Kinetic Approximations for the Incompressible Navier–Stokes Equations

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We present some recent results concerning a class of new kinetic approximations for the weak solutions to the Cauchy problem for the incompressible Navier Stokes equations in D space dimensions

$$\begin{cases} \partial_t U + \mathrm{div}(U \otimes U) + \nabla\phi = \nu\Delta U, \\ \nabla \cdot U = 0, \\ U(0,x) = U_0(x). \end{cases} \tag{1}$$

Our approximation is based on diffusive kinetic models with a finite number of velocities, which take the following general form. Fix the vector velocities $\lambda_i := (\lambda_{i1}, \ldots, \lambda_{iD})$, for $i = 1, \ldots, N$, and find a discrete kinetic (vector) distribution $\mathbf{f}_i = (f_i^0, f_i^1, \ldots, f_i^D) \in I\!R^{D+1}$, such that

$$\begin{cases} \partial_t \mathbf{f}_i + \frac{1}{\epsilon}\lambda_i \cdot \nabla_x \mathbf{f}_i = \frac{1}{\tau\epsilon^2}\left(\mathbf{M}_i(\rho, \epsilon\rho\mathbf{u}) - \mathbf{f}_i\right), \quad i = 1, \ldots, N \\ \mathbf{f}_i(x,0) = \mathbf{M}_i(\overline{\rho}, \epsilon\overline{\rho}U_0), \end{cases} \tag{2}$$

for some fixed positive constant $\overline{\rho} > 0$. Here we define the "macroscopic" variables

$$\rho := \sum_{i=1}^N f_i^0, \ \epsilon\rho u_l := \sum_{i=1}^N f_i^l.$$

for $l = 1, \ldots, D$. Moreover, we are using the discrete (vector) Maxwellian function $\mathbf{M}$, which is a smooth function of $D+1$ variables with $\mathbf{M}_i = (\mathbf{M}_i^0, \mathbf{M}_i^1, \ldots, \mathbf{M}_i^D) \in I\!R^{D+1}$, for $i = 1, \ldots, N$, and which verifies the following consistency conditions;

$$\sum_{i=1}^N M_i^0(\rho, \mathbf{q}) = \rho; \tag{3}$$

$$\sum_{i=1}^{N} M_i^l(\rho, \mathbf{q}) = \sum_{i=1}^{N} \lambda_{il} M_i^0(\rho, \mathbf{q}) = q_l; \tag{4}$$

$$\sum_{i=1}^{N} \lambda_{ij} M_i^l(\rho, \mathbf{q}) = \frac{q_j q_l}{\rho} + P(\rho)\delta_{jl}, \text{ (here take } P(\rho) = \kappa\rho^\gamma); \tag{5}$$

$$\sum_{i=1}^{N} \lambda_{ij} \lambda_{ik} \sum_r \partial_{q_r} M_i^l(\overline{\rho}, 0) u_r = \frac{\nu}{\tau} \delta_{jk} u_l \tag{6}$$

Our main goal is to prove the following result.

Theorem 1 (Bouchut & Natalini, 2002) *Assume that the BGK approximation has a bounded sequence of solutions* $(\rho^\epsilon, \mathbf{u}^\epsilon)$ *and the Maxwellian* $\mathbf{M}$ *is dissipative, which means that* $(M')^t \eta''$ *is symmetric, where* $\eta(\rho, q) = q^2/2\rho + \rho e(\rho)$, $de/d\rho = P(\rho)/\rho^2$, *and that the eigenvalues of* M' *are nonnegative. Then, as* $\varepsilon \to 0$, *and possibly passing to a sub-sequence, we have locally*

$$\begin{aligned}
&\rho^\epsilon - \overline{\rho} \to 0, && \text{in } L^2 : \\
&\mathbf{u}^\epsilon \rightharpoonup U, && \text{in } L^2 - (\text{weak}); \\
&\frac{P(\rho) - P(\overline{\rho})}{\overline{\rho}\varepsilon^2} \rightharpoonup \Phi, && \text{in } \mathcal{D}',
\end{aligned}$$

and (U, Φ) *are a (weak) solution to the Cauchy problem (1) for the incompressible Navier–Stokes equations.*

The key point of our construction is that these approximations are endowed with kinetic entropy functions, which yield useful energy inequalities. By using a suitably modified version of the compensated compactness, we are able to show the convergence of the approximated solutions. Also, from these approximations it is possible to generate numerical schemes, related to the Lattice BGK schemes, but with the supplementary and important features that they respect an H–Theorem and then they are nonlinearly stable. The properties and the accuracy of some special kinetic schemes will be presented and discussed in the talk, according to some further results obtained in collaboration with **M.F. Carfora** (IAC–CNR, Italy).

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