

High Order Numerical Methods for Convection Dominated PDEs

Chi-Wang Shu

Division of Applied Mathematics
Brown University
Providence, Rhode Island 02912, U. S. A.
E-mail: shu@cfm.brown.edu

ABSTRACT

In this talk we will present an overview of some recent progress in high order finite difference and finite volume weighted essentially non-oscillatory (WENO) and finite element discontinuous Galerkin (DG) methods for solving hyperbolic conservation laws and in general convection dominated PDEs, such as convection diffusion equations with small diffusion, and KdV type equations with small dispersion.

Among these three methods, the finite difference WENO method is the simplest to implement and the fastest in running time for multi-dimensional problems. Finite difference WENO schemes are available for orders of accuracy up to 11 (or even higher), but in most applications the fifth order version is a good choice. The method is a “blackbox” with no parameters to tune, and is extremely stable for strong shocks. It is also easy to implement the method on parallel machines with excellent parallel efficiency. The method is suitable especially for problems involving both shocks and complex smooth region structure in the solution. It has however very strong requirements on the smoothness of the meshes, hence is suitable only for problems on either rectangular geometry or domains covered by smooth curvilinear coordinates.

After a brief overview of the method, we will talk about the following recent progress: (1) Resolution study of using the method with different orders of accuracy on problems with discontinuities as well as complex smooth region structures. The double Mach flow problem and the Rayleigh-Taylor instability problem are used as examples. The conclusion is that, the ninth order WENO scheme uses only half the number of points in each direction to obtain a comparable resolution (by “eye-ball norm”) comparing with a fifth order WENO scheme, indicating that it does pay to use higher order methods for such problems. This is a joint work with Jing Shi and Yongtao Zhang; (2) Multi-domain finite difference WENO simulations. This is for the purpose of relaxing the requirement that the mesh be smooth everywhere. The computational domain is covered by several slightly overlapping rectangular domains and WENO interpolation with comparable order of accuracy to transfer information between subdomains. Numerical study indicate that the method is “essentially conservative”, meaning that the conservation error goes to zero with a mesh refinement, even for solutions with very strong shocks. The inter-domain interpolation is also very stable when shocks pass through the subdomain boundaries, and uniform high order accuracy is maintained. This is a joint work with Kurt Sebastian. (3) Simulations of the jet problem in astrophysics. This is a joint work with Carl Gardner, Youngsoo Ha and Anne Gelb.

The finite volume WENO method is based on the same designing principles as the

finite difference version in terms of the mechanism to achieve non-oscillatory solutions, but it is based on an integral version of the PDE. The finite volume WENO method is applicable to essentially arbitrary geometry and does not require any smoothness of meshes. Hence it is very suitable for the situation of complex geometry and adaptive computation. However, it is much more costly for multi-dimensional problems. After a brief overview of the method, we will talk about the recent joint work with Jing Shi and Changqing Hu on treating negative linear weights, which arise for high order finite volume WENO schemes.

The discontinuous Galerkin method is very similar to the finite volume WENO method. It is based on the same integral version of the PDE. The only difference is that, instead of relying on a reconstruction from cell averages to obtain a high order accuracy approximation as in finite volume WENO method, the DG method evolves all the degrees of freedoms for a high order polynomial in each cell, hence no reconstruction is needed. This certainly saves the reconstruction time but on the other hand increases the storage requirement as well as evolution time for all these degrees of freedoms. DG method relies on total variation bounded (TVB) limiters to control numerical oscillations for solutions with shocks. DG methods are especially suitable for parallel implementations and adaptivity, including both h adaptivity and p adaptivity (varying orders of accuracy in different cells, for which finite difference methods are difficult to do).

After a brief overview of the method, we will talk about the following recent progress: (1) Stable and accurate DG formulation for problems containing higher derivatives but are still convection dominated, such as convection diffusion equations with small diffusion coefficients or KdV type equations with small dispersions. This is a joint work with Jue Yan. (2) A post-processing technique which can effectively double the order of accuracy for the method on locally uniform meshes, with small extra computational cost. This is joint work with Bernardo Cockburn, Mitch Luskin and Endre Suli, with Jennifer Ryan, and with Jue Yan.