High Order Finite Difference Methods for Multiscale Complex Compressible Flows

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The classical way of analyzing finite difference schemes for hyperbolic problems is to investigate as many as possible of the following points

- Linear stability for constant coefficients
- Linear stability for variable coefficients
- Non-linear stability
- Stability at discontinuities

We will build a new numerical method, which satisfies all types of stability, by dealing with each of the points above step by step.

In addition to stability, there are other requirements for the numerical method. Such requirements can be

- Ability to accurately follow waves for long times.
- Ability to resolve turbulence, and other small scale phenomena.
- Efficient use of computational time.
- Efficient parallelization.

TVD schemes are too dissipative for turbulence simulations, and ENO schemes demand very high computational resource. We will here describe methods which can meet the above requirements, but which do not belong to the class of standard shock capturing schemes.

To assure a correct treatment of boundaries, we use finite difference operators having the so called summation by parts property (SBP). To improve nonlinear stability (or stability for linear variable coefficients) of the numerical computations we employ skew-symmetric splitting of the convective flux derivatives. As an alternative/complement to splitting, we use linear artificial dissipation of high order. Finally, for good resolution of shock waves we take the non-trivial artificial viscosity

[1]Part of this work was carried out while the author was a visiting scientist with RIACS, NASA Ames Research Center.

from a second order TVD scheme, and insert it into the method only near discontinuities. In order to switch on this viscosity only where it is needed, we use a special detection algorithm, based either on gradients or on wavelets.

This leads to the point of view, that inventing a new numerical method for hyperbolic conservation laws is no longer a problem of coming up with a clever formula relating the new time level to the old one, but rather a problem of designing a system by connecting known components together with switching mechanisms.

There is a large degree of freedom in how to connect the components. For example, the linear dissipation operator can be evaluated as a part of the residual, or added as a post processing step. If a Runge-Kutta method is used in time, the dissipation can be applied to each stage, or only at certain stages. There is also the choice of whether the linear dissipation is switched off when the TVD-dissipation is switched on, or if it is applied in full strength at all grid points.

The numerical method will be demonstrated on several examples to show its possibilities. In all problems below we use as the basic scheme, a sixth order centered difference operator, modified at boundaries to have the SBP property. Time integration is done with the classical fourth order Runge-Kutta method. Additonal examples and a more detailed discussion of the results will be given during the lecture.

Vortex convection

This problem serves as an illustration for non-linear stability, and long-time integration. The effects of entropy splitting and artificial dissipation will be demonstrated.

The solution consists of as single vortex which is translated with uniform velocity periodically around in a square. The vortex translation is an exact solution of the compressible Euler equations of gas dynamics. The solution is smooth for all times. Furthermore, since the boundary conditions are periodic, we do not need to involve boundary modifications of the operators. The problem will demonstrate in a very "clean" way, how numerical methods deal with non-linear effects.

In aeroacoustics, the interest is to follow weak waves for long times. The requirements of good accuracy after long integration times is very difficult to satisfy. Applications from rotorcraft machinery, typically require that a vortex is followed a distance which corresponds to $300 - 1000$ periods. In the computation of turbulence by direct numerical simulations, statistics of time accurate computations are taken for a very long time as well.

The computations show that on a given grid, non-linear instabilities destroy the solution after 5 periods when the pure centered scheme is used. This is illustrated in Fig. 1. With entropy splitting the break down comes after 60 periods instead, and the norm of the entropy is decreasing all the way to break down. After introducing an

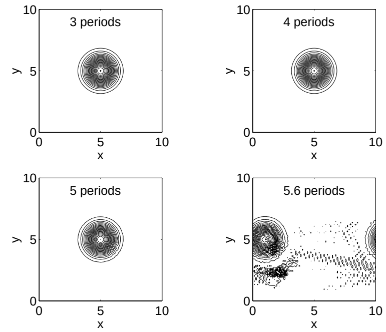

Figure 1: Density contours. Non-linear instability in vortex convection.

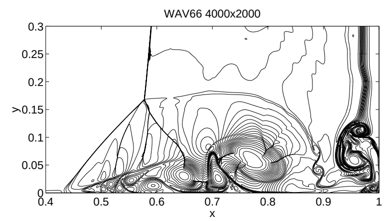

Figure 2: Density contours. Small scale structure in shock tube problem.

eight order linear dissipation operator the solution can be computed with accuracy up to around 200 periods. However, tuning of the dissipation strength is a very sensitive issue, and will be discussed in some detail.

Compressible Viscous Shock/Shear/Boundary-layer Interactions

For this problem, we increase the complexity of the numerical treatment, by adding a low order dissipation term at the shock waves that appear. The problem demonstrates benefits of high order accuracy.

The ideal gas compressible full Navier-Stokes equations with no slip BCs at the adiabatic walls are used. The fluid is at rest in a 2-D box $0 \leq x, y \leq 1$. A membrane with a shock Mach number of 2.37 located at $x = 1/2$ separates two different states of the gas. $\gamma = 1.4$, the Prandtl number is 0.73, and the Reynolds number is 1000.

The solution is shown in Fig. 2. The small scale features in the solution has turned out to be very sensitive to the numerical method used, unless the grid is very fine. In order to fully resolve the vortex structure, the 6th order method needed a grid of 8 million grid points in two space dimensions. Although strong shocks are present, TVD and ENO methods were not usable, either due to insufficient accuracy or due to too long computational time.