left(\rho v(e + v^2/2 + p/\rho)\right) &= 0, \end{aligned}$$

where ε denotes the volume fraction of gas, ρ the mixture density, p the pressure, v the velocity and e the specific inner energy of the mixture whereas the subsrcripts $\cdot_{\mathrm{G}}$ and $\cdot_{\mathrm{L}}$ denote respectively the gaseous and the liquid phase of a quantity.
The numerical simulation is based on a finite difference approach which is in the one dimensional framework obviously equivalent to a finite volume scheme. For the homogeneous system we make use of a modified HLL-Scheme. The influence of the source term is afterwards included by means of a splitting technique. The resulting semi discrete form of the governing equations represents a system of ordinary differential equations. This system can be approximated by well-known time stepping schemes of explicit type. We introduce a very simple explicit forward Euler approach. It is worth noting that the simulation is realized using a formulation of the system in conservative variables.
As mentioned our objective is the simulation of cavitational effects in the fluid dynamics. As a consequence we need to look seperately on the two phases of the fluid implying the necessity of one equation of state for each phase of the fluid. For the gaseous phase we employ the ideal gas law with a modified gas constant while for the liquid phase we use a formulation according to W. Wagner, S. Span and T. Bonsen in [1]. This poses some difficulties on the conversion of primitive and conservative variables as the formulation for the liquid phase is only existent implicitly. While the conversion from primitive to conservative variables can be done straightforwardly according to the definitions, we need to derive an algorithm for the opposite direction. In particular we will show that in every computational step we have to solve a nonlinear system of two equations for each computational cell to get the corresponding values of temperature and pressure for the current time level. In this context an iteration of two one dimensional bisections prove to give reliable results while Newton-like methods tend not to converge due to the scale of the gradients. Nevertheless, the method is highly time consuming and should be improved or replaced in the future.
Concerning the cavitation we need a model for the process of vaporization and condensation respective a model for the transition between the two phases of the fluid. The vaporization is considered as an isenthalpic process giving us an approach to define the massfraction of gas on a thermodynamic basis. Thus we get

$$\mu(p) = \frac{h(p_{\mathrm{evap}}(T)) - h'(p)}{h''(p) - h'(p)} + \mu_0 \quad \text{for} \quad p_{\mathrm{tripel}} < p < p_{\mathrm{evap}}(T),$$

with μ denoting the massfraction of the gas, μ_0 the massfraction of unsolved air, h the enthalpy of the mixture, h' the enthalpy on the liquid and h'' the enthalpy of the gas. As we chose to use a thermodynamic model for the mass fraction we have the possibility to use this definition as well for the mentioned source term Γ by defining it to be the material derivative of the mass fraction, i.e.

$$\Gamma := \rho\frac{d\mu}{dt} = \rho\frac{d\mu}{dp}\frac{dp}{dt} \approx \frac{\rho}{\Delta h + \frac{2\sigma}{3r_B}}\left(\frac{dh'}{dp} - \frac{1}{\rho_{\mathrm{L}}}\right)\frac{dp}{dt}$$

with $\Delta h := h'' - h'$, r_B the averaged model radius of the steam bubbles and σ the surface tension of the liquid. In the presentation we will put forward some studies concerning the parameters of unsolved air μ_0, the average model radius of the bubbles r_B and the surface tension σ.
At the end of the presentation we apply the developed system to the simulation of some examples which are of great interest for the study of cavitating pipe flows. After some typical examples we will look at the cavitational zone behind an infinitely

fast closing valve. By varying the parameters of the model we will use the results to study their influence on the cavitational process.

[1] W. Wagner, S. Span and T. Bonsen, *Wasser und Wasserdampf*, Springer Verlag, 1999
[2] T. Baur, *Auswirkungen der Diffusion gelöster Gase auf das Wachstum von Kavitationsblasen*, Master thesis at the Institute for Hydraulic Engineering and Water Ressources Management, Aachen University of Technology, Aachen, 1997.
[3] M. Beck, U. Iben, N. Mittwollen, H.-K. Iben and C.-D. Munz, *On solution of conservation equations in cavitated hydraulic pipelines*, 3rd International Symposium on Computational Technologies for Fluid/Thermal/Chemical Systems with Industrial Applications, 2001 ASME Pressure Vessels and Piping Conference July 22-26, Hyatt Regency, Atlanta, Georgia, USA, 2001.
[4] T. Hapke and H.-K. Iben, *Modellierung und Berechnung der Kavitation in instationären Leitungsströmungen*, MTZ, 59, 1998.
[5] J. Huhn and J. Wolf, *Zweiphasenströmung*, VEB Zweiphasenströmung, 1975.
[6] H. Nguyen-Schaefer and H. Sprafke, *Numerical study on interaction effects of bubbles induced by air-release and cavitation in hydraulic systems*, Tenth Bath International Fluid Power Workshop, Research Studies Press, John Wiley and Sons, 1998.
[7] G. Regner, *Blasendynamisches Kavitationsmodell für eine eindimensionale Strömungssimulation*, PhD thesis, Technical University Graz, 1999.
[8] P. Sprafke, *Numerische und experimentelle Untersuchungen des dynamischen Verhaltens hydraulischer Systeme unter Berücksichtigung von Kavitation und Luftausgasung*, PhD thesis, Universität Karlsruhe, Fakultät Maschinenbau, 1999.
[9] F. Wrona, *Numerische Umsetzung von homogenen Kavitationsmodellen*, Master thesis at the Institute for Aero and Gasdynamics, University of Stuttgart, 2001.

4.37 Friday, Session 1 (morning): Wave equations

Abstract Methods in Applications

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We prove the well-posedness of higher order abstract Cauchy problems and give a representation of the solution in its closed form. When it is not possible we give an expansion of the solution to the series. Many well-known formulas of mathematical physics can be obtained as particular cases of more general formulas. We show this for an example of the d'Alembert formula. We also prove that the Fourier method can be applied to more general problems than those already known.
It is well-known that a solution of the Cauchy problem

$$u''_{tt} = a^2 u''_{xx}, \qquad (t,x) \in [0,T] \times R, \tag{1}$$

$$u(0,x) = u_0(x), \quad u'_t(0,x) = u_1(x), \;\; x \in R, \tag{2}$$

where $a > 0$, is given by the d'Alembert formula

$$u(t,x) = \frac{1}{2}\left(u_0(x-at) + u_0(x+at) + \frac{1}{a}\int_{x-at}^{x+at} u_1(y)dy\right). \tag{3}$$

Consider the Cauchy problem

$$u''_{tt}(t,x) + a_1 u''_{tx}(t,x) + a_2 u''_{xx}(t,x) = 0, \quad (t,x) \in [0,T] \times R, \tag{4}$$

$$u(0,x) = u_0(x), \quad u'_t(0,x) = u_1(x), \;\; x \in R. \tag{5}$$

Denote by $\omega_{1,2}$ the roots (real and different) of the equation $\omega^2 + a_1\omega + a_2 = 0$. We show that a solution of problem (4)–(5) has the form

$$\begin{aligned} u(t,x) &= \tfrac{\omega_2}{\omega_2-\omega_1} u_0(x+\omega_1 t) + \tfrac{\omega_1}{\omega_1-\omega_2} u_0(x+\omega_2 t) \\ &- \tfrac{1}{\omega_2-\omega_1}\int_0^{x+\omega_1 t} u_1(y)dy - \tfrac{1}{\omega_1-\omega_2}\int_0^{x+\omega_2 t} u_1(y)dy \\ &= \tfrac{\omega_2}{\omega_2-\omega_1} u_0(x+\omega_1 t) + \tfrac{\omega_1}{\omega_1-\omega_2} u_0(x+\omega_2 t) \\ &+ \tfrac{1}{\omega_2-\omega_1}\int_{x+\omega_1 t}^{x+\omega_2 t} u_1(y)dy. \end{aligned} \tag{6}$$

In the case of problem (1)–(2) we have $\omega_1 = -a$ and $\omega_2 = a$, and formula (6) is, actually, (3). We get (6) as an application of the corresponding abstract result.
Let us consider, in the cylindrical domain $[0,T] \times G$, where $G \subset R^r, r \geq 2$ is a bounded domain, an initial boundary value problem for hyperbolic differential equations of the second order

$$\begin{aligned} &D_t^2 u(t,x) + \sum_{j=1}^r a_j D_j D_t u(t,x) - \sum_{s,j=1}^r D_s(b_{sj}(x) D_j u(t,x)) \\ &+ b(x)u(t,x) = f(t,x), \quad (t,x) \in [0,T] \times G, \end{aligned} \tag{7}$$

$$u(t,x') = 0, \quad (t,x') \in [0,T] \times \partial G, \tag{8}$$

$$u(0,x) = v_0(x), \quad u'_t(0,x) = v_1(x), \; x \in G, \tag{9}$$

where $D_t := \frac{\partial}{\partial t}$, $D_j := \frac{\partial}{\partial x_j}$, $x := (x_1, \dots, x_r)$, ∂G is a boundary of G. The corresponding spectral problem is

$$\begin{aligned}\lambda^2 u(x) + \lambda \textstyle\sum_{j=1}^r a_j D_j u(x) - \sum_{s,j=1}^r D_s(b_{sj}(x) D_j u(x))\\ +b(x)u(x) = 0, \quad x \in G,\end{aligned} \tag{10}$$

$$u(x') = 0, \quad x' \in \partial G. \tag{11}$$

Let the following conditions be satisfied: $a_j \in R$, $b_{sj} \in C^1(\overline{G})$, $b \in L_\infty(G)$, $b(x) \geq 0$, $\partial G \in C^2$; $b_{sj}(x) = b_{js}(x)$, b_{sj} are real-valued functions, $\sum_{s,j=1}^r b_{sj}(x)\sigma_s\sigma_j \geq \delta \sum_{j=1}^r \sigma_j^2, \quad \exists \delta > 0,\ x \in \overline{G},\ \sigma \in R^r$; $f \in W_p^1((0,T); L_2(G))$, where $p > 1$; $v_0 \in W_2^2(G; u|_{\partial G} = 0)$, $v_1 \in W_2^1(G; u|_{\partial G} = 0)$.
Then problem (7)–(9) has a unique solution $u \in C^2([0,T]; W_2^2(G), W_2^1(G), L_2(G))$ and the solution can be expanded to the series

$$u(t,x) = \textstyle\sum_{k=1}^\infty C_k e^{i\mu_k t} u_k(x), \tag{12}$$

where $\lambda_k = i\mu_k$ are eigenvalues (μ_k are real) and u_k are the corresponding eigenvectors of problem (10)–(11), $C_k = \frac{(Bv_0 - \lambda_k v_1, u_k) - \lambda_k \int_0^t e^{-\lambda_k \tau}(f(\tau), u_k)d\tau}{(Bu_k, u_k) + |\lambda_k|^2 (u_k, u_k)}$ and the series is convergent in the sense of the space $C^2([0,T]; W_2^2(G), W_2^1(G), L_2(G))$.
Here $(\cdot,\cdot)$ is a scalar product in $L_2(G)$, $Bu := -\sum_{s,j=1}^r D_s(b_{sj}(x)D_j u(x)) + b(x)u(x)$.
This result we also get as an application of some abstract theorem. So, these abstract theorems (which will be discussed, too) allow us to consider both the d'Alembert formula and the Fourier method (in the case when all $a_j = 0$, formula (12) is known as the Fourier method) for hyperbolic equations with mixed derivatives. In the latter case, the classical method of separation of variables does not work.

Analysis of 1D Template Matching Equations

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The Averaged Template Matching Equation (ATME) was recently derived in the context of computer vision and pattern matching [1], [2]. For one dimensional images defined on the unit interval a solution of ATME matches two images while minimizing the energy $\int_0^1 \int_0^1 u^2 + \alpha^2 u_x^2 \, dx \, dt$ for a real $\alpha \geq 0$. On the other hand a solution of the Template Matching Equation (TME) minimizes $\int_0^1 \int_0^1 u^2 \, dx \, dt$. The ATME is $v_t + uv_x + 2vu_x = 0$ with $v = u - \alpha^2 u_{xx}$ and hence is the Camassa-Holm equation while the TME is a Burgers like equation $u_t + 3uu_x = 0$. The ATME reduces to the TME for $\alpha = 0$.
We show that a variational derivation of these equations leads to a natural boundary condition in time. This condition has an exact solution and it leads to an initial condition parametrized by an undetermined constant. Thus the original boundary value problem in time becomes simplified. This derived initial condition is a delta function for TME and a function like $\exp(-|x|)$ for TME and these are centered at internal edges of an image. Since these initial conditions are not smooth we examine various possibilities for regularizing the equations or their initial conditions in order to study their solutions and the relationship between ATME and TME solutions. We also derive the form of the solutions for some of these regularizations. The relationship between ATME and TME is still open but we show that if the ATME regularizes a shock solution of TME then the shock profile is non monotonic, but the Rankine-Hugoniot conditions are satisfied.

References

[1] Hirani, A. N., Marsden, J. E. and Arvo, J. [2001] Averaged template matching equations, Springer Lecture Notes in Computer Science 2134, 528-543.

[2] Marsden, J. E. and S. Shkoller [2001] Global well-posedness for the Lagrangian averaged Navier-Stokes (LANS-) equations on bounded domains. Phil. Trans. R. Soc. Lond. A, 1449-1468.

Global Solutions For The Anisotropic Carrier and Kirchhoff Equations

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Introduction

We consider in the cylinder Q the following mixed problems:

$$u_{tt} - M_c(x,t,\|u(t)\|^2)\Delta u + g(x,t,u_t) = f(x,t), \tag{1.1}$$

$$u\,|_{\Gamma\times(0,T)} = 0,$$

$$u(x,0) = u_0(x),\; u_t(x,0) = u_1(x),\; x \in D$$

and

$$u_{tt} - \phi(x,t)M_k(\|\nabla u\|^2)\Delta u + g(x,t,u_t) = f(x,t), \tag{1.2}$$

$$u\,|_{\Gamma\times(0,T)} = 0,$$

$$u(x,0) = u_0(x),\; u_t(x,0) = u_1(x),\; x \in D,$$

where $x \in D \subset R^n, D$ is a bounded domain with a smooth boundary Γ; $Q = D \times (0,T)$.
Here $\phi(x,t), M_c(x,t,\lambda), M_k(\lambda)$ are positive smooth functions that ensure hyperbolicity of (1.1) and (1.2). When $M_k(\|\nabla u(t)\|^2)$ did not depend on x,t and $\phi(x,t) = 1,\ g(x,t,u_t) \equiv 0,$ (1.2)was derived by Kirchhoff [7] and had his name. When $M_c(x,t,\|u(t)\|^2)$ did not depend on x,t and $g(x,t,u_t) \equiv 0$, (1.1) was derived by Carrier [3] and had his name. Some authors call the Kirchhoff equation by Kirchhoff-Carrier equation that is not completely correct.
Both equations model small vibrations of elastic strings and were derived under the assumption that all the physical characteristics of the material of a string were constant in space and time (isotropic). In reality, materials are not completely uniform, their properties may vary from point to point and in time that justifies practicality of considering anisotropic equations (1.1),(1.2). Moreover, we admit internal nonlinear friction simulated by $g(x,t,u_t)$.
There is a series of papers ,[1,2,4,5,6,8,9,11], where global solvability of problems (1.1) and (1.2) was studied but only in isotropic case, that is the functions $M_c(\|u(t)\|^2),\ M_k(\|\nabla u(t)\|^2)$ did not depend on x,t and $\phi(x,t) \equiv 1$.
Here we prove the existence of strong solutions and the exponential decay of the energy without restrictions on a size of intial data for the problem (1.1) and under assumption of small initial data for the problem (1.2). We use the Galerkin method to prove the existence results and ideas of Nakao [10] for the stability results.

Carrier Equation

Existence of strong solutions.
Assumptions 2

2.1) $M \in C^1(Q \times R^+) \cap C(\bar{Q} \times R^+), R^+ = [0, \infty)$.
2.2) There exists a continuous function $\phi(\lambda)$ such that for all $(x,t) \in \bar{Q}$
$0 < m_0 \le \phi(\lambda) \le M(x,t,\|u(t)\|^2) \le C_1\phi(\lambda)$.
2.3) $\mid M_\lambda \lambda^{1/2} \mid \le k_0 M$.
2.4) $\mid M_t(x,t,\lambda) \mid \le C_2 M(x,t,\lambda)$.
2.5) $g(x,t,u_t) \in C^1(Q \times R^1)$, $\mid g \mid + \mid g_t \mid + \Sigma_{i=1}^n \mid g_{x_i} \mid \le C_3(1+ \mid u_t \mid^{\rho+1})$.
2.6) $g_\lambda(x,t,\lambda) \ge C_4 \mid \lambda \mid^\rho,\ g(x,t,0) = 0$.

Here $k_0, C_1, ..., C_4, m_0$ are positive constants, $\rho > 1$.

Theorem 2.1 *Let assumptions 2 hold;* $u_0 \in H^2(D) \cap H_0^1(D)$, $u_1 \in H_0^1(D) \cap L^{2\rho+2}(D)$. *Then for any* $f \in H^1(0,T;L^2(D))$ *there exists a unique strong solution to (2.1)-(2.3) from the class:*

$$u \in L^\infty(0,T;H^2(D) \cap H_0^1(D)),$$
$$u_t \in L^\infty(0,T;H_0^1(D)) \cap L^{\rho+2}(Q),$$
$$u_{tt} \in L^\infty(0,T:L^2(D)).$$

Energy decay.
Assumptions 3.

3.1) $M(x,t,\lambda) \in C^1(D \times R^+ \times R^+)$,
3.2) $g(x,t,u_t)u_t \ge \alpha u_t^2 + \beta \mid u_t \mid^{\rho+2}$,
3.3) $\mid g(x,t,u_t) \mid \le C_2(\mid u_t \mid + \mid u_t \mid^{\rho+1})$,
3.4) There exists a continuous function $\phi(\lambda)$ such that
$0 < m_0 \le \phi(\lambda) \le M(x,t,\lambda) \le C_1\phi(\lambda)$,
3.5) $\mid M_\lambda \mid \lambda^{1/2} \le k_0 M$, where $k_0 \le (\frac{\alpha\rho}{\rho-1})^{\rho-1} \frac{\beta}{C_1 4^{(2\rho-1)/\rho} (\mu_D)^{(2\rho-1)/2}}$,
3.6) $M_t(x,t,\lambda) \ge 0$,
3.7) $\int_t^{t+1} \|f(\tau)\|^2_{L^2(D)} d\tau \le C_3 e^{-\theta t}$,

where $\rho > 1; \theta, \alpha, \beta, C_1, C_2, C_3$ are positive constants; μ_D is the Lebesgue measure of D.

Theorem 2.2 *Let assumptions 3 hold;* $1 < \rho < \frac{4}{n-2}$ *if* $n > 2$, *and* $\rho > 1$ *when* $n = 1, 2$. *Then there exist positive constants* K *and* θ_1 *such that strong solutions to (1.1) satisfy the following inequality,*

$$\|u_t(t)\|^2 + \|u(t)\|^2_{H_0^1(D)} \le K e^{-\theta_1 t}.$$

Kirchhoff Equation

Existence of strong solutions
In this section we shall prove global existence and uniqueness of a strong solution to the problem (1.2)
Assumptions 4:

4.1 $\begin{cases} M \in \mathcal{C}^1([0,\infty)) \text{ and there exists a constant } M_0 \text{ such that:} \\ 0 < M_0 \le M(\lambda), \quad \text{for all } \lambda \ge 0; \end{cases}$

4.2 $\begin{cases} \varphi \in \mathcal{C}^1(\overline{\Omega} \times [0,\infty)) \text{ and there exist constants } \delta,\ \varphi_i, i = 0,1,2 \text{ such that:} \\ 0 < \varphi_0 \le \varphi(x,t) \le \varphi_1,\ \ 0 < \delta \le \alpha - \dfrac{|\varphi_t(x,t)|}{\varphi(x,t)} \text{ and} \\ \dfrac{\|\nabla\varphi(x,t)\|_{\mathbf{R}^n}}{\varphi(x,t)} \le \varphi_2, \ \text{ for all } (x,t) \in \overline{\Omega} \times [0,\infty). \end{cases}$

Theorem 3.1 *Let assumptions 3.2, 3.3, 4.1, 4.2 hold. If* $f \in H^1((0,\infty)\times D$, $u_0 \in H_0^1(D)\cap H^2(D)$, $u_1 \in H_0^1(D)\cap L^{2(\rho+1)}(D)$ *satisfy*

$$\|u_0\|_{H_0^1(D)\cap H^2(D)} + \|u_1\|_{H_0^1(D)\cap L^{2(\rho+1)}(D)} + \|f\|_{H^1((0,\infty)\times D)} < A_0,$$

then for A_0 *sufficiently small, there exists a unique solution to* (1.2) *in the class:*

$$u \in L^\infty([0,\infty); H^2(D)\cap H_0^1(D)),\ u' \in L^\infty([0,\infty); H_0^1(D)) \cap L^2([0,\infty); H_0^1(D)) \cap L^{\rho+2}(Q)),$$
$$u'' \in L^\infty([0,\infty); L^2(D)) \cap L^2(Q).$$

Energy Decay

Theorem 3.2 *Let assuptions 4 hold and*

$$\int_t^{t+1} \|f(\tau)\|^2_{L^2(D)} \le C_f e^{\theta t},$$

where C_f, $\theta > 0$. *Then there exist positive constants* C *and* K *such that*

$$\|u'(t)\|^2 + \|u(t)\|^2_{H_0^1(D)} \le Ce^{-Kt},\ t > 0,$$

for $n = 1, \cdots 4$ *and any* $\rho > 0$; *and if* $n \ge 5$, *then* $0 < \rho < \frac{8}{n-4}$.

References

[1nl] S. Bernstein, Sur une classe d'equations fonctionelles aux derivées partielles, *Isv. Acad. Nauk. SSSR Ser. Math.*, 4 (1940), pp. 17-26.

[2nl] E. H. Brito, The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Applicable Anal.*, 13 (1982), pp. 219-233.

[3nl] G. F. Carrier, On the non-linear vibration problem of the elastic string, *Quart. Appl. Math.*, 3 (1945), pp. 157-165.

[4nl] A. T. Cousin, C. L. Frota, N. A. Lar′kin and L. A. Medeiros, On the abstract model of the Kirchhoff-Carrier equation, *Communications in Applied Analysis*, 1, (1997), pp. 389-404.

[5nl] C. L. Frota, A. T. Cousin and N. A. Lar′kin, Existence of global solutions and energy decay for the Carrier equation with dissipative term, *Differential and Integral Equations*, Vol. 12, N. 4, (1999), pp. 453-469.

[6nl] R. Ikehata, A note on the global solvability of solutions to some nonlinear wave equations with dissipative terms, *Differential and Integral Equations*, 8 (1995), pp. 607-616.

[7nl] G. Kirchhoff, Vorlesungen über Mechanik, *Tauber Leipzig* (1883).

[8nl] S. Kouémou-Patcheu, Global existence and exponential decay estimates for a damped quasilinear equation, *Communications in Partial Differential Equations*, 22(11 & 12), (1997), pp. 2007-2024.

[9nl] L. A. Medeiros and M. Milla Miranda, On a nonlinear wave equation with damping, *Revista Matematica de la Universidade Complutense de Madri*, 3. (1990) pp. 213-231.

[10nl] M. Nakao, Convergence of solutions of the wave equation with a nonlinear dissipativa term to the steady state, *Mem. Facultty of Sci.*, Kyushu Univ., Ser. A, Vol. 30, N2,(1978), pp. 257-265.

[11nl] S. Pohozaev, On a class of quasilinear hyperbolic equations, *Math. Sbornik*, 96 (1974), pp. 152-166.

4.38 Friday, Session 2 (morning): Relaxation II

Large-Time Behavior of Solutions to the Three-Dimensional Euler Equations with Damping

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The Cauchy problem for the three-dimensional compressible Euler equations of isentropic gases with damping will be discussed. The main concern is the damping effect on the regularity and large-time behavior of the solutions. It will be proved that the size of the initial data in certain norms plays the key role. If the initial data is small in an appropriate norm, the damping can prevent the development of singularities in the smooth solution and the Cauchy problem has a unique global smooth solution. The global existence and decay of solutions are established. It is noticed that although the velocity decays exponentially, the density does not have this exponential decay. For large data, the solution will still develop singularities in a finite time. The related one-dimensional results will be reviewed.

Global existence and large time behavior of multidimensional Euler-Poisson equations

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The talk concerns the multidimensional Euler-Poisson equations with damping:

$$\begin{cases} n_t + \nabla \cdot (n\mathbf{u}) = 0, \\ (n\mathbf{u})_t + \nabla \cdot (n\mathbf{u} \bigotimes \mathbf{u}) + \nabla p(n) = n\nabla\Phi - \frac{n\mathbf{u}}{\tau}, \\ \Delta\Phi = n - b(x), \end{cases} \tag{1}$$

for $(x,t) \in R^N \times [0,+\infty), N = 2,3$. This system can be used to describe the hydrodynamic model for semiconductors in the isentropic case. The global existence of smooth solutions to the Cauchy problem for (1) have been established, provided the initial data are small perturbations of a given stationary solution. Moreover, the solution decays to the stationary solution exponentially fast as t tends to infinity.

One-dimensional stability of viscous shock and relaxation profiles

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Under the weak assumption of spectral stability, or stable point spectrum of the linearized operator about the wave, we establish sharp pointwise Green's function bounds and consequent linear and nonlinear stability for shock profiles of relaxation and real viscosity systems satisfying the dissipativity condition of Zeng/Kawashima. These include in particular compressible

Navier–Stokes and MHD equations, and essentially all standard relaxation models: in particular, the discrete kinetic models of Broadwell, Jin-Xin, Natalini, Bouchut, Platkowski–Illner, and the moment closure models of Grad, Levermore, Müller–Ruggeri. A consequence is stability of small-amplitude profiles of Broadwell and Jin-Xin models and of general real viscosity systems, for each of which spectral stability has been verified in other works. These are the first complete stability results for profiles of a real viscosity system, and the first for relaxation models with nonscalar equilibrium equations [3]. Our results apply also in principle to large-amplitude shocks, an important direction for future investigation.
As pointed out by Zeng [Ze.2], the two problems (relaxation/real viscosity) are essentially dual in the linear constant–coefficient case. A similar duality, at the structural level, may be seen in the variable-coefficient case [Z.1]. Consider, then, a general *hyperbolic relaxation model* of form

$$u_t + f(u,v)_x = 0,$$

$$v_t + g(u,v)_x = \tau^{-1} q(u,v), \tag{1}$$

u, $f \in R^n$, v, g, $q \in R^r$, where

$$Re\, \sigma\big(q_v(u, v^*(u))\big) < 0 \tag{2}$$

along a smooth equilibrium manifold defined by

$$q(u, v^*(u)) \equiv 0, \tag{3}$$

and τ is a (usually small) parameter determining relaxation time. Such models occur, for example, non-thermal equilibrium gas dynamics, traffic dynamics, and multiphase flow. System (1) can be viewed "to zeroth order" as a regularization of the associated *equilibrium*, or "relaxed" system of conservation laws

$$u_t + f^*(u)_x = 0, \tag{3}$$

$f^*(u) := f(u, v^*(u))$, which system may or may not be hyperbolic. To "first order," it may be approximated, at least formally, by an associated parabolic system of conservation laws

$$u_t + f^*(u)_x = \tau(B^*(u)u_x)_x, \tag{4}$$

where

$$B^*(u) = -f_v q_v^{-1}(g_u - g_v q_v^{-1} q_u + q_v^{-1} q_u (f_u - f_v q_v^{-1} q_u), \tag{5}$$

$g^* := g(u, v^*(u))$, is determined by Chapman–Enskog expansion as described in, e.g., [Wh,L.2]. Our interest is in behavior of solutions for a *fixed* relaxation time $\tau = 1$, as time t goes to infinity, a roughly analogous situation as pointed out by Liu [L.2]. A particularly interesting phenomenon, suggested by (3)–(4), is the existence of smooth *traveling front solutions*

$$(u,v)(x,t) = (\bar{u}, \bar{v})(x - st), \qquad \lim_{z\to\pm\infty} (\bar{u}, \bar{v}) = (u_\pm, v_\pm), \tag{6}$$

of (1), where by necessity $v_\pm = v^*(u_\pm)$ and $u_\pm$ corresponds to a shock solution of (3). See [L.2] for a treatment of existence in the general case $n = r = 1$, and [YoZ,FZe] for generalizations in case n or $r > 1$. Such traveling waves are known as *relaxation shocks* or *relaxation profiles*. The question of their stability was investigated in [L.2] in the general case $n = r = 1$, for which the equilibrium system is scalar, under the assumption of "weak," or small-amplitude shocks. This analysis has been generalized to strong shocks in the special case of the 2×2 Jin–Xin model [MaN]. For the 3×3 Broadwell model, a standard discrete kinetic model for which $n = 2$, $r = 1$, Szepessy and Xin [S,SX.2] have announced the result of linear and nonlinear stability of weak shocks, extending partial results of [KM,CaL]: to our knowledge, the only such result for the system case $n > 1$ occurring in most physical applications. However, up to now there was no complete analysis of stability for any other case, in particular for the case $n > 1$, $r >> 1$ arising through approximation of Boltzmann or Vlasov–Poisson equations by moment closure or discretization. For strong shocks, there was no complete analysis for *any* system such that n or $r > 1$.
Our main result here is as follows. A similar result is established for viscous profiles of systems with real viscosity.

[3] However, we note a previous HYP conference proceedings (Beijing) in which Szepessy and Xin sketch a quite different argument for stability of small-amplitude Broadwell shocks.

Theorem 1. *Let $(\bar{u}(x), \bar{v}(x))$ be a Lax type relaxation profile of a general relaxation model (1), satisfying mild additional hypotheses (H0)–(H3) to be described later, for which the linearized operator about the wave has stable point spectrum, and for which the corresponding ideal shock of (3) is hyperbolically stable. If, also, $(\bar{u}, \bar{v})$ is sufficiently weak, in the sense that $|(\bar{u}', \bar{v}')|_{L^\infty}$ is sufficiently small relative to the parameters of (1), and (1) is simultaneously symmetrizable, then, provided that initial perturbation $(u_0, v_0) - (\bar{u}, \bar{v})$ is bounded by ζ_0 in L_1 and H^2, for ζ_0 sufficiently small, the solution $(u, v)(x, t)$ of (1) with initial data (u_0, v_0) satisfies*

(7) $$|(u,v)(x,t) - (\bar{u}, \bar{v})(x - \delta(t))|_{L^p} \le C\zeta_0 (1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $2 \le p \le \infty$, for some $\delta(t)$ satisfying

(8) $$|\dot{\delta}(t)| \le C\zeta_0 (1+t)^{-\frac{1}{2}}$$

and

(9) $$|\delta(t)| \le C\zeta_0.$$

In particular, $(\bar{u}, \bar{v})$ is nonlinearly orbitally stable from $L^1 \cap H^2$ to L^p, for all $p \ge 2$. For weak shocks of simultaneously symmetrizable **discrete kinetic models***– i.e. models (1) for which f and g are linear– and data merely small in $L^1 \cap H^1$, we obtain (7) for $p = 2$, but only boundedness $|(u,v)(x,t) - (\bar{u}, \bar{v})(x - \delta(t))|_{L^\infty} \le C\zeta_0$, for $p = \infty$, and interpolated rates for p between 2 and ∞.*
For discrete kinetic models, not necessarily simultaneously symmetrizable, and data small in $W^{3,1} \cap W^{3,\infty}$, we obtain the rates (7) for all $1 \le p \le \infty$, for shocks of arbitrary strength.

This result is obtained using a modified version of the spectral arguments introduced in [ZH,Z.2] to treat the strictly parabolic case. Namely, we first obtain detailed pointwise bounds on the Green's function of the linearized perturbation equation by a detailed asymptotic analysis of the resolvent equation combined with stationary phase type estimates in the spectral resolution (Laplace inversion) formula describing the Green's function. However, substantial new complications arise from the need to estimate (at the linearized level), and control (at the nonlinear level) the contribution of the singular component of the Green's function, corresponding to hyperbolic nature of the high frequency ($\sim$ short time) behavior of equation (1).
Theorem 1 in particular shows that *strong spectral stability implies nonlinear stability*, for weak shocks of general simultaneously symmetrizable models, or for strong shocks of discrete kinetic models. As noted above, strong spectral stability follows for weak relaxation profiles of the Jin–Xin and Broadwell models from the results of [KM] and [Liu,Hu], respectively, thus yielding complete nonlinear stability results for these two cases. A very interesting open problem is spectral stability of weak shocks of general dissipative relaxation systems, as identified by Zeng/Kawashima [Kaw,Ze.2], or, likewise, spectral stability of strong shocks of special discrete kinetic models.

REFERENCES

[FreZe] H. Freistühler and Y. Zeng, *Shock profiles for systems of balance laws with relaxation,* preprint (1998).

[Hu] J. Humpherys, *Stability of Jin–Xin relaxation shocks,* preprint (2000).

[JX] S. Jin and Z. Xin, *The relaxation schemes for systems of conservation laws in arbitrary space dimensions,* Comm. Pure Appl. Math. 48 (1995), no. 3, 235–276.

[Kaw] S. Kawashima, *Systems of a hyperbolic–parabolic composite type, with applications to the equations of magnetohydrodynamics,* thesis, Kyoto University (1983).

[KM] S. Kawashima and A. Matsumura, *Asymptotic stability of traveling wave solutions of systems for one-dimensional gas motion,* Comm. Math. Phys. 101 (1985), no. 1, 97–127.

[Liu] H. Liu, *Asymptotic stability of relaxation shock profiles for hyperbolic conservation laws,* preprint (2000).

[L.1] T.-P. Liu, *Pointwise convergence to shock waves for viscous conservation laws,* Comm. Pure Appl. Math. 50 (1997), no. 11, 1113–1182.

[L.2] T.-P. Liu, *Hyperbolic conservation laws with relaxation,* Comm. Math. Phys. 108 (1987), no. 1, 153–175.

[LZe] T.-P. Liu and Y. Zeng, *Large time behavior of solutions for general quasilinear hyperbolic–parabolic systems of conservation laws*, AMS memoirs 599 (1997).

[LW] M.J. Lighthill and G.B. Whitham, *On kinematic waves. II. A theory of traffic flow on long crowded roads,* Proc. Roy. Soc. London. Ser. A. 229 (1955), 317–345.

[MaN] C. Mascia and R. Natalini, L^1 *nonlinear stability of traveling waves for a hyperbolic system with relaxation,* J. Differential Equations 132 (1996), no. 2, 275–292.

[MN] A. Matsumura and K. Nishihara, *On the stability of travelling wave solutions of a one-dimensional model system for compressible viscous gas*, Japan J. Appl. Math. 2 (1985), no. 1, 17–25.

[MR] I. Müller and T. Ruggeri, *Rational extended thermodynamics,* Second edition, with supplementary chapters by H. Struchtrup and Wolf Weiss, Springer Tracts in Natural Philosophy, 37, Springer-Verlag, New York (1998) xvi+396 pp. ISBN: 0-387-98373-2.

[PI] T. Platkowski and R. Illner, *Discrete velocity models of the Boltzmann equation: a survey on the mathematical aspects of the theory,* SIAM Rev. 30 (1988), no. 2, 213–255.

[S] A. Szepessy, *On the stability of Broadwell shocks,* Nonlinear evolutionary partial differential equations (Beijing, 1993), 403–412, AMS/IP Stud. Adv. Math., 3, Amer. Math. Soc., Providence, RI, 1997.

[SX.1] A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves,* Arch. Rat. Mech. Anal. 122 (1993) 53–103.

[SX.2] A. Szepessy and Z. Xin, *Stability of Broadwell shocks,* unpublished manuscript.

[Wh] G.B. Whitham, *Linear and nonlinear waves,* Reprint of the 1974 original, Pure and Applied Mathematics, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1999. xviii+636 pp. ISBN: 0-471-35942-4.

[YoZ] W.-A. Yong and K. Zumbrun, *Existence of relaxation shock profiles for hyperbolic conservation laws*, SIAM J. Appl. Math. 60 (2000), no. 5, 1565–1575 (electronic).

[Ze.1] Y. Zeng, L^1 *asymptotic behavior of compressible, isentropic, viscous* 1*-d flow*, Comm. Pure Appl. Math. 47 (1994) 1053–1092.

[Ze.2] Y. Zeng, *Gas dynamics in thermal nonequilibrium and general hyperbolic systems with relaxation,* Arch. Ration. Mech. Anal. 150 (1999), no. 3, 225–279.

[ZH] K. Zumbrun and P. Howard, *Pointwise semigroup methods and stability of viscous shock waves,* Indiana Mathematics Journal V47 (1998) no. 4, 741–871.

[Z.1] K. Zumbrun, *Multidimensional stability of planar viscous shock waves*, TMR Summer School Lectures: Kochel am See, May, 1999, to appear, Birkhauser's Series: Progress in Nonlinear Differential Equations and their Applications (2001), 207 pp.

[Z.2] K. Zumbrun, *Refined Wave–tracking and Nonlinear Stability of Viscous Lax Shocks*, to appear, Math. Anal. Appl. (2001); preprint (1999).

4.39 Friday, Session 3 (morning): Numerical methods III

Comparison of several difference schemes for the Euler equations in 1D and 2D

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Hyperbolic conservation laws, and the Euler equations of compressible fluid dynamics in particular, have been the subject of intensive research for at least the past five decades, and with good reason. The applications are many - aircraft design, stellar formation, weather prediction to name only a few. There are some theoretical results, however even if the theory were perfect the applications would not be possible without methods for obtaining approximate solutions. The unfortunate situation here is that rigorous error estimates for supposed approximate solutions are almost entirely nonexistent So, it is universally recognized that tests of methods on difficult problems are essential. Our concern here is to compare the behavior of some Eulerian methods to each other on problems that seem to us to be sufficiently difficult and representative to enable the reader to draw some conclusions about the applicability of these methods. The now classic work of this nature is the paper by Gary Sod [1].
We have chosen ten methods that we feel are representative of the different basic finite difference approaches to solving hyperbolic conservation laws:

A composite scheme called LWLFn consists of a cycle of $n-1$ time steps of some version of the Lax-Wendroff (LW) scheme followed by one step with Lax-Friedrichs (LF). The LF step acts as a consistent (with the differential equations) filter to reduce the oscillations of LW. Here we use the 2D version called CFLF [2].

A CFLF hybrid scheme is also as the previous one a combination of second order LW and first order LF type schemes. The numerical flux is given by a weighted average of LF diffusive and LW oscillatory fluxes. The weight is chosen in such a way that the scheme is second order on smooth solutions, but becomes sufficiently dissipative in shocks. We use the Harten weight [3] with the CF and LF numerical fluxes [2].

Centered scheme with limiter uses discontinuous limited piecewise linear reconstruction fron cell averages to get fluxes at cell edges. It uses neither dimensional splitting nor eigenvector decomposition nor any overt Riemann solver. We use the 2D version of this nonoscillatory central scheme by Jiang and Tadmor [4].

Positive scheme by Liu and Lax [5] is an explicit two-level method giving the new value of the solution vector as a linear combination of values at the previous time with coefficients that are positive symmetric matrices summing to the identity. It uses eigenvector decomposition and limiting.

Clawpack wave propagation scheme is a sophisticated flux splitting scheme developed by LeVeque [6], based on advection and wave propagation ideas. From the many options permitted by Clawpack [7] we use the nonsplit version with monotonized centered limiter using the Roe Riemann solver with 4 waves (separate shear and entropy waves).

Weighted average flux (WAF) scheme developed by Toro [8] defines the flux at the cell boundary as a spatial weighted average over the states of an approximate Riemann solver. A limiter is employed in the computation of the weights. We use the 2D nonsplit version from the Numerica library [9] with HLLC approximate Riemann solver using the Rankine-Hugoniot condition for evaluating the middle fluxes.

WAF scheme with HLLE speeds is a WAF split scheme using the HLLC approximate Riemann solver with Einfeldt [10] speeds and fluxes evaluated at the middle states.

Weighted essentially nonoscillatory (WENO) scheme is a weighted combination of essentially nonoscillatory schemes on several stencils. Weights used with upwind biased spatial differencing produce high order accuracy for smooth flows but become low order and dissipative for shocks. We use the fifth order version from [11] applying the WENO procedure after eigenvector decomposition with Runge-Kutta time integration.

Component wise WENO scheme is the third order variant of WENO applied directly to the conserved variables without eigenvector decomposition.

Piecewise parabolic (PPM) scheme is a third order Godunov method using piecewise parabolic limited reconstruction to obtain states to use in the Riemann problems defining the fluxes [12]. Dimensional splitting is used.

These finite difference methods are compared on a set of 1D and 2D problems. Most of the 1D problems, except the last one, are Riemann problems (five of them, tests *1,2,4,5,6*, are from [8]) for which we have exact solutions and we can compare errors of the different numerical schemes. 1D problems include:

Test 1 is Toro's [8] version of Sod's Riemann problem [1] with a sonic point in the rarefaction.

Test 1-tvj is the same problem as Test 1 but with a large jump in transverse velocity (all other 1D tests have zero transverse velocity everywhere).

Test 2 develops a near vacuum in the central state with both density and pressure very close to zero, but the internal energy is far from zero.

1D-Noh is the classical 1D Noh problem [13]. The solution of this problem consists of two infinite strength shocks moving out from the center, leaving a constant density and pressure state behind.

Test 3a is a variant of Toro's Test 3 [8] with a stationary contact.

Test 4 has two strong shocks with a strong contact wave.

Test 5 has only one wave, a stationary contact.

Test 6 has also only one wave, a slowly moving contact.

Peak produces a very narrow strong peak in density between the shock and contact and rarefaction with small change in density and large change in velocity.

Blast is the classical Woodward-Collela blast wave problem [12] including interaction of two Riemann problems with reflecting boundary conditions.

The accuracy of the numerical schemes is numerically determined for the smooth exact 2D solution ($\rho(x,y,t) = 1 + 0.2\sin(\pi(x+y-t(u+v))),\ u,v,p$ constants) with periodic boundary conditions.
The first set of 2D test problems consists of six cases from the collection of 2D Riemann problems proposed by [14] and used by others [5], namely, cases *3,4,6,12,15* and *17* from [5] (which are configurations *3, 4, B, F, G, K* from [14]). These 2D Riemann problems are solved on a square divided into four quadrants. The initial conditions, constant states in each quadrant, were chosen in such a way that each of four 1D interfaces between the quadrants defines 1D Riemann problem whose solution includes only one wave, a shock, contact or rarefaction.
The second set of 2D problems consists of four tests:

2D Noh is a classical test [13] with exact solution being an infinite strength circularly symmetric shock reflecting from the origin. Behind the shock (i.e. inside the circle) the density is 16, the velocity is 0 and the pressure is $16/3$. The shock speed is 1/3 and ahead of the shock, that is for $\sqrt{x^2+y^2} > t/3$, the density is $(1+t/\sqrt{x^2+y^2})$ while velocity and pressure remain the same as initially, i.e. velocity is directed towards the origin and the pressure is zero.

Rayleigh-Taylor instability is a physical phenomenon appearing when a layer of heavier fluid is placed on top of a layer of lighter fluid. This is the only problem having a source in the Euler equations. A gravitational source term appears in the momentum equation. The typical mushroom structure results.

Implosion is defined on a square box with reflecting boundary conditions. Initially the gas inside a smaller square, placed in the center of the box and rotated by $\pi/4$, has smaller density and pressure than the rest of the box. At later stages of this problem a rich sructure of waves reflected from the boundaries develops.

Explosion problem [8] starts with higher density and pressure circular area inside a lower density and pressure gas. An unstable contact develops at later times.

For all 2D problems, except the 2D Noh problem, we no longer have exact solutions available, so a definitive objective evaluation of the validity of the solutions obtained is not possible.

We have applied a representative set of ten finite difference schemes approximating Euler equations to a suite of problems in one and two dimensions. It is clear that some methods appear to work better than others on a specific problem, but no one scheme has shown itself to be superior on all of them, which should come as no surprise. For more details see [15].

References

[1] G. A. Sod. A survey of several finite difference schemes for hyperbolic conservation laws. *J. Comp. Phys.*, 27:1–31, 1978.

[2] R. Liska and B. Wendroff. Composite schemes for conservation laws. *SIAM J. Numer. Anal.*, 35(6):2250–2271, 1998.

[3] A. Harten. The artificial compression method for computation of shocks and contact discontinuities: Iii self adjusting hybrid schemes. *Mathematics of Computation*, 32:363–389, 1978.

[4] G.S. Jiang and E. Tadmor. Non-oscillatory central schemes for multidimensional hyperbolic conservation laws. *SIAM Journal on Scientific Computing*, 19:1892–1917, 1998.

[5] P. D. Lax and X.-D. Liu. Solution of two dimensional Riemann problem of gas dynamics by positive schemes. *SIAM J. on Scientific Comp.*, 19(2):319–340, 1998.

[6] R.J. LeVeque. Wave propagation algorithms for multi-dimensional hyperbolic systems. *J. Comp. Phys.*, 131:327–353, 1997.

[7] R.J. LeVeque. Clawpack version 4.0 user's guide. Technical report, University of Washington, Seatle, 1999. `http://www.amath.washington.edu/~claw/`.

[8] E.F. Toro. *Riemann Solvers and Numerical Methods for Fluid Dynamics*. Springer-Verlag, Berlin, Heidelberg, 1997.

[9] E.F. Toro. Numerica a library of source codes for teaching, research and applcations:, 2000. `http://www.numeritek.com/numerica_software.html`.

[10] B. Einfeldt. On godunov-type methods for gas dynamics. *SIAM J. Numer Anal.*, 25:294–318, 1988.

[11] Guan-Shan Jiang and Chi-Wang Shu. Efficient implementation of weighted eno schemes. *J. Comp. Phys.*, 126:202–228, 1996.

[12] P. Woodward and P. Colella. The numerical simulation of two-dimensional fluid flow with strong shocks. *J. Comp. Phys.*, 54:115–173, 1984.

[13] W. F. Noh. Errors for calculations of strong shocks using an artificial viscosity and artificial heat flux. *J. Comp. Phys.*, 72:78–120, 1987.

[14] C. W. Schulz-Rinne, J. P. Collins, and H. M. Glaz. Numerical solution of the Riemann problem for two-dimensional gas dynamics. *SIAM J. Sci. Comput.*, 14:1394–1414, 1993.

[15] R. Liska and B. Wendroff. Comparision of several difference schemes on 1d and 2d test problems for the euler equations. Technical Report LA-UR-01-6225, Los Alamos National Laboratory, Los Alamos, 2001. Available at Conservation Laws Preprint Server at `http://www.math.ntnu.no/conservation/2001/049.html`.

Evolution Galerkin Methods for Nonlinear Multidimensional Hyperbolic Systems

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The subject of this contribution is the analysis of new multidimensional high-resolution finite volume evolution Galerkin methods for systems of nonlinear hyperbolic conservation laws. In the last decade emphasis has been put on the development of genuinely multidimensional finite volume schemes, see, e.g. [1], [2], [7], approach of LeVeque [6] is also related. In multidimensional flows, there are in general no longer a finite number directions of propagation of information along the bicharacteristics, but rather infinitely many. This has to be taken into account in order to design a reliable multidimensional scheme. Instead of solving one-dimensional Riemann problems in normal directions to cell interfaces the finite volume evolution Galerkin schemes are based on a genuinely multidimensional approach. The approximate solution at cell interfaces is computed by means of an approximate evolution operator using all of the inifinitely many bicharacteristics explicitly into account. This is a novel feature of our method and a genuine generalization of Godunov's ideas.
The approach described has been fully exploited for linear hyperbolic systems such as the wave equation system or the Maxwell equations, see [2], [3], [4], [8].
The main objective of this contribution is to present a generalization to nonlinear hyperbolic systems. We have particularly in mind the system of the Euler equations of gas dynamics and the shallow water equations. In order to derive an approximate evolution operator for a nonlinear hyperbolic system it is suitable to linearise the system by freezing the Jacobi matrices of nonlinear flux functions at some suitable physical states. Generally, due to the advection terms one gets rather complex configurations with slanted Mach cones. Integrals along the Mach cones are evaluated exactly or by means of numerical quadratures. Second order resolution is obtained with a conservative piecewise bilinear recovery and the second order midpoint rule for the time integration. The schemes are studied theoretically as well as experimentally. We have proven the second order error estimates for the FVEG schemes applied to linearized hyperbolic systems. Using the Simpson rule for time integration and a biquadratic recovery for space approximation third order schemes are derived, see [8]. Numerical experiments which confirm the accuracy and multidimensional behaviour of the FVEG schemes will be presented. We will also discuss appropriate implementations of reflecting as well as absorbing boundary conditions and demonstrate them by video.

References

[1] M. Fey. Multidimensional upwinding, Part II. Decomposition of the Euler equations into advection equations, *J. Comp. Phys.*, **143**:181-199, 1998.

[2] M. Lukáčová - Medviďová, K.W. Morton, G. Warnecke. Evolution Galerkin methods for hyperbolic systems in two space dimensions. *MathComp.*, **69**:1355–1384, 2000.

[3] M. Lukáčová - Medviďová, K.W. Morton, G. Warnecke. High-resolution finite volume evolution Galerkin schemes for multidimensional conservation laws, *Proceedings of ENUMATH'99*, World Scientific Publishing Company, Singapore, 1999.

[4] M. Lukáčová - Medviďová, G. Warnecke. Lax-Wendroff type second order evolution Galerkin methods for multidimesnional hyperbolic systems, *East-West Journal*, **8(2)**:127–152, 2000.

[5] M. Lukáčová-Medviďová, K.W. Morton, G.Warnecke. Finite volume evolution Galerkin methods for Euler equations of gas dynamics, *Inter J. Num. Methods in Fluids*, to appear.

[6] R.J. LeVeque. Wave propagation algorithms for multi-dimensional hyperbolic systems, *J. Comp. Phys.*, **131**:327–353, 1997.

[7] S. Noelle. The MoT-ICE: a new high-resolution wave-propagation algorithm for multi-dimensional systems of conservation laws based on Fey's Method of Transport, *J. Comp. Phys.*, to appear.

[8] Y. Zahaykah. Evolution Galerkin schemes and discrete boundary conditions for multidimensional first order systems *PhD thesis, University of Magdeburg*, 2001.

Non-Oscillatory Schemes for Scalar Conservation Laws

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We consider a class of Godunov-type schemes for solving scalar conservation laws in one space dimension. There are two main steps in such schemes: evolution and projection. In the original Godunov scheme, the projection is onto piecewise constant functions – the cell averages. In the general Godunov-type method, the projection is onto piecewise polynomials. Many well-known methods are non-oscillatory, however, non-oscillation (or essential non-oscillation) is, in general, not sufficient to prove convergence of such methods to the *entropy solution* or derive error estimates. For example, the original MinMod, UNO, ENO, and WENO methods are known to be numerically robust, at least for piecewise smooth initial data, but theoretical results about convergence are still missing. We introduce the notion of Weakly Non-Oscillatory (WNO) schemes, which generalizes the classical concept of non-oscillation. For example, any Godunov-type scheme with non-oscillatory evolution and projection is WNO. We restrict our attention to WNO Godunov-type methods with exact evolution. Our main result is a convergence theorem and an error estimate for a subclass of WNO Godunov-type schemes which includes simple modifications of MinMod and UNO. In the case of a linear flux and a Lipschitz initial data, we show that the modified MinMod scheme coincides with the original MinMod scheme, and hence, our error estimate is valid for the original MinMod scheme in this case.

4.40 Friday, Session 4 (morning): Adaptive methods IV

Adaptive Computation of Inherently Unresolved Evolution Problems

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Adaptive mesh refinement has established itself as a very useful technique in the description of phenomena that exhibit local features requiring much greater resolution than the overall background. New meshes are typically generated based upon a predictor step that provides information on where additional resolution is required. Initial data for the new, finer meshes are usually obtained by low order polynomial interpolation from previous, coarser meshes. This approach assumes that enough computational resources are available to ensure full resolution of the phenomena of interest. In many situations this is not feasible. For example, turbulent flow exhibits scales of motion over several decades and full resolution would be prohibitive. One is then led to the question of how to establish criteria for adaptive refinement based upon models of the subgrid phenomena. This paper presents an analysis of grid placement and proper interpolation technique in the context of Burgers turbulence which is of interest in cosmology and interface dynamics. The adaptive techniques derived from this study are shown to be useful for hydrodynamic turbulence also. Simulations in 1 to 3 dimensions are presented to verify the analysis.

Multi-level techniques in 2D numerical simulations

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High Resolution shock Capturing methods combine sharp numerical profiles at shocks and contact discontinuities with high accuracy in smooth regions. Some HRSC schemes tend to be quite expensive due to numerical flux evaluations, making numerical simulations in multidimensions a painfully slow task, unless a large computing facility is available.
Since flux evaluations of a HRSC scheme are only absolutely necessary in the neighborhood of discontinuities, or in compression regions, where discontinuities are ready to be formed, we have developed in [1] a tool that reduces these number of expensive computations. Classical test cases for the 2D Euler equations shown in [1] demonstrate that the application of this technique to 2D simulations of systems of conservation laws reduces drastically the cpu time keeping high quality numerical solutions.

With the aid of this technique, we shall present high resolution simulations of high Mach number shock-vortex interactions. In particular, we investigate the production of vorticity and acoustic waves and compare our results to the physical theory. The combined use of a particular Riemann solver [2] and the technique developed in [1] allows us to obtain high resolution simulations at Mach numbers larger than those recorded in the literature [3].

[1] G. Chiavassa and R. Donat, *Point Value Multiscale Algorithm for 2D Compressible Flows*, SIAM J. Sci. Comp., Vol. 23,(3), 2001.
[2] R. Donat and A. Marquina: *Capturing Shock Reflections: An Improved Flux Formula*, JCP, Vol 125, (42), 1996
[3] J.L. Ellzey, M.R. Henneke, J.M. Picone and E.S. Oran, *The interaction of a shock with a vortex: Shock distortion and the production of acoustic waves*, Phys. Fluids 7(1), 1995

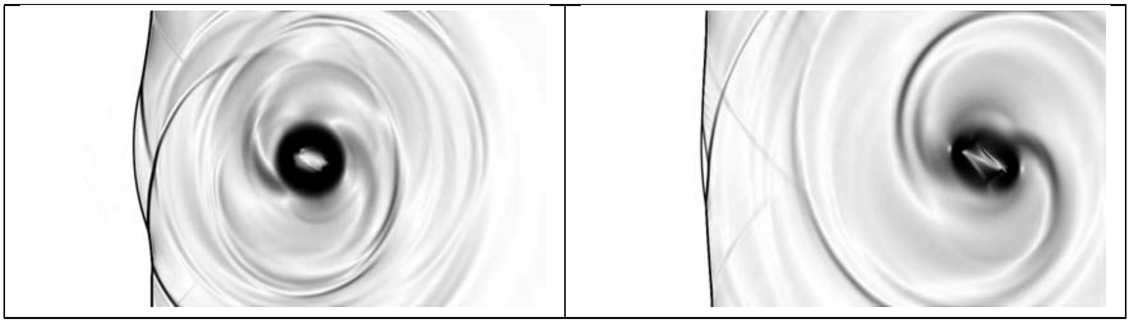

Figure 1: *Creation of acoustic waves for weak shock-vortex interaction at two different times. Numerical Schlieren images obtained with the pressure.*

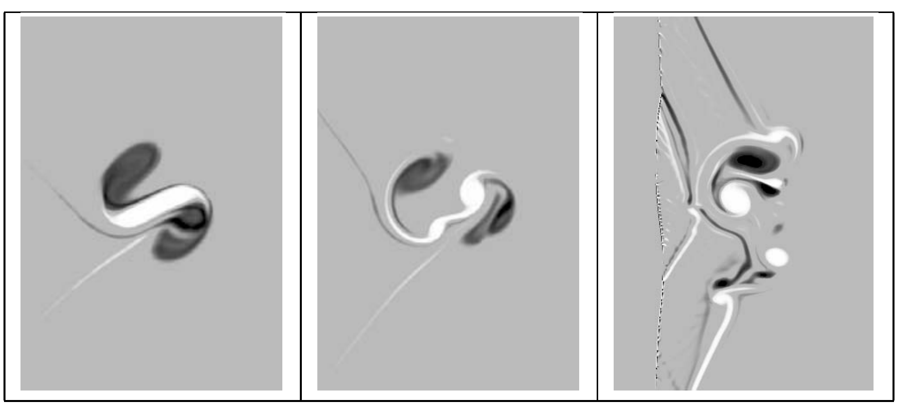

Figure 2: *Visualization of the vorticity after the shock-vortex interaction for different shock Mach numbers M: from left to right: $M = 1.5$, $M = 2$ and $M = 7$. White color represents positive vorticity and black negative.*

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