Adjoint equations in the presence of shocks

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Motivated by applications in design optimisation and error analysis, this paper is concerned with the formulation and discretisation of adjoint equations when there are shocks in the underlying solution to the original nonlinear hyperbolic p.d.e.

The theory is presented for the model problem of a scalar unsteady one-dimensional p.d.e. with a convex flux function. It is shown that the analytic formulation of the adjoint equation requires the imposition of an interior boundary condition along any shock. Looking ahead to three-dimensional flow applications, it would be extremely difficult in practice to apply such interior boundary conditions in a numerical approximation. The question then is whether it is possible to construct a convergent numerical approximation to the adjoint equation without explicitly enforcing this interior boundary condition.

An adjoint discretisation is defined by requiring the adjoint equations to give the same value for the linearised functional as a linearisation of the original nonlinear discretisation. It is proved that applying this technique to a class of explicit discretisations of the original p.d.e. yields a consistent approximation of the adjoint p.d.e. if the original solution is smooth. If the discretisation is also stable then the computed adjoint solution will converge to the analytic solution.

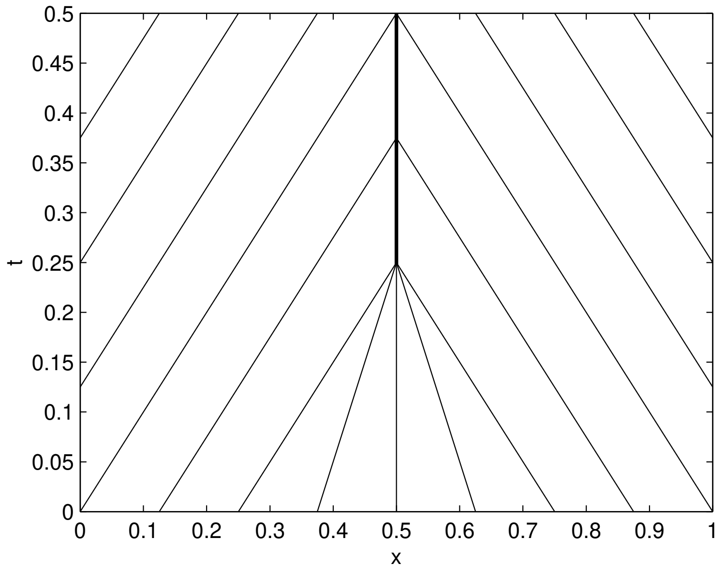

Figure 1: Characteristics with shock forming along $x = 0.5$

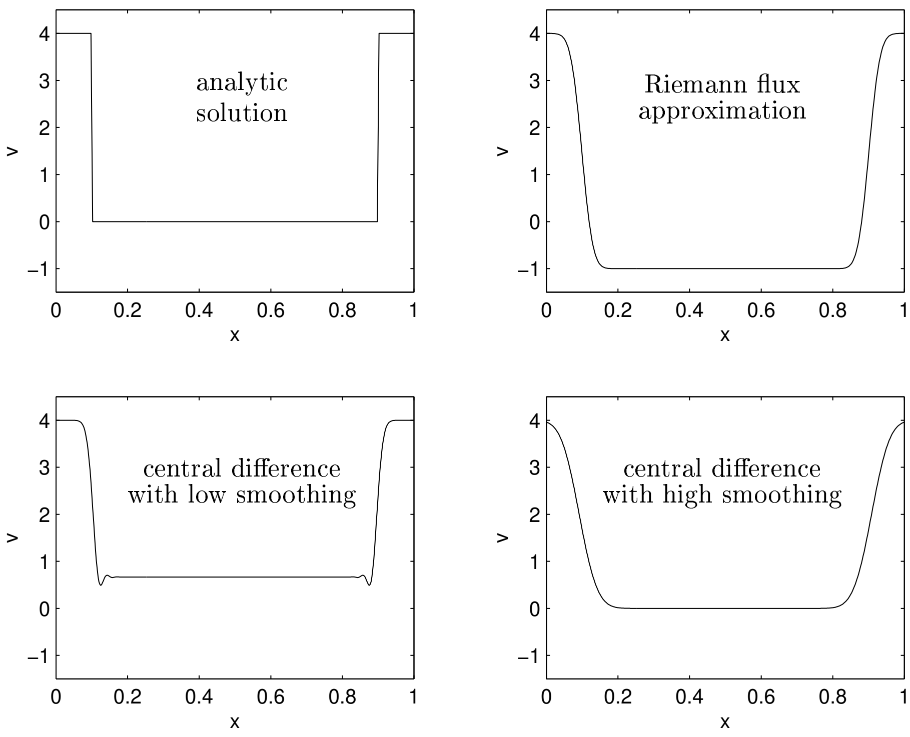

Figure 2: Analytic adjoint solution and three numerical approximations

To investigate whether convergence is obtained when there is a shock, numerical results are obtained for the Burgers equation with the initial and boundary conditions which yield the solution shown in Figure 1.

Figure 2 shows the analytic adjoint solution and three numerical approximations at time $t = 0.1$. All four correspond to the linearisation of an output functional which is the integral of a function of the nonlinear solution at the final time $t = 0.5$. Although the discontinuities are still a bit smeared, the numerical solutions are reasonably grid-converged, and it is evident that the first order Riemann solver and the central difference approximation with a very low level of numerical smoothing both yield results which differ significantly from the analytic solution in the region $0.1 < x < 0.9$. Only the central difference approximation with a high level of smoothing gives a computed adjoint which is in good agreement with the analytic solution.

The paper explains why it is generally the case that a convergent approximation requires that the number of grid points across the shock must increase as the mesh spacing and timestep are reduced. However, the numerical evidence suggests the numerical error decays exponentially with the number of points across the shock, so very accurate results can be obtained with just a few points in the shock.