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1 Introduction

The Ninth International Conference on Hyperbolic Problems: Theory, Numerics and Applications, is being held at Caltech from March 25 to 29, 2002, (Hyp2002). This is one of the highest quality and most successful conference series in applied mathematics. This series originated in 1986 at Saint-Etienne, with an earlier focus towards the theoretical aspects of hyperbolic conservation laws. As computers became more powerful in the late eighties, and as the development of modern innovative numerical algorithms has made a considerable impact during the nineties, the scope of the Hyp series was expanded to its present format.

In recent years, many effective numerical methods that have been originally developed in the context of of Computational Fluid Dynamics, have found new applications outside their traditional areas. These applications include material sciences, multiphase/multicomponent flows, combustion/detonation, incompressible fluid flows with free boundaries, PDE-based imaging processing, etc.

There have been many new developments in hyperbolic and nonlinear evolution PDEs in recent years. We are very pleased to see that the Hyp2002 has brought together leading researchers from different disciplines to address the theoretical, modeling, and computational issues in solving hyperbolic PDEs and more generally, nonlinear evolution equations arising from different areas of applications. To better integrate across thee various aspects of theory, numerics and applications, we highlighted three main themes for Hyp2002:

1. Fundamental Theory and Numerical Analysis.
2. Multiscale analysis, modeling and simulation.
3. Applications: Geophysics, Materials Science, Free Boundary Problems.

More than one hundred fifty abstracts are assembled in this book, reflecting the high-level quality expected at Hyp2002, with lectures covering a diverse range of theory, numerics and applications.

We believe that the conference will provide a forum to exchange and to stimulate new ideas from different disciplines, and to formulate new challenging problems that will have important physical and industrial impacts. A special effort has been made to attract young participants as well as women and minority participants. Over 30 young participants are provided with financial support to attend the Hyp2002 conference.

Finally, we would like to take this opportunity to thank Andrew Westhead for his dedicated work in assembling this book of abstracts, and to acknowledge the partial support from the National Science Foundation, Office of Naval Research, Institute of Pure and Applied Mathematics at UCLA, DFG Priority Research Program on Analysis and Numerics for Conservation Laws in Germany, Center of Integrative Multiscale Modeling and Simulation at Caltech, Division of Engineering and Applied Science at Caltech and Caltech ASCI Center.

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Note that where multiple authors are listed, the presenter is marked by an asterisk*.

2 Abstracts of plenary speakers

Viscosity Solutions of Nonlinear Hyperbolic Systems

Alberto Bressan

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Consider a strictly hyperbolic $n \times n$ system of conservation laws in one space dimension:

$$u_t + f(u)_x = 0\,. \tag{1}$$

Here $u = (u_1, \ldots, u_n)$ is the vector of *conserved quantities*, while the components of $f = (f_1, \ldots, f_n)$ are the *fluxes*. For small BV initial data, the global existence of weak solutions was proved in the fundamental paper of Glimm [5], under the additional assumption

(H) Each characteristic field is either *linearly degenerate* or *genuinely nonlinear*.

The uniqueness and stability of entropy weak solutions has later been established by the author together with several collaborators. See [3] for a comprehensive account of these results.
Since the pioneering work of Glimm, the construction of BV solutions was always achieved by piecing together solutions of several Riemann problems. A priori BV and stability estimates were based on the careful analyis of interaction between elementary waves (shocks or centered rarefactions). Bounds on the total variation are obtained by means of a *wave interaction potential*, controlling the production of new oscillations. In this connection, the hypothesis (H) is a simplifying assumption that guarantees that that every Riemann problem can be solved in terms of n elementary waves. At the price of considerable technicalities, this assumption has been relaxed in subsequent works by Liu [6] concerning existence of solutions and by Ancona and Marson [1], concerning stability.
Very recently, uniform BV and stability estimates have been obtained by S. Bianchini and the author in [2] also for vanishing viscosity approximations:

$$u_t + A(u)u_x = v^\varepsilon\, u_{xx}\,, \tag{2}$$

where $A(u) = Df(u)$ is the $n \times n$ Jacobian matrix. The key ingredients in our proof are:

• The local decomposition of the gradient u_x as a sum of gradients of viscous travelling waves, based on a center manifold construction.

• The a-priori bounds on source terms (producing new oscillations), based on the introduction of 4 Lyapunov functionals, which we call: *transversal wave interaction, length, area* and *energy functionals.*

This result marks the first time where uniform BV and stability estimates are obtained without any reference to Riemann problems. The well posedness of the Cauchy problem can now be proved for the whole class of strictly hyperbolic systems, without any reference to the assumptions (H).

Theorem. *Consider the Cauchy problem for the hyperbolic system with viscosity*

$$u_t + A(u)u_x = v^\varepsilon\, u_{xx} \qquad\qquad u(0,x) = \bar u(x)\,. \tag{3}$$

Assume that the matrices $A(u)$ are strictly hyperbolic, smoothly depending on u in a neighborhood of a compact set $K \subset \mathbb{R}^n$. Then there exist constants C, L, L' and $\delta > 0$ such that the following holds. If

$$\textit{Tot.Var.}\{\bar u\} < \delta\,, \qquad\qquad \lim_{x\to-\infty} \bar u(x) \in K\,, \tag{4}$$

then for each $v^\varepsilon > 0$ the Cauchy problem (3)$^\varepsilon_v$ has a unique solution u^{v^ε}, defined for all $t \geq 0$. Adopting a semigroup notation, this will be written as $t \mapsto u^{v^\varepsilon}(t,\cdot) \doteq S^{v^\varepsilon}_t\,\bar u$. In addition, one has:

BV bounds :

$$\textit{Tot.Var.}\left\{S_t^{v^\varepsilon}\bar{u}\right\} \leq C\,\textit{Tot.Var.}\{\bar{u}\}\,. \tag{5}$$

$\mathbf{L}^1$ stability :

$$\left\|S_t^{v^\varepsilon}\bar{u} - S_t^{v^\varepsilon}\bar{v}\right\|_{\mathbf{L}^1} \leq L\left\|\bar{u}-\bar{v}\right\|_{\mathbf{L}^1}, \tag{6}$$

$$\left\|S_t^{v^\varepsilon}\bar{u} - S_s^{v^\varepsilon}\bar{u}\right\|_{\mathbf{L}^1} \leq L'\left(|t-s| + \left|\sqrt{v^\varepsilon t} - \sqrt{v^\varepsilon s}\,\right|\right). \tag{7}$$

Convergence: *As $v^\varepsilon \to 0+$, the solutions u^{v^ε} converge to the trajectories of a semigroup S such that*

$$\left\|S_t\bar{u} - S_s\bar{v}\right\|_{\mathbf{L}^1} \leq L\left\|\bar{u}-\bar{v}\right\|_{\mathbf{L}^1} + L'\,|t-s|\,. \tag{8}$$

These vanishing viscosity limits can be regarded as the unique **vanishing viscosity solutions** *of the hyperbolic Cauchy problem*

$$u_t + A(u)u_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)\,. \tag{9}$$

Ĭn the conservative case $A(u) = Df(u)$, every vanishing viscosity solution is a weak solution of

$$u_t + f(u)_x = 0, \qquad\qquad u(0,x) = \bar{u}(x)\,, \tag{10}$$

satisfying the Liu admissibility conditions.

Assuming, in addition, that each field is genuinely nonlinear or linearly degenerate, the vanishing viscosity solutions coincide with the unique limits of Glimm and front tracking approximations.

We remark that the BV setting is essential, in order to achieve the uniqueness and stability of vanishing viscosity limits. For a strictly hyperbolic 3×3 system with linearly degenerate fields, a Cauchy problem having two distinct $\mathbf{L}^\infty$ solutions (both obtained as limits of vanishing viscosity approximations) can be found in [4].

The talk will present the main ideas involved in the BV estimates for the viscous approximations (2), and discuss the various interaction functionals. The possibility of obtaining similar BV bounds and stability estimates for other approximations (relaxations, numerical schemes) will also be briefly commented.

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Mathematical questions along the flow of a river

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Motivated by computational aspects of Saint-Venant s shallow water system (usual for many applications like rivers flow, tidal waves, but also narrow tubes) conducted at INRIA/M3N, we consider some mathematical and algorithmic questions for hyperbolic systems with a topogra-phy driven source term. We also revisit some classical questions in the numerical analysis of finite volume methods such as: what are sharp CFL conditions for E-schemes (after Tadmor's seminal paper), why TVD bounds on the approximate solutions ARE NOT necessary for $h^{1/2}$ convergence rates.

Global Propagation of Regular Nonlinear Hyperbolic Waves

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By means of introducing the concept of weak linear degeneracy and the method of normalized coordinates, a complete result is presented on the global existence and the blow-up mechanism of C^1 solution to the Cauchy problem for general first order quasilinear hyperbolic systems in 1-D case with small and decaying initial data as follows:
1. The Cauchy problem admits a unique global C^1 solution for all $t \in \mathbb{R}$ if and only if the system is weakly linear degenerate (WLD).
2. If the system is not WLD and the corresponding index α is an integer ≥ 0, then for a large class of initial data, the first order derivatives of the C^1 solution to the Cauchy problem must blow up in a finite time with a sharp estimate on the life-span and the formation of singularity is due to the envelope of Characteristics of the same kind.
This result impies all previous results obtained by F. John, T. P. Liu and L. Hörmander and can be applied to the following physical situations: the system of nonlinear elasticity, the system of 1-D gaz dynamics without convexity, the system of the motion of elastic strings and the system of finite amplitude plane waves for hyperelastic materials such as Hadamard-Green material, neo-Hookean material, St. Venant-Kirchhoff material and Odgen material etc.

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Macroscopic limits of kinetic models revisited

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Introduction

A fluid, a plasma or a solid-state device consists of a large number of particles which interact among themselves and with their environment. The large scale (or macroscopic) behaviour of the system is, to a large extent, determined by the nature or the elementary particle interactions at the microscopic level. The study of the interplay between the microscopic and macroscopic worlds has been the subject of an intense research for more than a century (starting from the works of Boltzmann and Maxwell). Recently, this field of research has received an increased interest driven by the quest for new models accompanying the development of new technologies. In this talk, after an introduction to the field, we shall present some new developments in this area and illustrate them by examples pertaining with space and plasma devices technologies.

Summary of the talk

We shall start with a basic introduction to kinetic models, which will constitute our microscopic level of physical description. In a kinetic model, the state of the ensemble of particles is described by a phase-space particle density $f(x, v, t)$, which describes the number of particles at a given location x, with a given velocity v at time t. The equation for f models how particles evolve, subject to force fields and interactions among themselves or with their surrounding (obstacles, boundaries, other species of molecules, radiation, etc). This leads to the so-called Boltzmann equation. Here, we shall mainly deal with particle interactions with their surroundings.
The kinetic distribution function f is related with more conventional continuum (or macroscopic) variables, like the number density $n(x, t)$ or the temperature $T(x, t)$ through an integration of f with respect to the velocity variable (velocity moments). One central question is to try to deduce evolution equations for these variables from that of the distribution function. This question has been solved, at least formally, for conventional models (like Euler, Navier-Stokes equations or Drift-Diffusion equations). However, many mathematical questions remain unanswered and are the subject of an intense research activity.
Nevertheless, in this talk, we shall take another direction, and discuss how to extend these techniques in order to derive new models. Indeed, new technologies using advanced fluid mechanics, solid-state or plasmas technologies require the design of new physical models, the standard ones lacking of physical accuracy. The main reason for this discrepancy is a question of scale. The continuum models are derived on the assumption that the scale of the system is much larger than the scale of microscopic interactions. Many high technology devices have a smaller size, which is intermediate between what can be considered as a macroscopic scale, and the microscopic scale of elementary particle interactions. This is the problem of mesoscale modeling.
We shall present an approach to derive mesoscale models. In most physical cases, the collision operator (the mathematical object which describes the particle interactions) exhibits a multiscale behaviour: it forces the evolution of certain quantities to be faster than other ones. A typical example is the interaction between electrons and ions in a plasma. Because of the mass ratio (ions are typically 10^3 to 10^5 times heavier than electrons), electron momentum is modified much faster than electron energy. At large scales, one can consider that both momentum and energy will have relaxed towards those of the ions. However, on shorter time or length scales, it is very likely that mean electron momentum will be close to that of the ions, but not the mean energy.
When full relaxation towards thermodynamical equilibrium has occured (under the influence of collisions), the system is amenable to a description by a reduced set of variables, namely the macroscopic ones (like density, or temperature). This leads to the conventional continuum models. However, when incomplete relaxation has occured, which often happens at the mesoscale, then modeling of the system requires to keep track of some of the microscopic variables, but not necessarily all of them. For instance, in the above example, it is enough to keep track of the energy distribution function of the electrons, but not of their angular velocity distribution since this will be isotropic as a result of collisions against the ions. In this way, intermediate descriptions between the microscopic kinetic model and the macroscopic continuum models are obtained.
The aim of this talk will be to develop a mathematical apparatus which describes how such intermediate models can be obtained. More specifically, we shall focus on the derivation of the *SHE* and *Energy-Transport* models (SHE is an acronym coming from the physics literature for 'Spherical Harmonics Expansion, but no such expansion is needed in current derivations of this model). Both are systems of diffusion equations which can describe various physical situations. Various examples and

numerical applications will be borrowed from plasma device technology. As far as possible, the status of the mathematical theory of these models will be precised.
This talk will present an overview of work done in collaboration with various authors: N. Ben Abdallah, J. P. Boeuf, F. Deluzet, L. Garrigues, S. Gnieys, A. Jngel, V. Latocha, D. Levermore, B. Lucquin-Desreux, S. Mancini, A. Mellet, F. Poupaud, C. Schmeiser, R. Talaalout, M. H. Vignal.

Heterogeneous Multiscale Methods

Bjorn Engquist

Princeton University, USA

In many applications the difference in temporal and spatial scales in the differential equations pose serious challenge to numerical simulations. We shall describe a framework for computations on the macroscale but where the forces and fluxes are computed on the microscale. This means that macroscale computations can be done without the knowledge of effective equations but also without the high computational complexity of a full microscale simulation. Examples will be given from homogenization theory for hyperbolic problems with oscillatory solutions.

Recent Progress in the Mathematical Analysis of Vortex Sheets

Sijue Wu

University of Maryland, USA

I will give an overview of the 2-D incompressible Euler equations with vortex sheet initial data and present some recent results concerning the Birkhoff-Rott equation. One of the issues I will discuss is the behavior of vortex sheet solution in general after the singularity formation.

Array imaging, time reversal and communications in random media

George C. Papanicolaou

Department of Mathematics, Stanford University, USA

I will present an exposition of the mathematical problems that arise in using arrays of transducers for imaging and communications in random media. The key to understanding their performance capabilities is the phenomenon of statistically stable super-resolution in time reversal, which I will explain carefully. Signals that are recorded, time reversed and re-emitted by the array into the medium tend to focus on their source location with much tighter resolution when there is multipathing because of random inhomogeneities. I will explain how this super-resolution enters into array imaging and communications when there is multipathing.

High Order Finite Difference Methods for Multiscale Complex Compressible Flows

Björn Sjögreen

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The classical way of analyzing finite difference schemes for hyperbolic problems is to investigate as many as possible of the following points

- Linear stability for constant coefficients
- Linear stability for variable coefficients
- Non-linear stability
- Stability at discontinuities

We will build a new numerical method, which satisfies all types of stability, by dealing with each of the points above step by step.
In addition to stability, there are other requirements for the numerical method. Such requirements can be

- Ability to accurately follow waves for long times.
- Ability to resolve turbulence, and other small scale phenomena.
- Efficient use of computational time.
- Efficient parallelization.

TVD schemes are too dissipative for turbulence simulations, and ENO schemes demand very high computational resource. We will here describe methods which can meet the above requirements, but which do not belong to the class of standard shock capturing schemes.
To assure a correct treatment of boundaries, we use finite difference operators having the so called summation by parts property (SBP). To improve nonlinear stability (or stability for linear variable coefficients) of the numerical computations we employ skew-symmetric splitting of the convective flux derivatives. As an alternative/complement to splitting, we use linear artificial dissipation of high order. Finally, for good resolution of shock waves we take the non-trivial artificial viscosity from a second order TVD scheme, and insert it into the method only near discontinuities. In order to switch on this viscosity only where it is needed, we use a special detection algorithm, based either on gradients or on wavelets.
This leads to the point of view, that inventing a new numerical method for hyperbolic conservation laws is no longer a problem of coming up with a clever formula relating the new time level to the old one, but rather a problem of designing a system by connecting known components together with switching mechanisms.
There is a large degree of freedom in how to connect the components. For example, the linear dissipation operator can be evaluated as a part of the residual, or added as a post processing step. If a Runge-Kutta method is used in time, the dissipation can be applied to each stage, or only at certain stages. There is also the choice of whether the linear dissipation is switched off when the TVD-dissipation is switched on, or if it is applied in full strength at all grid points.
The numerical method will be demonstrated on several examples to show its possibilities. In all problems below we use as the basic scheme, a sixth order centered difference operator, modified at boundaries to have the SBP property. Time integration is done with the classical fourth order Runge-Kutta method. Additonal examples and a more detailed discussion of the results will be given during the lecture.
Vortex convection
This problem serves as an illustration for non-linear stability, and long-time integration. The effects of entropy splitting and artificial dissipation will be demonstrated.
The solution consists of as single vortex which is translated with uniform velocity periodically around in a square. The vortex translation is an exact solution of the compressible Euler equations of gas dynamics. The solution is smooth for all times. Furthermore, since the boundary conditions are periodic, we do not need to involve boundary modifications of the operators. The problem will demonstrate in a very “clean” way, how numerical methods deal with non-linear effects.

In aeroacoustics, the interest is to follow weak waves for long times. The requirements of good accuracy after long integration times is very difficult to satisfy. Applications from rotorcraft machinery, typically require that a vortex is followed a distance which corresponds to 300 – 1000 periods. In the computation of turbulence by direct numerical simulations, statistics of time accurate computations are taken for a very long time as well.

The computations show that on a given grid, non-linear instabilities destroy the solution after 5 periods when the pure centered scheme is used. This is illustrated in Fig. 1. With entropy splitting the break down comes after 60 periods instead, and the norm of the entropy is decreasing all the way to break down. After introducing an eight order linear dissipation operator the solution can be computed with accuracy up to around 200 periods. However, tuning of the dissipation strength is a very sensitive issue, and will be discussed in some detail.

Compressible Viscous Shock/Shear/Boundary-layer Interactions

For this problem, we increase the complexity of the numerical treatment, by adding a low order dissipation term at the shock waves that appear. The problem demonstrates benefits of high order accuracy.

The ideal gas compressible full Navier-Stokes equations with no slip BCs at the adiabatic walls are used. The fluid is at rest in a 2-D box $0 \leq x, y \leq 1$. A membrane with a shock Mach number of 2.37 located at $x = 1/2$ separates two different states of the gas. $\gamma = 1.4$, the Prandtl number is 0.73, and the Reynolds number is 1000.

The solution is shown in Fig. 2. The small scale features in the solution has turned out to be very sensitive to the numerical method used, unless the grid is very fine. In order to fully resolve the vortex structure, the 6th order method needed a grid of 8 million grid points in two space dimensions. Although strong shocks are present, TVD and ENO methods were not usable, either due to insufficient accuracy or due to too long computational time.

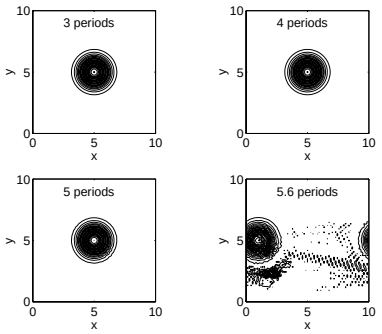

Figure 1: Density contours. Non-linear instability in vortex convection.

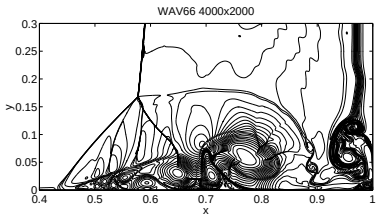

Figure 2: Density contours. Small scale structure in shock tube problem.

Boltzmann equation and conservation laws

Tai-Ping Liu

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Conservation laws, the Euler and Navier-Stokes equations for gas dynamics, are derived through Hilbert and Chapman-Enskog expansions. With Shih-Hsien Yu and others, we devise a macro-micro decompositon to rewrite the Boltzmann equation into fluid and non-fluid parts. The conservation laws become part of Boltzmann equation. The decomposition is useful for the study of nonlinear waves. It has been used to verify the positivity of the Boltzmann shock profiles. The H-theorem has also its fluid part representation. This natural representation is useful for the energy method. Our energy method yields a elementary virification of the time-asymptotic stability of Maxwellian states. In this talk, we will start with basic properties of Boltzmann equation and relates these to the theory for hyperbolic and viscous conservation laws.

High Order Numerical Methods for Convection Dominated PDEs

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In this talk we will present an overview of some recent progress in high order finite difference and finite volume weighted essentially non-oscillatory (WENO) and finite element discontinuous Galerkin (DG) methods for solving hyperbolic conservation laws and in general convection dominated PDEs, such as convection diffusion equations with small diffusion, and KdV type equations with small dispersion.

Among these three methods, the finite difference WENO method is the simplest to implement and the fastest in running time for multi-dimensional problems. Finite difference WENO schemes are available for orders of accuracy up to 11 (or even higher), but in most applications the fifth order version is a good choice. The method is a "blackbox" with no parameters to tune, and is extremely stable for strong shocks. It is also easy to implement the method on parallel machines with excellent parallel efficiency. The method is suitable especially for problems involving both shocks and complex smooth region structure in the solution. It has however very strong requirements on the smoothness of the meshes, hence is suitable only for problems on either rectangular geometry or domains covered by smooth curvilinear coordinates.

After a brief overview of the method, we will talk about the following recent progress: (1) Resolution study of using the method with different orders of accuracy on problems with discontinuities as well as complex smooth region structures. The double Mach flow problem and the Rayleigh-Taylor instability problem are used as examples. The conclusion is that, the ninth order WENO scheme uses only half the number of points in each direction to obtain a comparable resolution (by "eye-ball norm") comparing with a fifth order WENO scheme, indicating that it does pay to use higher order methods for such problems. This is a joint work with Jing Shi and Yongtao Zhang; (2) Multi-domain finite difference WENO simulations. This is for the purpose of relaxing the requirement that the mesh be smooth everywhere. The computational domain is covered by several slightly overlapping rectangular domains and WENO interpolation with comparable order of accuracy to transfer information between subdomains. Numerical study indicate that the method is "essentially conservative", meaning that the conservation error goes to zero with a mesh refinement, even for solutions with very strong shocks. The inter-domain interpolation is also very stable when shocks pass through the subdomain boundaries, and uniform high order accuracy is maintained. This is a joint work with Kurt Sebastian. (3) Simulations of the jet problem in astrophysics. This is a joint work with Carl Gardner, Youngsoo Ha and Anne Gelb.

The finite volume WENO method is based on the same designing principles as the finite difference version in terms of the mechanism to achieve non-oscillatory solutions, but it is based on an integral version of the PDE. The finite volume WENO method is applicable to essentially arbitrary geometry and does not require any smoothness of meshes. Hence it is very suitable for the situation of complex geometry and adaptive computation. However, it is much more costly for multi-dimensional

problems. After a brief overview of the method, we will talk about the recent joint work with Jing Shi and Changqing Hu on treating negative linear weights, which arise for high order finite volume WENO schemes.

The discontinuous Galerkin method is very similar to the finite volume WENO method. It is based on the same integral version of the PDE. The only difference is that, instead of relying on a reconstruction from cell averages to obtain a high order accuracy approximation as in finite volume WENO method, the DG method evolves all the degrees of freedoms for a high order polynomial in each cell, hence no reconstruction is needed. This certainly saves the reconstruction time but on the other hand increases the storage requirement as well as evolution time for all these degrees of freedoms. DG method relies on total variation bounded (TVB) limiters to control numerical oscillations for solutions with shocks. DG methods are especially suitable for parallel implementations and adaptivity, including both h adaptivity and p adaptivity (varying orders of accuracy in different cells, for which finite difference methods are difficult to do).

After a brief overview of the method, we will talk about the following recent progress: (1) Stable and accurate DG formulation for problems containing higher derivatives but are still convection dominated, such as convection diffusion equations with small diffusion coefficients or KdV type equations with small dispersions. This is a joint work with Jue Yan. (2) A post-processing technique which can effectively double the order of accuracy for the method on locally uniform meshes, with small extra computational cost. This is joint work with Bernardo Cockburn, Mitch Luskin and Endre Suli, with Jennifer Ryan, and with Jue Yan.

High-Resolution Methods for Wave Propagation in Heterogeneous Media

Randall J. LeVeque

University of Washington, USA

Wave propagation problems in heterogeneous media are modeled by hyperbolic systems of equations with spatially-varying coefficients or flux functions. These problems arise in numerous applications involving the propagation of acoustic, elastic, or electromagnetic waves, for example. I will discuss the use of high-resolution finite volume methods for such problems, and the development of approximate Riemann solvers for nonlinear examples. I will also present some results on wave propagation in nonlinear periodicly layered media, where dispersive effects arising from the heterogeneity lead to the appearance of solitary waves and perhaps solitons.

Finite volume methods based on Riemann solvers are a natural choice for such problems. Each grid cell is assumed to consist of a single material and the Riemann problem at the interface between two grid cells is solved by determining the waves propagating into each cell, based on the consituitive relations describing the two materials. This Riemann solution can be directly interpreted in terms of reflection and transmission of propagating waves. These methods, when combined with appropriate wave limiters, can yield high-resolution results on problems with sharp interfaces. Multidimensional generalizations can also deal with problems where interfaces are not aligned with the grid.

For nonlinear problems (e.g., nonlinear elasticity in a heterogeneous medium), the exact Riemann solution may not be easy to compute. I will discuss a general approach to developing approximate Riemann solutions for generalized Riemann problems having a discontinuity in the flux function as well as in the data. This general approach to solving conservation laws with spatially varying flux functions is based on decomposing the flux difference into eigenvectors of an approximate Jacobian matrix. This approach can also be related to a generalization of relaxation schemes.

These methods have been applied to nonlinear elasticity equations in a rapidly-varying periodic layered medium and yield some surprising results. For some choices of nonlinear material properties, a pulse breaks up into solitary waves that appear to interact as solitons.

Nonlinear Boundary Layers of the Boltzmann Equation

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We discuss the half-space problem of the Boltzmann equation,

$$\begin{cases} \xi_1 F_x = Q(F,F), & x \in (0,\infty),\ \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_0(\xi), & \xi_1 > 0, (\xi_2,\xi_3) \in \mathbb{R}^2, \\ F \to M_\infty(\xi) \quad (x \to \infty), & \xi \in \mathbb{R}^3, \end{cases} \tag{1}$$

where the unknown $F = F(x,\xi)$ is the mass density distribution of gas particles at position $x \in (0,\infty)$ with velocity $\xi = (\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3$, while ξ_1 is the x-component of ξ and Q, the *collision operator*, is a quadratic integral operator in ξ whose integral kernel is determined by the interaction potential of the gas particle. So far, our result is proved only for the hard ball gas, but the same result seems to hold for general cutoff hard potentials.

The second equation in (1) is the Dirichlet boundary condition at the boundary $x = 0$. The Dirichlet data $F_0(\xi)$ is assigned only for incoming particles from the boundary, i.e. for $\xi_1 > 0$. Physically this is natural because we can control only the incoming distribution but not the outgoing($\xi_1 < 0$) distribution. Mathematically, this is a well posed boundary condition. It is known that assigning the outgoing distribution makes the problem (1) ill-posed.

The third equation of (1) specifies the far field. This is the Dirichlet boundary condition at $x = \infty$, and is assigned for all $\xi \in \mathbb{R}^3$. Then, two remarks are to follow. One is that the far field M_∞ cannot be arbitrary but must be a zero of Q, that is, a *Maxwellian*,

$$M_\infty(\xi) = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} \exp\Big(-\frac{|\xi - u_\infty|^2}{2T_\infty}\Big), \tag{2}$$

and $\rho_\infty > 0, u_\infty = (u_{\infty,1}, u_{\infty,2}, u_{\infty,3}) \in \mathbb{R}^3$, and $T_\infty > 0$ are the only quantities which we can control. By a shift of ξ_2, ξ_3, we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$, and then, the sound speed and Mach number of the equilibrium state described by (2) are given by

$$c_\infty = \sqrt{\frac{5}{3}T_\infty}, \qquad \mathcal{M}^\infty = \frac{u_{\infty,1}}{c_\infty},$$

respectively. The other remark is that since the outgoing distribution at $x = \infty$ (i.e. for $\xi_1 > 0$) is assigned, the problem (1) may become ill-posed and hence only conditionally solvable. Indeed, we will show that the solvability condition changes with $\mathcal{M}^\infty$ as follows.

(a) If $\mathcal{M}^\infty < -1$, the problem (1) admits a unique smooth solution for any F_0 sufficiently close to M_∞.

(b) If $\mathcal{M}^\infty > -1$, such a solution exists only for F_0 close to M_∞ and satisfying certain admissible conditions. The set of admissible F_0 forms a smooth manifold whose co-dimension is 1 for the case $0 < \mathcal{M}^\infty < -1$, 4 for $0 < \mathcal{M}^\infty < 1$ and 5 for $\mathcal{M}^\infty > 1$, respectively.

The problem (1) arises in the theory of the kinetic boundary layer, the analysis of the condensation-evaporation and so on. The corresponding linearized problem has been studied by many authors, e.g. [2],[3],[4],[5], mainly in the context of the classical Milne and Kramers problems and hence with auxiliary conditions on boundary fluxes. In [6], an existence theorem was established for the nonlinear case with the specular boundary condition but the method of proof does not apply to other boundary conditions, especially the Dirichlet condition. Recently, nonlinear existence and stability theorems have been established for the discrete velocity model of the Boltzmann equation [7], [9]. Our result is the first existence theorem on the full nonlinear problem. Furthermore, it provides a new aspect of the linearized problem and also a partial proof of the numerical results established in [1], [8], on (1) with F_0 fixed to be the standard Maxwellian. The talk will include the details on these points as well as the idea of proof and the stability of our stationary solution.

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3 Abstracts of invited speakers

On unique continuation for the nonlinear Schrödinger equations

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Joint work with Carlos E. Kenig and Luis Vega

This talk is concerned with uniqueness properties of solutions of nonlinear Schrödinger equation of the form

$$i\partial_t u + \Delta u + F(u,\overline{u}) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{1}$$

More precisely, we shall consider the following question :
Q : Let $u_1,\ u_2$ be solutions of the equation (1) with $(x,t) \in \mathbb{R}^n \times [0,1]$, belonging to an appropriate class X and such that for some domain $D \subset \mathbb{R}^n,\ D \neq \mathbb{R}^n$

$$u_1(x,0) = u_2(x,0), \ \text{ and } \ u_1(x,1) = u_2(x,1), \quad \forall\, x \in D. \tag{2}$$

Is $u_1 \equiv u_2$?
Before stating our results, we shall comment on previous related results.
For the case of the k-generalized KdV

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \ \ k \in \mathbb{Z}^+, \tag{3}$$

it was shown in [4] that if

$$u \in C([0,1] : H^4(\mathbb{R})) \cap C^1([0,1] : H^1(\mathbb{R})), \tag{4}$$

are real solutions of the equation (3) such that for some $a \in \mathbb{R}$

$$u(x,0) = u(x,1) = 0, \quad \forall\, x > a, \ (\forall\, x < a), \tag{5}$$

then $u \equiv 0$.
Concerning the equation (1), in [10] B-Y. Zhang answered question Q in the case

$$n = 1, \ \ F = \alpha|u|^2 u, \ \alpha \in \mathbb{R}, \ \ u_2 \equiv 0, \ \ D = (-\infty, a) \ \ (\text{or } D = (a,\infty)), \tag{6}$$

for some $a \in \mathbb{R}$. The proof in [10] is based on the inverse scattering theory (IST). It is not clear to us if in the case (6) the IST can be applied to obtain the desired result for any pair of solutions.
Other unique continuation results have been obtained under analyticity assumptions on the data, and under appropriate assumptions on the form of the non-linearity

$$F = F(u,\overline{u},\nabla_x u, \nabla_x \overline{u}), \tag{7}$$

see [3] and references therein, (since the equation for the difference of two solutions does not necessarily preserve the form of the non-linearity, it is not clear that such results extend to pairs of (analytic) solutions), or under analitycity assumptions on the non-linearity F, without analyticity of the data, but under the stronger assumption that $supp\, u(\cdot,t)$ is compact for all $t \in [0,1]$, (see [1]).
Our main result is the following.

Theorem 1

Let $u_1,\ u_2 \in C([0,1] : H^s(\mathbb{R}^n))$, $s \geq max\{n/2^+; 2\}$ be two solutions of the equation

$$i\partial_t u + \Delta u + F(u,\overline{u}) = 0, \tag{8}$$

where $F \in C^{[s]+1}(\mathbb{C} : \mathbb{C})$ with

$$|F(u,\overline{u})| \leq c(|u|^{p_1} + |u|^{p_2}), \ \ p_1,\, p_2 > 1, \tag{9}$$

and

$$|\nabla F(u,\overline{u})| \leq c(|u|^{p_1-1}+|u|^{p_2-1}), \quad p_1,\, p_2 > 1, \tag{10}$$

If there exists Γ a convex cone strictly contained in a half-space such that

$$u_1(x,0) = u_2(x,0), \quad u_1(x,1) = u_2(x,1), \quad \forall\, x \notin \Gamma + y_0, \quad y_0 \in \mathbb{R}^n. \tag{11}$$

Then $u_1 \equiv u_2$.

Remarks

a) In the one dimensional case our assumption on the complement of $\Gamma + y_0$, i.e. $(\Gamma + y_0)^c = D$ reduces to a semi-line (a,∞) (or $(-\infty, a)$). Also we observe that the class of nonlinearities F considered is very general. In particular, it does not contain any analyticity hypothesis on F.

b) In contrast to our approach in [4], here we do not rely on estimates of the type found in [5], [6]. In fact, the proof given below for Theorem 1 can be slightly modified to obtain a different proof of the results in [4] without using the results in [5], [6].

c) As in [4], we need an appropriate local Carleman estimate. In [4], for the case of the generalized KdV equation, we used a unique continuation result due to Saut-Scheurer [7]. For the equations considered here, we will apply the local unique continuation results of V. Isakov [2].

d) We do not know if the result of Theorem 1 is still valid for the case where Γ_{x_0} is just a semispace. This question seems to be related to problems considered in [11].

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Asymptotic behaviour of semiconductor models

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We review recent results on the asymptotic behavior of semiconductor models including hydrodynamical systems and drift-diffusion models. The derivation of the mathematical models from the semi-classical Boltzmann equation in terms of the moment method is highlighted, and the mathematical analysis of the large- time-asymptotic behavior of both classical solutions (for small data in d- dimensions) and entropy weak solutions (small shock strength, in one dimension) is given on spatially bounded domains or whole space for hydrodynamical equations. The quasi-neutral limit of the drift-diffusion system is then carried for initial data with uniformly bounded entropy and without. Finally, open problems in the field are discussed.

Nonreflecting Boundary Conditions for Wave Propagation in Unbounded Media

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The simulation of waves in unbounded media arises in many applications from acoustics, electromagnetics, or elasticity. Typically, the local phenomenon of interest contains complicated geometric features, inhomogeneity, and possibly nonlinear effects. Modern numerical methods can handle complicated geometry, inhomogeneous media, and nonlinearities. However, they require an artificial boundary $\mathcal{B}$, which truncates the unbounded exterior domain and restricts the region of interest to a finite computational domain Ω. It then becomes necessary to impose a boundary condition at $\mathcal{B}$, which ensures that the solution in Ω coincides with the restriction to Ω of the solution in the unbounded region. Usually various approximate boundary conditions are used, such as the Bayliss-Turkel [1] or Engquist-Majda [2] boundary conditions, which produce some spurious reflection. To eliminate spurious reflection from the artificial boundary, we have devised *exact nonreflecting boundary conditions* for the wave equation [3,4], Maxwell's equations [5], and the elastic wave equation [6,7]. These boundary conditions are *local in time* and involve only first derivatives of the solution. Therefore, they are easy to use with standard finite difference or finite element methods. Numerical examples demonstrate the improvement in accuracy over standard methods.

The accurate simulation of waves at high frequencies or the detailed representation of small scale geometric features requires the use of adaptive mesh strategies. Then, explicit time integrators become prohibitively expensive because of the stringent CFL condition; hence, implicit methods, such as Crank-Nicolson, are typically used, yet they require the solution of a large linear system of equations at every time step. Because of the nonreflecting boundary condition, this linear system is no longer symmetric, unlike the situation in bounded domains. However, it is possible to reformulate the discretized equations by decoupling the additional unknowns needed on the artificial boundary from the interior unknowns [8]. As a consequence the symmetry and positive definiteness of the linear system are restored while the additional computational effort due to the nonreflecting boundary condition becomes negligible.

For multiple scattering problems the use of a single artificial boundary surrounding all scatterers involved becomes prohibitively expensive in memory requirement. Instead, it is necessary to enclose each scatterer within a single separate computational domain. Clearly waves that leave a certain domain, Ω_1, will impinge upon a different domain, Ω_2, at later times; hence they are no longer purely outgoing waves. To transfer the time-retarded information from Ω_1 to Ω_2 an analytical representation of the solution in the unbounded medium becomes necessary. This analytical representation is inherent to the exact nonreflecting boundary conditions described above and thus naturally leads to *exact transmission boundary conditions* for multiple scattering problems.

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Taxis Equations from the Diffusion Limit of Transport Equations

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There are two major approaches used to describe the motion of biological organisms: (i) a space-jump process in which individuals move by random jumps in space, and (ii) a velocity-jump process in which discontinuous changes in the speed or direction of an individual are generated by a stochastic process. The former leads to a renewal equation in which the kernel governs the waiting time between jumps and the redistribution after a jump, and these determine the type of partial differential equation that describes the asymptotic behavior of the evolution. In this talk we discuss velocity-jump processes, and in particular, the diffusion approximation to the transport equation

$$\frac{\partial}{\partial t}p(x,v,t) + v\cdot\nabla p(x,v,t) = -\lambda p(x,v,t) + \lambda\int_V T(v,v')p(x,v',t)dv' \tag{1}$$

describing such a process. Here $p(x,v,t)$ denotes the density of particles at spatial position $x \in \Omega \subset \boldsymbol{R}^n$, moving with velocity $v \in V \subset \boldsymbol{R}^n$ at time $t \geq 0$. Here λ is the (constant) turning rate and $1/\lambda$ is a measure of the mean run length between velocity jumps. In general λ may be space dependent and depend on internal and external variables as well. The turning kernel $T(v,v')$ gives the probability of a velocity jump from v' to v if a jump occurs, and implicit in the above formulation is the assumption that the choice of a new velocity is independent of the run length. The turning kernel may also be space dependent. When applied to the bacterium *E. coli*, the kernel T includes a bias, and the turning frequency must depend on the extracellular signal, as transduced through the signal transduction and motor control system. When applied to the amoeboid cell *Dictyostelium discoideum*, which uses both run length control and taxis, both the turning kernel and the turning rate must depend indirectly on the extracellular distribution of the signaling substance.

The backward equation that corresponds to (1) has been derived from the underlying stochastic velocity-jump process by Stroock to describe the motion of bacteria, and in a more general framework by Papanicolaou. In this talk we discuss the general assumptions on the turning kernel T which ensure that the turning operator defined by (1) is positive in an appropriate sense, and the positivity in turn guarantees that a diffusion limit of the jump process exists. We introduce the parabolic scaling and formally derive the parabolic limit equation. Since the parabolic limit is the outer solution in singular perturbations terms, these higher approximations depend only on the initial values for the parabolic limit problem. We derive several equivalent conditions on the turn angle distribution under which the diffusion matrix is a scalar multiple of the identity, and show that the

diffusion constant depends on the second eigenvalue of the turning operator. We shall also discuss signal-dependent turning rates and redistribution kernels, and give an example of an order one, anisotropic perturbation of the redistribution kernel that nonetheless leads to a scalar diffusion matrix. We introduce several different classes of $\mathcal{O}(\epsilon)$ perturbations of the turning kernel and turning rate and show how the chemotactic velocity and sensitivity are obtained from more fundamental and measurable properties of the motion. This leads to a variety of different types of signal dependence of turning rates and kernels for which the jump process is asymptotically described by the Patlak–Keller–Segel–Alt chemotaxis equation. We also discuss an open problem connected with the incorporation of internal dynamics that describe the signal transduction process, and in particular give examples in which diffusion limits do not exist.
Finally, we shall discuss some new computational techniques that couple the direct use of Monte Carlo methods with a large-time-step algorithm to produce an efficient scheme that can be used when the macroscopic equations are not known.

Spectral Stability of Small Shock Waves

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In this talk, spectral stability of small amplitude shock waves associated with simple modes in systems of viscous conservation laws is shown to be a direct consequence of the spectral stability of shock waves in scalar viscous conservation laws. This relationship is established through a precise description of the behaviour of Evans functions in the zero amplitude limit. The eigenvalue problem is studied via flows it induces on suitable Grassmann manifolds. An appropriate scaling allows to exploit the slow-fast nature of the problem. The stability theorem covers nonlinearities of arbitrary finite order, extending previous results on the second order ("genuinely nonlinear") case.
The applicability of the method is not restricted to the abovementioned specific situation.

Regularity in kinetic formulations via averaging lemmas

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This talk represents joint works with Benoit Perthame, DMA, ENS and Luis Vega, University of Bilbao, Spain.

A new class of averaging lemmas is developed, they are directly motivated by the question of regularity for different nonlinear equations or variational problems which admit a kinetic formulation. In particular they improve the known regularity for systems like $\gamma = 3$ in isentropic gas dynamics or in some variational problems arising in thin micromagnetic films. They also allow to obtain directly the best known regularizing effect in multidimensional scalar conservation laws.
The new ingredient here is to use velocity regularity for the solution to the transport equation under consideration. The proof is based on a decomposition of the density and the K-method of real interpolation but it can be done completely in the real space without any use of Fourier transform.

Two-Dimensional Riemann Problems: Analysis of Solution Structures

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In this talk a brief overview of problems and methods used to study the structure of solutions for a class of two-dimensional Riemann problems will be presented. The speaker will focus on the analysis of models arising in gas dynamics (the steady and the unsteady transonic small disturbance equations, the nonlinear wave equation) especially on the treatement of *nonlinear waves* and their interaction with a nontrivial subsonic region.
Since the interaction between the supersonic and subsonic flow occurs either through a transonic shock, through a rarefaction or via a sonic curve, different techniques need to be used to analyze the solution in each case. An overview of the techniques and a comparison between the methods will be given.
In the end, it will be shown how these methods can be used in the treatement of nonlinear waves' structures in compressible Euler equations (isentropic or adiabatic) where linearly degenerate modes are present. The reduced (self-similar) system is of mixed (elliptic-hyperbolic) type (density satisfies a degenerate elliptic equation, whereas vorticity satisfies a transport equation). In the low-velocity regime the mixed system decouples (the nonlinear wave equations) and the structure of both the nonlinear and the linearly degenerate waves can be analyzed. A similarity between the structure of the decoupled system and the fully coupled equations (arising from the compressible Euler equations) will be emphasized thereby hinting how the techniques presented in the first half of the lecture will be useful in the analysis of the structure of self-similar solutions of the full set of compressible Euler equations.
Collaborators: Barbara Lee Keyfitz, University of Houston, Eun Heui Kim, CalState Long Beach, Gary Lieberman, Iowa State University, Dragan Mirkovic, University of Houston.

Adjoint equations in the presence of shocks

Mike B. Giles

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Motivated by applications in design optimisation and error analysis, this paper is concerned with the formulation and discretisation of adjoint equations when there are shocks in the underlying solution to the original nonlinear hyperbolic p.d.e.
The theory is presented for the model problem of a scalar unsteady one-dimensional p.d.e. with a convex flux function. It is shown that the analytic formulation of the adjoint equation requires the imposition of an interior boundary condition along any shock. Looking ahead to three-dimensional flow applications, it would be extremely difficult in practice to apply such interior boundary conditions in a numerical approximation. The question then is whether it is possible to construct a convergent numerical approximation to the adjoint equation without explicitly enforcing this interior boundary condition.
An adjoint discretisation is defined by requiring the adjoint equations to give the same value for the linearised functional as a linearisation of the original nonlinear discretisation. It is proved that applying this technique to a class of explicit discretisations of the original p.d.e. yields a consistent approximation of the adjoint p.d.e. if the original solution is smooth. If the discretisation is also stable then the computed adjoint solution will converge to the analytic solution.
To investigate whether convergence is obtained when there is a shock, numerical results are obtained for the Burgers equation with the initial and boundary conditions which yield the solution shown in Figure 1.
Figure 2 shows the analytic adjoint solution and three numerical approximations at time $t = 0.1$. All four correspond to the linearisation of an output functional which is the integral of a function of the nonlinear solution at the final time $t = 0.5$. Although the discontinuities are still a bit smeared, the numerical solutions are reasonably grid-converged, and it is evident that the first order Riemann solver and the central difference approximation with a very low level of numerical smoothing both yield results which differ significantly from the analytic solution in the region $0.1 < x < 0.9$. Only the central difference approximation with a high level of smoothing gives a computed adjoint which is in good agreement with the analytic solution.

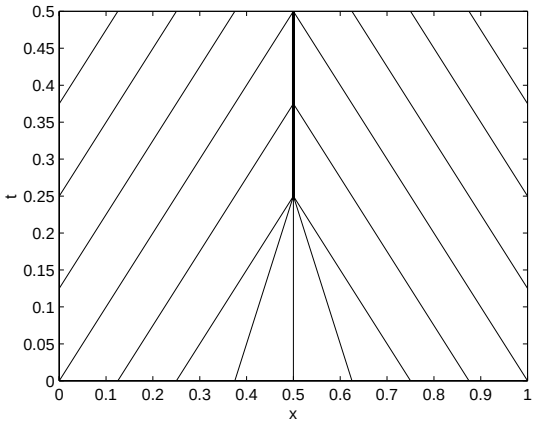

Figure 1: Characteristics with shock forming along $x = 0.5$

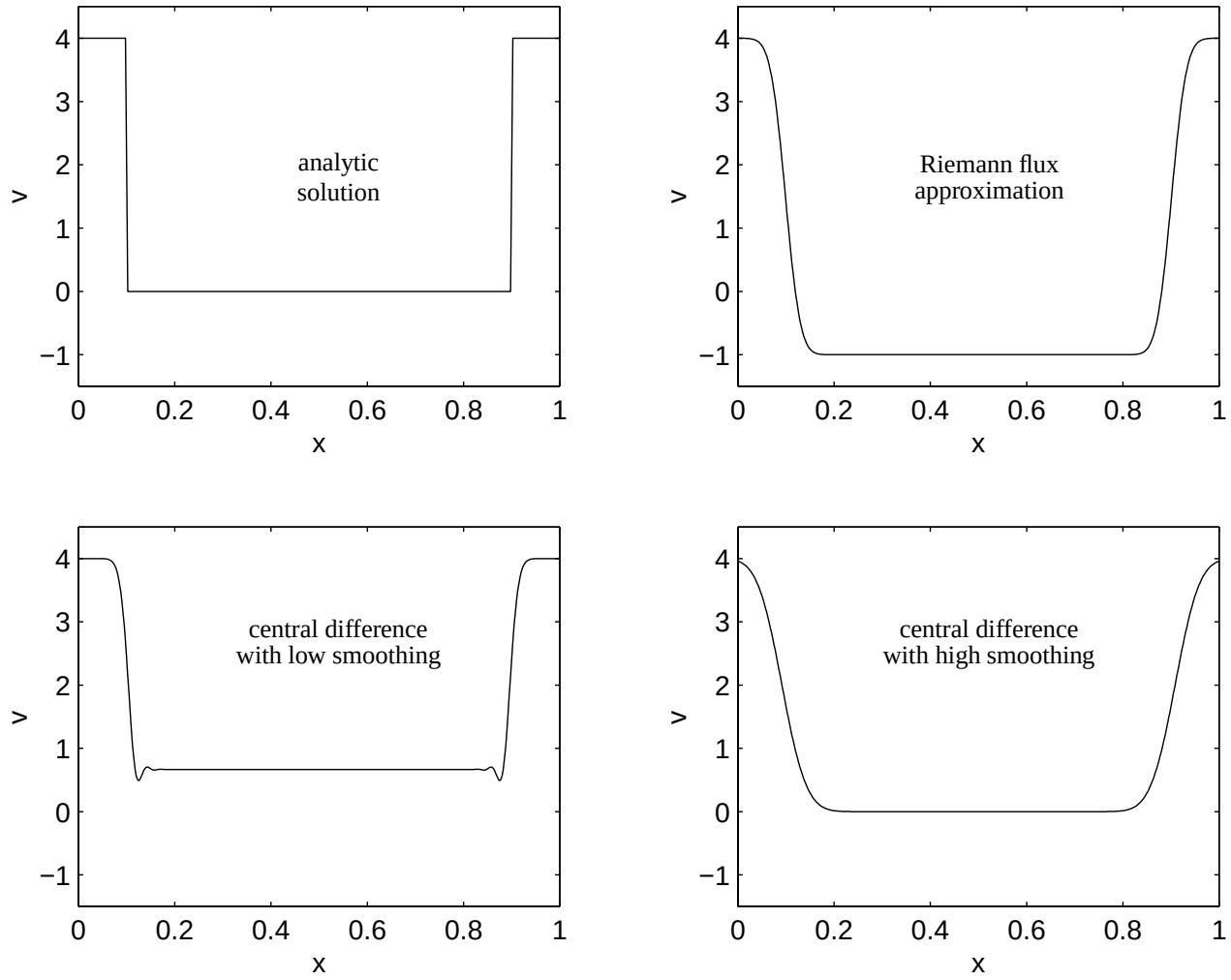

Figure 2: Analytic adjoint solution and three numerical approximations

The paper explains why it is generally the case that a convergent approximation requires that the number of grid points across the shock must increase as the mesh spacing and timestep are reduced. However, the numerical evidence suggests the numerical error decays exponentially with the number of points across the shock, so very accurate results can be obtained with just a few points in the shock.

Implicit-Explicit Runge-Kutta schemes for hyperbolic systems with relaxation

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Several physical phenomena of great importance for applications are described by hyperbolic systems with relaxation terms, for example we mention discrete kinetic theory of rarefied gases, hydrodynamical models for semiconductors, viscoelasticity, linear and nonlinear waves, multiphase and phase transitions, radiations hydrodynamics, etc.
In one space dimension these systems have the form

$$\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \quad x \in \mathbb{R}, \tag{1}$$

where $U = U(x,t) \in \mathbb{R}^N$, $F : \mathbb{R}^N \to \mathbb{R}^N$, the Jacobian matrix $F'(U)$ has real eigenvalues and admits a basis of eigenvectors $\forall\, U \in \mathbb{R}^N$, and $\varepsilon > 0$ is is called *relaxation parameter*.
The development of efficient numerical schemes for such systems is challenging, since in many applications the relaxation time varies from values of order one to very small values if compared to the time scale determined by the characteristic speeds of the system. In this second case the hyperbolic system with relaxation is said to be stiff and typically its solutions are well approximated by solutions of a suitable reduced set of conservation laws called equilibrium system [4].
Usually it is extremely difficult, if not impossible, to split the problem in separate regimes and to use different solvers in the stiff and non stiff regions. Thus one has to use the original relaxation system in the whole computational domain. The construction of scheme that works for all ranges of the relaxation time, using coarse grids that do not resolve the small relaxation time, has been studied mainly in the context of upwind methods using a method of lines approach combined with suitable operator splitting techniques [3, 7] and more recently in the context of central schemes [8, 9].
A very general and commonly used approach to the solution of this problem is based on splitting methods. A simple splitting consists in solving separately a non-stiff system of conservation laws without source

$$\partial_t U + \partial_x F(U) = 0, \tag{2}$$

applying an explicit scheme and, using an implicit scheme, a stiff system of ODEs for the source terms

$$\partial_t U = \frac{1}{\varepsilon} R(U). \tag{3}$$

This splitting is restricted to first order accuracy in time, nevertheless its simple structure presents several advantages. In fact some properties of the solution are maintained (positivity, TVD property, other physically relevant properties), consistency with the equilibrium system in the limit of small relaxation times can be easily checked (asymptotic preservation) and in many cases the implicit scheme for the stiff system of ODEs can be explicitly solved thanks to some conservation properties of the system.
Higher order splitting can be constructed using suitable combinations of the two previous steps [6, 11]. Unfortunately all these higher order extensions present a severe loss of accuracy when the source term is stiff [7]. Second order Runge-Kutta splitting which maintain the accuracy in the stiff limit have been constructed recently [3, 7].
In this talk we will present a unified approach of Runge-Kutta splitting schemes which provides a framework for the derivation of more general, accurate and efficient schemes. In particular, we show that these schemes are strictly related with the recently

developed implicit-explicit (IMEX) Runge-Kutta schemes [1, 2]. An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the source terms and an explicit one for the flux in the form

$$U^{(i)} = U^n + h\sum_{j=1}^{i-1}\tilde{a}_{ij}\partial_x F(U^{(j)}) + h\sum_{j=1}^{\nu} a_{ij}\frac{1}{\varepsilon}R(U^{(j)}), \quad (4)$$

$$U^{n+1} = U^n + h\sum_{i=1}^{\nu}\tilde{w}_i\partial_x F(U^{(i)}) + h\sum_{i=1}^{\nu} w_i\frac{1}{\varepsilon}R(U^{(i)}). \quad (5)$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (a_{ij})$ are $\nu \times \nu$ matrices such that the resulting scheme is explicit in F, and implicit in R.

Since the simplicity and efficiency of solving the algebraic equations corresponding to the implicit part of the discretization at each step is of paramount importance it is natural to consider diagonally implicit Runge-Kutta (DIRK) schemes for the souce terms ($a_{ij} = 0$, for $j > i$).

We show that most of the splitting schemes can be written in the formalism of IMEX Runge-Kutta schemes, where the implicit solver is a DIRK scheme. Similarly it is easy to write an IMEX Runge-Kutta scheme in splitting form. In particular, we derive general conditions that guarantee the asymptotic preserving property, i.e. the consistency of the scheme with the equilibrium system, and show that the implicit step can be solved, in many cases, every time we use a DIRK scheme. Accuracy, stability and TVD properties of these schemes are studied both analytically and numerically. Finally several applications of second and third order schemes obtained using ENO and WENO space discretizations [5] will be presented.

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Critical Thresholds and Conditional Stability for Euler Equations and Related Models

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When dealing with the questions of time regularity for Euler-related equations, one encounters several limitations with the classical stability analysis. Among others issues, we mention that
(i) the stability analysis does not tell us how large perturbations are allowed before losing stability, say with the incompressible Navier-Stokes equations;
(ii) the steady solution may be only conditionally stable due to the weak dissipation in the system, say in certain Euler-Poisson models.
In order to address these difficulties we propose a new notion of critical threshold (CT), which serves to describe the conditional stability for a class of Euler type equations.
We first discuss this remarkable CT phenomena associated with the Euler-Poisson equations, where the answer to questions of global smoothness vs. finite time breakdown depends on whether the initial configuration crosses an intrinsic, $O(1)$ critical threshold.
We investigate various one-dimensional problems with or without forcing mechanisms as well as multi-dimensional isotropic models with geometrical symmetry. The critical thresholds for these essentially 1-D problems are shown to depend on the relative size of the initial velocity slope and the initial density.
We then extend our discussion of the CT phenomena for multi-dimensional systems of the form $\partial_t u + u \cdot \nabla u = F$, which show up in different contexts dictated by the different modeling of F's. Here we utilize a novel description for the spectral dynamics of the (possibly complex) eigenvalues, $\lambda = \lambda(\nabla u)$ which are shown to be governed by the Ricatti-like equation $\lambda_t + u \cdot \nabla \lambda + \lambda^2 = \langle l, \nabla F r \rangle$. Restricting attention to restricted Euler-Poisson equations driven by localized forcing we identify the set of their $[n/2]$ global invariants, which in turn yield (i) sufficient conditions for finite time breakdown, and (ii) characterization of a large class of 2-dimensional initial configurations leading to global smooth solutions.
Moreover, the critical thresholds for 2D REPs are shown to depend on the relative size of three quantities at initial time: density, divergence and the spectral gap $\lambda_2 - \lambda_1$.
This lecture reflects recent joint investigations with Professor Eitan Tadmor.

Patch Dynamics for Multiscale Hyperbolic PDEs

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There are important systems that need to be modeled on relatively long time or space scales, but the dynamics can only be advanced on short time or space scales. These systems with multiple, separated scales include coupling molecular dynamics with the macroscale behavior of a material or using a Boltzmann particle model to predict large scale patterns in a fluid flow. We construct an efficient approach to bridge these scales by using locally averaged properties of the small time and short spatial scales to advance and predict the dynamics of the long time and space scale dynamics. First we advance an underlying

microscale model for a short time in small regions, called a patches, defined at the nodes on a grid covering a macroscopic spatial region. The microscale boundary conditions at the edges of the patches are used for the patches to communicate with each other across the macroscopic spatial gaps between the patches. We do not assume that there is an explicit evolution equation for the long time or spatial scales, but we do assume that the solution of the macroscale dynamics is smoothly varying in the gaps between the microscale patches being modeled. We construct a bi-directional map between the micro and macro scales, the map to reconstructing the microscale field from the macroscale only preserves the statistical properties of the field. Because this map between the fine and coarse scales is not fully invertible, the approach is only appropriate where the coarse field representation is sufficient to address the questions. When the underlying model is based on conservation laws, then the patch method must be modified to be fully conservative when material in the gaps between the patches is not been modeled.

Wave Interactions and Numerical Approximation for Two–Dimensional Scalar Conservation Laws

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This talk is concerned with the approximation of solutions to multidimensional scalar conservation laws by high resolution numerical schemes. Our approach here is based on the comparison of the numerical approximation with analytic solutions. Such solutions are obtained in two cases, to be discussed in detail: The Burgers equation and the Guckenheimer equation. In both cases, the "Riemann–type" problems to be studied are genuinely two–dimensional, leading to non–trivial wave interactions, which nonetheless can be obtained analytically. As already pointed out by Lindquist , " These solutions to two–dimensional Riemann problems also supply a set of problems for testing of finite difference schemes. The richness of structure of these solutions lends itself to this purpose". We note right away that a nonlinear one–dimensional conservation law can be "rotated" in the (x, y) plane, thus forming a "two–dimensional" problem. This, however, cannot lead to the kind of wave interactions discussed here, and will not be considered (even though the consideration of such problems is important in testing the basic features of a numerical scheme).

There are three ingredients in the present talk: (a) Analytic solutions involving wave interactions due to the two–dimensional geometry. (b) A high–resolution scheme for one–dimensional conservation laws. (c) A "spatial splitting" technique which enables us to convert the one–dimensional scheme into a two–dimensional one. The point here is to try and study the "mutual interaction" of these ingredients. In particular, while (a)–(b) seem to be well studied, the interaction between (b) and (c) is not yet fully understood . This interaction is influenced by (at least) two factors, namely, the adaptivity of the particular one–dimensional scheme to "spatial splitting" and the geometric complexity of the problem. The latter includes also the interplay between a Cartesian grid and strong curvilinear waves .

We consider the initial value problem (IVP) for the equation,

$$u_t + f(u)_x + g(u)_y = 0, \tag{1}$$

$$u(x, y, 0) = \phi(x, y), \quad (x, y) \in \Re^2, \tag{2}$$

where $u(x, y, t)$ is a real (scalar) function and $f(u)$, $g(u)$ are real smooth flux functions.

A "Riemann type" problem for (1) is the IVP where $\phi(x, y)$ is finitely valued and homogeneous of order zero,

$$\phi(x, y) = u_0(\theta), \quad \theta = \arg(x, y)(= \arctan \frac{y}{x}), \tag{3}$$

and $u_0(\theta)$ is piecewise constant in $[0, 2\pi]$ with finitely many jumps.

Recall that, for any initial function $\phi \in L^\infty(\Re^2)$, there exists a unique (weak) solution $u(x, y, t)$ to (1)–(2). The entropy condition (which includes already the fact that u is indeed a weak solution) can be described as follows.

Let $U(s)$ be a real convex function and $F(s)$ and $G(s)$ functions such that

$$F'(s) = U'(s)f'(s), \quad G'(s) = U'(s)g'(s). \tag{4}$$

Then, in the sense of distributions,

$$U(u)_t + F(u)_x + G(u)_y \leq 0. \tag{5}$$

The initial value (2) is attained in the sense that

$$u(x,y,t) \to \phi(x,y) \quad \text{in } L^1_{loc}(\Re^2), \quad \text{as } t \to +0. \tag{6}$$

When the initial data is given by (3), the uniqueness implies that the solution is "self–similar", namely,

$$u(x,y,t) = u(x/t, y/t, 1), \quad t > 0. \tag{7}$$

The solutions to the Riemann–type problem (1)–(3) display a rich variety of wave patterns, some of which are far from being "evident". Our intention in this talk is to show that this variety can serve as a basis for the investigation of "fine points and subtleties" pertinent to high resolution schemes. We first discuss the detailed structure of the solutions. For the Burgers equation we have $f(u) = g(u) = \frac{1}{2}u^2$. Even in this rather elementary case, we demonstrate various possibilities of wave interactions. Next we describe the solution for the "Guckenheimer equation", where $f(u) = \frac{1}{2}u^2$ and $g(u) = \frac{1}{3}u^3$.
This equation was first studied by Guckenheimer (1975) . Here we take the initial data

$$u_0(\theta) = \begin{cases} 0, & 0 < \theta < \frac{3\pi}{4}, \\ 1, & \frac{3\pi}{4} < \theta < \frac{3\pi}{2}, \\ -1, & \frac{3\pi}{2} < \pi < 2\pi. \end{cases} \tag{8}$$

The structure of the soluition can be described in the ($\xi = x/t, \eta = y/t$) plane as follows. Outside of a large disk we obtain three shocks:
(a) A shock emanating from $y = 0$ and moving at speed $\frac{1}{3}$ in the positive y direction (note that u^3 is concave on $[-1, 0]$). In the (ξ, η) plane it is given by $\eta = \frac{1}{3}$.
(b) A standing shock along $\xi = 0$ $(\eta < 0)$.
(c) A shock emanating from the line $x + y = 0$. In the (ξ, η) plane it is given by $\xi + \eta = \frac{5}{6}$.
The interaction of these three shocks in a disk around $(0, 0)$ form a very complex wave pattern, which can be described as follows (see attached Figure).
At a certain point $(0, b)$, $0 < b < \frac{1}{3}$, the shock (b) bifurcates into a centered rarefaction wave (CRW) whose leading characteristic is a sonic shock, across which the solution v jumps from -1 to a (still unknown) value $\tilde{v}$. Then v increases across the rarefaction from $\tilde{v}$ to 1. The rarefaction wave modifies shock (c) . We show numerical results based on Godunov's (first-order) scheme and on the GRP (Generalized Riemann Problem) second-order scheme. The latter scheme is actually an "algorithm", allowing various analytic extensions of the Godunov scheme (such as MUSCL, which results from a suitable approximation of the solution to the generalized Riemann problem).
Next, we show some results of simulations of duct (nozzle) flows, performed by the GRP scheme. All the above results are included in the upcoming monograph,
M. Ben-Artzi and J. Falcovitz, Generalized Riemann Problems in Computational Fluid Dynamics (Cambridge Univ. Press, 2002).

Stability of Weak Oblique Shock Front

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When a stationary supersonic flow passes a wedge with its angle less than a critical value, there will be a shock front attached at the head of the wedge. According to the Rankine-Hugoniot condition and the entropy condition two shocks are possibly appeared. However, only one of them is physically admissible. By the analysis of the formation and the global construction of shock, as well as the asymptotic behavior of the same supersonic flow passing a modified wall, we find that only the weak one is stable. Therefore, such a stability can be regarded as a criterion to single out a physically reasonable solution together with the entropy condition.

The Riemann function and oscillations in hyperbolic systems

Athanasios E. Tzavaras

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The fundamental solution for the equation describing entropies for systems of two conservation laws is determined in terms of the Riemann function. It gives rise to a "universal" entropy pair that can be cut in the directions of Riemann invariants to produce singular entropy pairs. The goal of the talk is to describe how to use the singular pairs to obtain simplified proofs of well known results of compensated compactness. Moreover, the rapport between the singular pairs and the fundamental solution may be used to obtain a new formula for the coupling of oscillations between two general characteristic fields. (joint work with B. Perthame)

Kinetic Approximations for the Incompressible Navier–Stokes Equations

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We present some recent results concerning a class of new kinetic approximations for the weak solutions to the Cauchy problem for the incompressible Navier Stokes equations in D space dimensions

$$\begin{cases} \partial_t U + \operatorname{div}(U \otimes U) + \nabla \phi = \nu \Delta U, \\ \nabla \cdot U = 0, \\ U(0,x) = U_0(x). \end{cases} \tag{1}$$

Our approximation is based on diffusive kinetic models with a finite number of velocities, which take the following general form. Fix the vector velocities $\lambda_i := (\lambda_{i1}, \ldots, \lambda_{iD})$, for $i = 1, \ldots, N$, and find a discrete kinetic (vector) distribution $\mathbf{f}_i = (f_i^0, f_i^1, \ldots, f_i^D) \in \mathbb{R}^{D+1}$, such that

$$\begin{cases} \partial_t \mathbf{f}_i + \frac{1}{\epsilon} \lambda_i \cdot \nabla_x \mathbf{f}_i = \frac{1}{\tau \epsilon^2} \left(\mathbf{M}_i(\rho, \epsilon \rho \mathbf{u}) - \mathbf{f}_i \right), \quad i = 1, \ldots, N \\ \mathbf{f}_i(x,0) = \mathbf{M}_i(\overline{\rho}, \overline{\rho} U_0), \end{cases} \tag{2}$$

for some fixed positive constant $\overline{\rho} > 0$. Here we define the "macroscopic" variables

$$\rho := \sum_{i=1}^N f_i^0, \ \epsilon \rho u_l := \sum_{i=1}^N f_i^l.$$

for $l = 1, \ldots, D$. Moreover, we are using the discrete (vector) Maxwellian function $\mathbf{M}$, which is a smooth function of $D+1$ variables with $\mathbf{M}_i = (\mathbf{M}_i^0, \mathbf{M}_i^1, \ldots, \mathbf{M}_i^D) \in \mathbb{R}^{D+1}$, for $i = 1, \ldots, N$, and which verifies the following consistency conditions;

$$\sum_{i=1}^N M_i^0(\rho, \mathbf{q}) = \rho; \tag{3}$$

$$\sum_{i=1}^{N} M_i^l(\rho, \mathbf{q}) = \sum_{i=1}^{N} \lambda_{il} M_i^0(\rho, \mathbf{q}) = q_l; \tag{4}$$

$$\sum_{i=1}^{N} \lambda_{ij} M_i^l(\rho, \mathbf{q}) = \frac{q_j q_l}{\rho} + P(\rho)\delta_{jl}, \text{ (here take } P(\rho) = \kappa\rho^\gamma); \tag{5}$$

$$\sum_{i=1}^{N} \lambda_{ij}\lambda_{ik} \sum_r \partial_{q_r} M_i^l(\overline{\rho}, 0) u_r = \frac{\nu}{\tau}\delta_{jk} u_l \tag{6}$$

Our main goal is to prove the following result.

Theorem 1 (Bouchut & Natalini 2002) *Assume that the BGK approximation has a bounded sequence of solutions $(\rho^\epsilon, \mathbf{u}^\epsilon)$ and the Maxwellian $\mathbf{M}$ is dissipative, which means that $(M')^t\eta''$ is symmetric, where $\eta(\rho, q) = q^2/2\rho + \rho e(\rho)$, $de/d\rho = P(\rho)/\rho^2$, and that the eigenvalues of M' are nonnegative. Then, as $\varepsilon \to 0$, and possibly passing to a sub-sequence, we have*

$$\rho^\epsilon - \overline{\rho} \to 0, \qquad \text{in } L^2:$$

$$\mathbf{u}^\epsilon \rightharpoonup U, \qquad \text{in } L^2 - (weak);$$

$$\frac{P(\rho) - P(\overline{\rho})}{\overline{\rho}\varepsilon^2} \rightharpoonup \Phi,, \qquad \text{in } \mathcal{D}',$$

and (U, Φ) are a (weak) solution to the Cauchy problem (1) for the incompressible Navier–Stokes equations.

The key point of our construction is that these approximations are endowed with kinetic entropy functions, which yield useful energy inequalities. By using a suitably modified version of the compensated compactness, we are able to show the convergence of the approximated solutions. Also, from these approximations it is possible to generate numerical schemes, related to the Lattice BGK Schemes, but with the supplementary and important features that they respect an H–Theorem and then they are nonlinearly stable. The properties and the accuracy of some special kinetic schemes will be presented and discussed in the talk, according to some further results obtained in collaboration with **M.F. Carfora** (IAC–CNR, Italy).

From Ghost-cell to Conservative Front Tracking

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Front tracking method combines Lagrangian particle propagation of the interface and Eulerian finite difference solvers for fluid in subdomains separated by an interface. By separating different finite difference solver in each subdomain and avoiding interpolation of physical quantities across a tracked interface, it maintains a sharp discontinuity of the physical quantities across the interface. In many physical problems, such discontinuity plays an important role in obtaining correct dynamical measurement. A good example is the simulation of the two and three dimensional Rayleigh-Taylor instability with randomly perturbed interface. The prevention of numerical diffusion across the fluid interface has retained the correct buoyancy acceleration and gives a much closer agreement with the experiment.
To solve the conservation laws in fluid physics

$$u_t + f(u)_x = 0,$$

the front tracking method consists of three key components. They are: (1) the Riemann solution for the interface propagation, (2) the finite difference solution in each subdomain bounded by the interface, and (3) the coupling of the Riemann solution and the finite difference solution at the interface. The Riemann solution not only calculates the normal velocity of the interface but also advances the states on both sides of the interface which satisfy the Rankine-Hugoniot condition. For many years since 1980, the interface-interior coupling in the front tracking method has used ghost-cells whose states are extrapolated from the

Riemann solution at the interface. For example, suppose the position of the interface is between two cells j and $j+1$, to update the state of the cell j which is in the left of the interface, a second order Godunov-type scheme has

$$U_j^{n+1} = U_j^n - \frac{\Delta t}{\Delta x}\left(F^n_{j+1/2,L} - F^n_{j-1/2}\right).$$

Since the cells $j+1$ and $j+2$ are in the right side of the interface, the extrapolated states $\overline{U^n_{j+1}}$ and $\overline{U^{n+1}_{j+1}}$ must be used to calculate

$$F^n_{j+1/2,L} = f(U^n_{j-1}, U^n_j, \overline{U^n_{j+1}}, \overline{U^{n+1}_{j+1}})$$

while in the right side of the interface, the flux at the same location $F^{\,n}_{j+1/2,R}$ must be calculated through

$$F^n_{j+1/2,R} = f(\overline{U^n_{j-1}}, \overline{U^n_j}, U^n_{j+1}, U^{n+1}_{j+1})$$

Since $F^n_{j+1/2,L} \neq F^n_{j+1/2,R}$, conservation cannot be achieved near the interface.
Our new scheme for cells near the tracked interface is based on the integral form of Euler's equation over a cell with both fixed and moving boundaries.

$$\frac{\partial}{\partial t}\int_V u dV + \int_{S_F} f_n(u) dS + \int_{S_M} (f_n(u) - u v_n) dS = 0 \ .$$

where v_n is the normal velocity of the moving boundary. We define $f_n(u) - uv_n$ as the dynamic flux in contrast to $f_n(u)$, the stationary flux. To update the states of the two cells j and $j+1$ near the interface, we use

$$\overline{\Delta x}^{n+1}_j U^{n+1}_j = \overline{\Delta x}^n_j U^n_j - \Delta t\left(F^{n,L}_I - F^n_{j-1/2}\right)$$

and

$$\overline{\Delta x}^{n+1}_{j+1} U^{n+1}_{j+1} = \overline{\Delta x}^n_{j+1} U^n_{j+1} - \Delta t\left(F^n_{j+3/2} - F^{n,R}_I)\right)$$

here $F^{n,L}_I = f(u^n_L) - s^n u^n_L$ and $F^{n,R}_I = f(u^n_R) - s^n u^n_R$ are dynamic fluxes on the left and right sides of the interface, respectively. Conservation is achieved because $F^{n,L}_I = F^{n,R}_I$ due to Rankine-Hugoniot condition. When the interface crosses the cell center during a time step, cell merge is needed to ensure numerical stability.
The extension of the conservative front tracking into higher dimensional spaces requires the construction of time-space cells and the merge of neighboring finite volume cells at the old and new time steps. In this presentation we will give a detailed account on the implementation of the conservative algorithm in two dimensional space.
1 Presented by Xiaolin Li

Partial Differential Equations for Fluid Flow and Computer Graphics

Ronald Fedkiw

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Scientists, engineers and mathematicians have used numerical techniques for partial differential equations to simulate physical phenomena for many years. More recently, these numerical techniques have worked their way into a variety of new areas including computer graphics. Some key techniques will be discussed including the Level Set Method for tracking interfaces and discontinuities, the Ghost Fluid Method for accurate modeling of boundary conditions at these interfaces and discontinuities, and Vorticity Confinement as a method of removing excess numerical dissipation on coarse grids. Example simulations of smoke, fire and water will be shown. Time permitting; we will also discuss numerical methods for cloth.

BV solutions for the semidiscrete upwind scheme

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We consider the semidiscrete upwind scheme

$$u_t(t,x) + \frac{1}{\epsilon}\Big(f\big(u(t,x)\big) - f\big(u(t,x-\epsilon)\big)\Big) = 0. \tag{1}$$

We prove that if the initial data of (1) have small total variation, then the solution $u^\epsilon(t)$ has uniformly bounded BV norm, independent of t, ϵ. This implies that as $\epsilon \to 0$, $u^\epsilon(t)$ converges to a weak solution of the corresponding hyperbolic system,

$$u_t + f(u)_x = 0.$$

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Hyperbolic Models of Traffic Flow

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In this talk, I will first briefly describe a model introduced a few years ago with A. Aw ([1]: "Resurrection of ...", SIAM J. Appl. Math., 2000) in which we replaced the Payne-Whitham (PW) class of "second order" (= 2 equations) continuous models of traffic flow based on the gas dynamics system by a new system. In the PW model, the way the drivers adjust their velocity depends on the x-derivative of the pressure $P(\rho)$, whereas in our model it depends on the Lagangian time-derivative of $P(\rho$, and we have shown in [1]that this modification suppresses *all* the severe inconsistencies of the PW model, such as cars going backward in some cases !!!

$$\begin{aligned} \partial_t\rho + \partial_x(\rho v) &= 0 \\ \partial_t(\rho w) + \partial_x(v\rho w) &= 0\ ,\ w := v + P(\rho). \end{aligned} \tag{1}$$

I will then describe - joint paper with Aw, Klar and Materne - how this heuristic model can be discretized either in Eulerian coordinates with Lagrangian cells or, see also J. Greenberg, in Lagrangian mass coordinates by the Godunov scheme, call it (God), with - surprisingly - uniform BV-estimates, due to the very special nature both of the system, which admits coinciding shocks and rarefaction waves, and of the discretization. Moreover, this Godunov approximation turns out to be the natural

explicit time-discretization of the microscopic "Follow-the-Leader" models.

$$\begin{aligned} \dot{x}_i &= v_i, \\ \dot{v}_i &= C\frac{v_{i+1}-v_i}{(x_{i+1}-x_i)^{\gamma+1}}, \end{aligned} \tag{2}$$

where $x_i(t), v_i(t), i = 1, \ldots,$ are location and speed of the vehicles at time t.
In fact, the model (1) introduced in [1] can be *rigorously* viewed as the fluid limit of this microscopic model, with an appropriate function P. The precise relations between the three levels are the following :

(i) Start with the fully discrete system, i.e. the Godunov approximation of the Lagrangian version (1') of (1). In the natural "hyperbolic" scaling, i.e. when we "make a zoom" $(x', t') := \varepsilon(x, t)$, where x is the Lagrangian "mass"" coordinate, then the solution constructed by the Godunov scheme converges to an entropy weak solution of (1'), with uniform BV-estimates.

(ii) Now, starting again with the fully discrete system (God), make the same scaling, but only in time, with a fixed Δx. Then the solution converges to the solution of the "Follow-the-leader" model (3), which therefore inherits the same uniform BV and L^∞ estimates, and turns out to be the semi-discretization of (1').

(iii) In turn, start now with system (3), and let Δx tend to 0. Then, at least for a sub-sequence (see other talks in this Conference for uniqueness results, the solution of (3) converges to an (the) entropy solution of (1'), which is therefore the fluid limit of (2), without passing through any kinetic description.

Finally, I will describe recent attempts to describe oscillations in experimental data.

4 Abstracts of contributed speakers

4.1 Monday, Session 1 (morning): Navier-Stokes equations

Dynamics of Singularity Surfaces for Compressible Viscous Flows in 2D

David Hoff

Indiana University, USA

We prove the global existence of piecewise smooth solutions of the Navier-Stokes equations of compressible, barotropic flow in two space dimensions. We show that discontinuities in the fluid density and in the divergence of the fluid velocity persist for all time, and are concentrated on smooth curves which are convected with the flow, exactly as predicted by elementary considerations based on the Rankine-Hugoniot conditions. We show that the strengths of these discontinuities decay exponentially in time, more rapidly for larger acoustic speeds and smaller viscosities. We also derive lower bounds on the strengths of discontinuities, thus showing that solutions do indeed remain discontinuous for all time. The rather remarkable and counterintuitive conclusion is therefore that, not only does viscosity fail to completely smooth out singularities in the initial data, but also that, since convective singularities are not present in solutions of the corresponding Euler equations for inviscid, barotropic flow, these singularities actually result from the presence of viscosity. We also show that the curves in physical space across which solutions are discontinuous remain reasonably smooth for all time, even though the velocity fields which convect them are not continuously differentiable. This difficulty is circumvented by a somewhat delicate analysis of tangential regularity, based on new estimates for Newtonian potentials.

Global Existence for the Full Multi-dimensional Compressible Navier-Stokes Equations with Large Symmetric Data

Helge Kristian Jenssen* and David Hoff

Indiana University, USA

We establish global existence of a weak solution to the Navier-Stokes equations for a compressible, heat-conducting fluid in space dimension $d = 2$ or 3. The initial data for the density, velocity and temperature are assumed to be either spherically symmetric ($d = 3$) or cylidrically symmetric ($d = 2$). The symmetry assumption implies that the problem is essentially one-dimensional away from the origin. To deal with the crucial multi-dimensional effects caused by the singularity at the origin, we combine techniques developed for the exterior cylider problem, Frid & Shelukin [2], and for the spherically symmetric barotropic case, Hoff [1]. Making use of a priori estimates and higher order energy estimates we obtain a global solution of the equations in Lagrangian coordinates. An interesting aspect of the analysis is the new phenomena related to the dependence on temperature. In particular the possible formation of a vacuum region and its regularity is more involved than in the barotropic case.

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4.2 Monday, Session 1 (afternoon): Incompressible flows

Stability of Plane Couette flow

Gunilla Kreiss

KTH, Sweden

Consider the Navier-Stokes equations modeling a viscous, incompressible fluid between two infinite, parallel moving plates. The stationary solution, consisting of parallel motion with a linear velocity profile, is linearly stable for all Reynolds numbers, R. It has been shown that for sufficiently small perturbations the flow is nonlinearly stable, (Romanov, 1973). Thus there is a threshold value for the size of the perturbations. Below this value all perturbations decay to zero. Above it some perturbations do not decay and may eventually lead to turbulence.
The question of how the threshold depends on the Reynolds number is addressed in this talk. We consider a dependence of the form $R^{-\beta}$. By direct numerical simulations and asymptotic analysis $\beta = 1.25$ and $\beta = 1$, respectively, has been found. We will derive an upper bound of the threshold by proving nonlinear stability. We use a resolvent bound and keep track of the R dependence. Previously this approach yielded a threshold with $\beta = 5.25$. The resolvent bound was based on computations in the Fourier domain.
In this talk we discuss improvements along two lines. Firstly, we have found that with a norm, where the components of the velocity are weighted compared to each other, the resolvent bound grows slower with R. This yields a sharper threshold bound. Secondly, we have strengthened the computations of the resolvent. By Fourier transform the resolvent equation is reduced to one dimensional boundary value problems with three parameters for the normal velocity and for the normal vorticity, respectively. We show estimates of the resolvent outside a bounded region in the parameter space. Thus a numerically determined resolvent is needed only for a bounded region of the parameters space.

Numerical simulations of incompressible flows by Finite Pointset Method

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A Finite Pointset Method (FPM) is a particle method for solving fluid dynamic equations. It is a Lagrangian and mesh free method. Therefore, this method is suitable for flows with complex geometry, rapid change of geometry in time, and flows with free surface flows. This method is similar to the method of smoothed particle hydrodynamics (SPH) except the approximation of spatial derivatives and the treatment of boundary conditions. In FPM, the spatial derivatives at an arbitrary point are approximated from the clouds of neighboring points with the help of weighted least squares method.
The method describes as follows: A fluid domain is filled by finite number of particles. The boundary is also filled by boundary particles and the boundary conditions are prescribed on boundary particles. On each particle, we assigned the fluid quantities like density, velocity, pressure and so on. Particles move with their own fluid velocities and carry all fluid informations with them. Boundary particles move with boundary velocity. The conservation laws is expressed in Lagrangian form. First, the spatial derivatives of conservation laws are approximated for each particle. Then the partial differential equations reduce to a system of ordinary differential equations. Finally, the system of ordinary differential equations are solved by a simple time integration scheme.
In order to solve the incompressible flow, it is necessary to solve the pressure Poisson equation. The main focus of the talk is to show the Poisson solver in grid free framework. The Poisson equation is solved by mesh less method. No direct approximation of the Poisson equation is required. It is a explicit scheme and unconditionally stable. The approximation is also based on the weighted least squares method, where the Poisson equation is used as constraint.
The incompressible Euler or Navier-Stokes equations are solved by a projection method. The numerical solutions are compared with the analytical solutions. It is found that a quite good agreement between the exact and numerical solutions. The scheme gives quite accurate results. Some simulations of free surface flows will also be presented.

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The small obstacle limit of 2D incompressible and inviscid exterior flow

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Consider Ω a bounded, connected and simply connected domain of the plane with smooth boundary Γ and $\Omega_\varepsilon \equiv \varepsilon\Omega$ with boundary Γ_ε. We consider a family $u^\varepsilon = u^\varepsilon(x,t)$ of incompressible flows which, for each fixed $0 < \varepsilon < \varepsilon_0$ satisfy the two-dimensional Euler equations on the exterior of Ω_ε, with velocity tangent to Γ and satisfying the initial conditions (i) the initial vorticity $\omega^0(x) = \text{ curl } u^\varepsilon(x,0)$ is independent of ε and has support disjoint from Ω_ε and (ii) the circulation around Γ^ε of the velocity u^ε is a real constant γ which also does not depend on ε. The purpose of the work presented in this talk is to identify the asymptotic behavior of the sequence $\{u^\varepsilon\}$. We have proved that, for $\gamma = 0$, u^ε converges to a flow u satisfying the incompressible two dimensional Euler equations in the full plane with initial vorticity ω^0 and that, for $\gamma \neq 0$, any weak limit of the sequence $\{u^\varepsilon\}$ satisfies a modification of the Euler equations which, in the vorticity formulation, consists of the usual transport equation, with the addition of a point vortex background.

The question of the limit behavior of incompressible flow in the exterior of a small obstacle is a very natural one, although the authors have not found reference to concerns regarding this particular asymptotic regime in the fluid mechanics literature. The ideal flow assumption is physically inappropriate to model the behavior of the flow near the obstacle, due to boundary layer effects, so that the whole issue of small obstacle asymtotics could be more meaningfully posed for the Navier-Stokes equations. However, given that there are no rigorous results in either the inviscid or the viscous cases, it makes sense to begin the analysis in the simpler case of ideal flow. This already contains a substantial fraction of the difficulties inherent to the general problem. Most of the work on time-dependent, incompressible exterior flow has been in the nature of well-posedness through energy methods. The present research, however, adopts the point of view of *vortex dynamics*. This means understanding 2D flow in terms of the description of the dynamics of vorticity, an approach which has recently been very fruitful. Our problem can be thought of in terms of vortex dynamics as attempting to describe the effect of a small obstacle on vortex motion, which is the initial motivation of this work, in a way we will make more precise. Furthermore, this work relies on the vortex dynamics point of view in an essential way for its more technical development.

Our interest in the small obstacle limit stems from the work of C. Marchioro [3] on confinement of vorticity for ideal, exterior flow. Confinement of vorticity is another quintessentially vortex dynamical line of research. We make a brief description of the problem of confinement of vorticity. Let $\omega = \omega(x,t)$ be a (classical) solution of the full plane 2D Euler equations such that $\omega(x,0)$ is compactly supported. The problem of confinement of vorticity is to obtain control over the growth of the diameter of the support of $\omega(\cdot,t)$. Current research on this subject originates with a result obtained by C. Marchioro in [2]. Marchioro proved that the solution of the incompressible 2D Euler equations in the full plane, with bounded, nonnegative initial vorticity with support contained in the ball $B(0;R_0)$, will have, at time t, its support contained in a ball of radius $R(t) = (bt + R_0^3)^{1/3}$,

for some constant $b \geq 0$. The state-of-the-art confinement result in the full plane has almost fourth root exponent, and it is due to Ph. Serfati. Among a number of results inspired on Marchioro's Theorem, we highlight his study of confinement in an exterior domain. For the exterior of a disk the corresponding result is cube root confinement and almost square root confinement for a general exterior domain, see [3]. We will not get into the issue of why the confinement result is sensitive to the presence and geometry of the domain, but we observe that confinement is connected to the way a obstacle influences very distant particles, called far-field effects. The scaling behavior of the incompressible Euler equations makes the problem of describing the influence of a very distant obstacle for a very long time naturally associated to observing the effect of a vanishingly small nearby obstacle for some time, which is the object of this work. In this context, the influence of the precise shape of the small domain in the vortex motion is of particular interest.
From the technical standpoint, this article makes use of the techniques of weak convergence methods for the asymptotic analysis. Such methods have often been developed for proving existence of weak solutions, but they are well suited for studying singular limits in general. The basic ingredients of the present proof are a collection of *a priori* estimates obtained mainly through exhaustive use of explicit formulas for the Green's function for the exterior domain, and strong compactness of approximate velocities obtained by using a parametrized div-curl Theorem introduced in [1]

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Convergence of Nonlinear Schrödinger-Poisson Systems to the Compressible Euler equations

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The combined semi-classical and quasineutral limit in the bipolar defocusing nonlinear Schrödinger-Poisson system in the whole space is proven. The electron and current densities, defined by the solution of the Schrödinger-Poisson system, converge to the solution of the compressible Euler equation with nonlinear pressure. The corresponding Wigner function of the Schrödinger-Poisson system converges to a solution of a nonlinear Vlasov equation. The proof of these results is based on estimates of a modulated energy functional and on the Wigner measure method.

4.3 Monday, Session 2 (morning): Riemann Problems I

The p-system with Large Data

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We describe recent progress towards proving global existence of solutions to the p-system of isentropic gas dynamics. The Cauchy data may be large in amplitude and in total variation. We first provide an exact solution of the Riemann problem with arbitrary Riemann data. Using this, we present an exact analysis of the general Glimm interaction of two waves of arbitrary strength. As a corollary we give precise conditions for the formation and annihilation of a vacuum. In particular it is shown that a vacuum appears only if it is embedded in the data (which is then necessarily discontinuous). Finally we describe a modified front tracking scheme for constructing large amplitude solutions. Our guiding principle is that in order to consider solutions with large data, one has to treat waves and their interactions exactly, and eschew the use of asymptotic interactions in which amplitudes (and total variation) are small. Nevertheless, the hyperbolic features of the system on which Glimm functionals are based persist.

Admissibility of Shock Waves and Uniqueness of the Riemann Problem

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We study the admissibility of shock waves and uniqueness of the Riemann problem for a general 2×2 hyperbolic system of conservation laws in one space dimension:

$$U_t + F(U)_x = 0, \quad (x,t) \in \boldsymbol{R} \times \boldsymbol{R}_+$$

with initial data having large amplitude. We assume that that the characteristic fields are *strictly separated*, that is: there exist two disjoint (open) conical neighborhoods such that the first characteristic field is confined to one of these neighborhoods and the second characteristic field to the other. We show that, together with some technical assumptions, the viscous profile exists for states $U_-,\ U_+$ not necessarily close and there exists at most one solution to the Riemann problem for general Riemann data U_L, U_R. These results will generalize admissibility and uniqueness theorems of Smoller and Liu, by giving descriptions free from particular choice of rectangular coordinates.

4.4 Monday, Session 2 (afternoon): Convergence results

Adjoint Error Analysis

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In a variety of fields, numerical methods are vital for obtaining approximate solutions to partial differential equations that describe the phenomena of interesting systems. In many cases, the principal quantitative objective of these simulations is the estimation of an integral functional of the PDE solution (e.g. the drag on an aircraft at transonic cruise, the radar cross-section of a glider, the electrostatic free energy of a biomolecule in water, or the flux of fossil fuels through a porous medium).
In the present work, a method is presented for obtaining functional estimates that have twice the order of accuracy of the computed PDE solution on which they are based. This superconvergence property is achieved via error analysis that uses an adjoint PDE to relate the local errors in approximating the original PDE solution to the corresponding global errors in the functional of interest. The talk will emphasize current work on functionals of steady and unsteady solutions to hyperbolic PDEs.

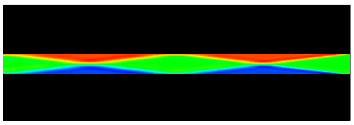

Figure 1: Representative adjoint behavior for uniform two-dimensional supersonic Euler flow

A Posteriori Error Estimate for Front-Tracking for Nonlinear Systems of Conservation Laws

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We demonstrate an a posteriori error estimate in the L^1 norm for front-tracking approximations for hyperbolic systems of nonlinear conservation laws. Extending the L^1-stability result of Bressan, Liu, and Yang we use their L^1-equivalent functional for pairs of front-tracking approximations and identify the leading order contribution to the numerical error. This leading term is closely related to the residual and determines an a posteriori bound of the error for conservation laws. We demonstrate the estimate for the front-tracking approximations of Risebro, which are extensions to systems of Dafermos' polygonal approximations.

A New Approach to Divergence Cleaning in Magnetohydrodynamic Simulations

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For the simulation of electrically conducting fluid flow the equations of magnetohydrodynamics (MHD) have to be solved. These combine the Euler equations of gas dynamics and the Maxwell equations. They are a system of nonlinear conservation laws for the fluid density ρ, momentum $\rho\mathbf{u}$, magnetic field $\mathbf{B} = (B_x, B_y, B_z)^T$, and total energy density e. $\mathbf{u} = (u_x, u_y, u_z)^T$ denotes the fluid velocity. Ignoring higher order effects like viscosity and resistivity we obtain the system

$$\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0\,, && \text{(1a)}\\
\partial_t (\rho \mathbf{u}) + \nabla \cdot \left[\rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}|\mathbf{B}|^2 \right) \mathcal{I} - \mathbf{B}\mathbf{B}^T \right] &= 0\,, && \text{(1b)}\\
\partial_t \mathbf{B} + \nabla \cdot (\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T) &= 0\,, && \text{(1c)}\\
\partial_t e + \nabla \cdot \left[\left(e + p + \frac{1}{2}|\mathbf{B}|^2 \right) \mathbf{u} - \mathbf{B}(\mathbf{u}\cdot\mathbf{B}) \right] &= 0\,. && \text{(1d)}
\end{aligned}$$

The additional constraint

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(1e)}$$

ensures a divergence–free magnetic field. This system has to be closed by an equation of state for the fluid pressure p. For the work presented here we assume a perfect fluid so that $p = (\gamma - 1)\rho\varepsilon$ with a constant adiabatic exponent $\gamma > 1$. The internal energy ε is given by $\varepsilon = \frac{e}{\rho} - \frac{1}{2}|\mathbf{u}|^2 - \frac{1}{2\rho}|\mathbf{B}|^2$. The scheme presented in our talk can be also used for more complex pressure laws.

For the conservative quantities $\mathbf{U} = (\rho, \rho\mathbf{u}, \mathbf{B}, e)^T$ we have to prescribe suitable initial and boundary conditions. In particular the constraint (1e) requires that the initial data for the magnetic field has zero divergence. Equation (1c) can be rewritten in the equivalent form $\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$. Since the divergence of a curl vanishes identically the divergence of the magnetic field will stay zero for all time if it is zero initially. Hence numerical methods are usually based only on the hyperbolic evolution equations (1a) — (1d).

In simulations divergence errors can be introduced by approximation errors if the divergence constraint (1e) is not explicitly taken into account. These errors can then increase with time and will result in unphysical solutions: Magnetic field lines can have wrong topologies leading to plasma transport *orthogonal* to the magnetic field. This effect is discussed among others by Brackbill and Barnes [BB80]. Even if the overall solution is not strongly affected by local divergence errors, they generally cause severe problems for the stability of the scheme.

In our numerical scheme we use a standard finite–volume approach with approximate Riemann solvers for the computation of the flux functions across cell interfaces [DRW99, DKRW01]. This requires an approximate solution of of (1a) — (1d) in one space dimension. In 1d the evolution equation for the normal magnetic field B_x (1c) and the divergence constraint (1e) read

$$\partial_t B_x = 0 \quad \text{and} \quad \partial_x B_x = 0. \qquad (2)$$

Therefore B_x may vary neither in space nor time. On the other hand the normal magnetic field component are in general discontinuous across a cell interface. Thus the value of B_x which we use for the solution of the one–dimensional Riemann problem is not constant as required by (2). This leads to unphysical numerical fluxes and the loss of stability in the two– and three–dimensional schemes.

In [DKK+02] we extend a correction method developed for the Maxwell equations in [MOS+00]. This method is based on a modified equation for the magnetic field and introduces a new unknown function ψ. The new equation for the magnetic field which replaces (1c) and the equation for the auxiliary function ψ are given by

$$\begin{aligned}
\partial_t \mathbf{B} + \nabla \cdot \left(\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \right) + \nabla\psi &= 0\,, && \text{(3)}\\
\mathcal{D}(\psi) + \nabla \cdot \mathbf{B} &= 0\,, && \text{(4)}
\end{aligned}$$