Critical Thresholds and Conditional Stability for Euler Equations and Related Models

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When dealing with the questions of time regularity for Euler-related equations, one encounters several limitations with the classical stability analysis. Among others issues, we mention that

(i) the stability analysis does not tell us how large perturbations are allowed before losing stability, say with the incompressible Navier-Stokes equations;

(ii) the steady solution may be only conditionally stable due to the weak dissipation in the system, say in certain Euler-Poisson models.

In order to address these difficulties we propose a new notion of critical threshold (CT), which serves to describe the conditional stability for a class of Euler type equations.

We first discuss this remarkable CT phenomena associated with the Euler-Poisson equations, where the answer to questions of global smoothness vs. finite time breakdown depends on whether the initial configuration crosses an intrinsic, $O(1)$ critical threshold.

We investigate various one-dimensional problems with or without forcing mechanisms as well as multi-dimensional isotropic models with geometrical symmetry. The critical thresholds for these essentially 1-D problems are shown to depend on the relative size of the initial velocity slope and the initial density.

We then extend our discussion of the CT phenomena for multi-dimensional systems of the form $\partial_t u + u \cdot \nabla u = F$, which show up in different contexts dictated by the different modeling of F's. Here we utilize a novel description for the spectral dynamics of the (possibly complex) eigenvalues, $\lambda = \lambda(\nabla u)$ which are shown to be governed by the Ricatti-like equation $\lambda_t + u \cdot \nabla \lambda + \lambda^2 = \langle l, \nabla F r \rangle$. Restricting attention to restricted Euler-Poisson equations driven by localized forcing we identify the set of their $[n/2]$ global invariants, which in turn yield (i) sufficient conditions for finite time breakdown, and (ii) characterization of a large class of 2-dimensional initial configurations leading to global smooth solutions.

Moreover, the critical thresholds for 2D REPs are shown to depend on the relative size of three quantities at initial time: density, divergence and the spectral gap $\lambda_2 - \lambda_1$.

This lecture reflects recent joint investigations with Professor Eitan Tadmor.