On unique continuation for the nonlinear Schrödinger equations

Joint work with Carlos E. Kenig and Luis Vega

§1. ABSTRACT

This talk is concerned with uniqueness properties of solutions of nonlinear Schrödinger equation of the form

$$i\partial_t u + \Delta u + F(u, \overline{u}) = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{1}$$

More precisely, we shall consider the following question :

Q : Let u_1, u_2 be solutions of the equation (1) with $(x,t) \in \mathbb{R}^n \times [0,1]$, belonging to an appropriate class X and such that for some domain $D \subset \mathbb{R}^n$, $D \neq \mathbb{R}^n$

$$u_1(x,0) = u_2(x,0), \text{ and } u_1(x,1) = u_2(x,1), \quad \forall\, x \in D. \tag{2}$$

Is $u_1 \equiv u_2$?

Before stating our results, we shall comment on previous related results.

For the case of the k-generalized KdV

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}, \quad k \in \mathbb{Z}^+, \tag{3}$$

it was shown in [KePoVe] that if

$$u \in C([0,1] : H^4(\mathbb{R})) \cap C^1([0,1] : H^1(\mathbb{R})), \tag{4}$$

are real solutions of the equation (3) such that for some $a \in \mathbb{R}$

$$u(x,0) = u(x,1) = 0, \quad \forall\, x > a, \ (\forall\, x < a), \tag{5}$$

then $u \equiv 0$.

Concerning the equation (1), in [Zh2] B-Y. Zhang answered question Q in the case

$$n = 1, \quad F = \alpha |u|^2 u, \ \alpha \in \mathbb{R}, \quad u_2 \equiv 0, \quad D = (-\infty, a) \ \ (\text{or } D = (a, \infty)), \tag{6}$$

for some $a \in \mathbb{R}$. The proof in [Zh2] is based on the inverse scattering theory (IST). It is not clear to us if in the case (6) the IST can be applied to obtain the desired result for any pair of solutions.

Other unique continuation results have been obtained under analyticity assumptions on the data, and under appropriate assumptions on the form of the non-linearity

(7) $$F = F(u, \overline{u}, \nabla_x u, \nabla_x \overline{u}),$$

see [Ha] and references therein, (since the equation for the difference of two solutions does not necessarily preserve the form of the non-linearity, it is not clear that such results extend to pairs of (analytic) solutions), or under analyticity assumptions on the non-linearity F, without analyticity of the data, but under the stronger assumption that $supp\, u(\cdot, t)$ is compact for all $t \in [0,1]$, (see [Bo]).

Our main result is the following.

Theorem 1.

Let $u_1,\, u_2 \in C([0,1] : H^s(\mathbb{R}^n)),\ s \geq max\{n/2^+; 2\}$ be two solutions of the equation

(8) $$i\partial_t u + \Delta u + F(u, \overline{u}) = 0,$$

where $F \in C^{[s]+1}(\mathbb{C} : \mathbb{C})$ with

(9) $$|F(u, \overline{u})| \leq c(|u|^{p_1} + |u|^{p_2}), \quad p_1,\, p_2 > 1,$$

and

(10) $$|\nabla\, F(u, \overline{u})| \leq c(|u|^{p_1 - 1} + |u|^{p_2 - 1}), \quad p_1,\, p_2 > 1,$$

If there exists Γ a convex cone strictly contained in a half-space such that

(11) $$u_1(x,0) = u_2(x,0), \quad u_1(x,1) = u_2(x,1), \quad \forall\, x \notin \Gamma + y_0, \quad y_0 \in \mathbb{R}^n.$$

Then $u_1 \equiv u_2$.

Remarks

a) In the one dimensional case our assumption on the complement of $\Gamma + y_0$, i.e. $(\Gamma + y_0)^c = D$ reduces to a semi-line (a, ∞) (or $(-\infty, a)$). Also we observe that the class of nonlinearities F considered is very general. In particular, it does not contain any analyticity hypothesis on F.

b) In contrast to our approach in [KePoVe], here we do not rely on estimates of the type found in [KeRuSo], [KeSo]. In fact, the proof given below for Theorem 1 can be slightly modified to obtain a different proof of the results in [KePoVe] without using the results in [KeRuSo], [KeSo].

c) As in [KePoVe], we need an appropriate local Carleman estimate. In [KePoVe], for the case of the generalized KdV equation, we used a unique continuation result due to Saut-Scheurer [SaSc]. For the equations considered here, we will apply the local unique continuation results of V. Isakov [Is].

d) We do not know if the result of Theorem 1 is still valid for the case where Γ_{x_0} is just a semispace. This question seems to be related to problems considered in [Zu].

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