Implicit-Explicit Runge-Kutta schemes for hyperbolic systems with relaxation

Lorenzo Pareschi

Department of Mathematics, University of Ferrara, ITALY.

e-mail: `pareschi@dm.unife.it`

Several physical phenomena of great importance for applications are described by hyperbolic systems with relaxation terms, for example we mention discrete kinetic theory of rarefied gases, hydrodynamical models for semiconductors, viscoelasticity, linear and nonlinear waves, multiphase and phase transitions, radiations hydrodynamics, etc.

In one space dimension these systems have the form

$$\partial_t U + \partial_x F(U) = \frac{1}{\varepsilon} R(U), \quad x \in \mathbb{R}, \tag{1}$$

where $U = U(x,t) \in \mathbb{R}^N$, $F : \mathbb{R}^N \to \mathbb{R}^N$, the Jacobian matrix $F'(U)$ has real eigenvalues and $\varepsilon > 0$ is the relaxation time.

The development of efficient numerical schemes for such systems is challenging, since in many applications the relaxation time varies from values of order one to very small values if compared to the time scale determined by the characteristic speed of the system. In this second case the hyperbolic system with relaxation is said to be stiff and typically its solutions are well approximated by solutions of a suitable reduced set of conservation laws called equilibrium system [4].

Usually it is extremely difficult, if not impossible, to split the problem in separate regimes and to use different solvers in the stiff and non stiff regions. Thus one has to use the original relaxation system in the whole computational domain. The construction of scheme that works for all ranges of the relaxation time, using coarse grids that do not resolve the small relaxation time, has been studied mainly in the context of upwind methods using a method of lines approach combined with suitable operator splitting techniques [3, 7] and more recently in the context of central schemes [8, 9].

A very general and commonly used approach to the solution of this problem is based on splitting methods. A simple splitting consists in solving separately a non-stiff system of conservation laws without source ($R(U) \equiv 0$) applying an explicit scheme and, using an implicit scheme, a stiff system of ODEs ($F(U) \equiv 0$) for the source terms.

This splitting is restricted to first order accuracy in time, nevertheless its simple structure presents several advantages. In fact some properties of the solution are maintained (positivity, TVD property, other physically relevant properties), consistency with the equilibrium system in the limit of small relaxation times can be easily checked (asymptotic preservation) and in many cases the implicit scheme for the stiff system of ODEs can be explicitly solved thanks to some conservation properties of the system.

Higher order splitting can be constructed using suitable combinations of the two previous steps [6, 11]. Unfortunately all these higher order extensions present a

severe loss of accuracy when the source term is stiff [7]. Second order Runge-Kutta splitting which maintain the accuracy in the stiff limit have been constructed recently [3, 7].

In this talk we will present a unified approach of Runge-Kutta splitting schemes which provides a framework for the derivation of more general, accurate and efficient schemes. In particular, we show that these schemes are strictly related with the recently developed implicit-explicit (IMEX) Runge-Kutta schemes [1, 2]. An IMEX Runge-Kutta scheme consists of applying an implicit discretization to the source terms and an explicit one for the flux in the form

$$U^{(i)} \;\;=\;\; U^n + h \sum_{j=1}^{i-1} \tilde{a}_{ij} \partial_x F(U^{(j)}) + h \sum_{j=1}^{\nu} a_{ij} \frac{1}{\varepsilon} R(U^{(j)}), \qquad (2)$$

$$U^{n+1} \;\;=\;\; U^n + h \sum_{i=1}^{\nu} \tilde{w}_i \partial_x F(U^{(i)}) + h \sum_{i=1}^{\nu} w_i \frac{1}{\varepsilon} R(U^{(i)}). \qquad (3)$$

The matrices $\tilde{A} = (\tilde{a}_{ij})$, $\tilde{a}_{ij} = 0$ for $j \geq i$ and $A = (a_{ij})$ are $\nu \times \nu$ matrices such that the resulting scheme is explicit in F, and implicit in R.

Since the simplicity and efficiency of solving the algebraic equations corresponding to the implicit part of the discretization at each step is of paramount importance it is natural to consider diagonally implicit Runge-Kutta (DIRK) schemes for the souce terms ($a_{ij} = 0$, for $j > i$).

We show that most of the splitting schemes can be written in the formalism of IMEX Runge-Kutta schemes, where the implicit solver is a DIRK scheme. Similarly it is easy to write an IMEX Runge-Kutta scheme in splitting form. In particular, we derive general conditions that guarantee the asymptotic preserving property, i.e. the consistency of the scheme with the equilibrium system, and show that the implicit step can be solved, in many cases, every time we use a DIRK scheme. Accuracy, stability and TVD properties of these schemes are studied both analytically and numerically. Finally several applications of second and third order schemes obtained using ENO and WENO space discretizations [5] will be presented.

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