Nonlinear Boundary Layers of the Boltzmann Equation

Seiji Ukai[†], Tong Tang[‡] and Shih-Hsien Yu[‡]

[†]Department of Applied Mathematics, Yokohama National University
79-5 Tokiwadai, Hodogaya, Yokohama 240-8501, Japan

[‡]Department of Mathematics, City University of Hong Kong
83 Tat Chee Avenue, Kowloon Tong, Hong Kong

We discuss the half-space problem of the Boltzmann equation,

$$
(1) \qquad \begin{cases} \xi_1 F_x = Q(F, F), & x \in (0, \infty),\ \xi \in \mathbb{R}^3, \\ F|_{x=0} = F_0(\xi), & \xi_1 > 0, (\xi_2, \xi_3) \in \mathbb{R}^2, \\ F \to M_\infty(\xi) \quad (x \to \infty), & \xi \in \mathbb{R}^3, \end{cases}
$$

where the unknown $F = F(x, \xi)$ is the mass density distribution of gas particles at position $x \in (0, \infty)$ with velocity $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, while ξ_1 is the x-component of ξ and Q, the *collision operator*, is a quadratic integral operator in ξ whose integral kernel is determined by the interaction potential of the gas particle. So far, our result is proved only for the hard ball gas, but the same result seems to hold for general cutoff hard potentials.

The second equation in (1) is the Dirichlet boundary condition at the boundary $x = 0$. The Dirichlet data $F_0(\xi)$ is assigned only for incoming particles from the boundary, i.e. for $\xi_1 > 0$. Physically this is natural because we can control only the incoming distribution but not the outgoing($\xi_1 < 0$) distribution. Mathematically, this is a well posed boundary condition. It is known that assigning the outgoing distribution makes the problem (1) ill-posed.

The third equation of (1) specifies the far field. This is the Dirichlet boundary condition at $x = \infty$, and is assigned for all $\xi \in \mathbb{R}^3$. Then, two remarks are to follow. One is that the far field M_∞ cannot be arbitrary but must be a zero of Q, that is, a *Maxwellian*,

$$
(2) \qquad M_\infty(\xi) = \frac{\rho_\infty}{(2\pi T_\infty)^{3/2}} \exp\Big(-\frac{|\xi - u_\infty|^2}{2T_\infty}\Big),
$$

and $\rho_\infty > 0, u_\infty = (u_{\infty,1}, u_{\infty,2}, u_{\infty,3}) \in \mathbb{R}^3$, and $T_\infty > 0$ are the only quantities which we can control. By a shift of ξ_2, ξ_3, we can assume without loss of generality that $u_{\infty,2} = u_{\infty,3} = 0$, and then, the sound speed and Mach number of the equilibrium state described by (2) are given by

$$
c_\infty = \sqrt{\frac{5}{3} T_\infty}, \qquad \mathcal{M}^\infty = \frac{u_{\infty,1}}{c_\infty},
$$

respectively. The other remark is that since the outgoing distribution at $x = \infty$ (i.e. for $\xi_1 > 0$) is assigned, the problem (1) may become ill-posed and hence only conditionally solvable. Indeed, we will show that the solvability condition changes with $\mathcal{M}^\infty$ as follows.

(a) If $\mathcal{M}^\infty < -1$, the problem (1) admits a unique smooth solution for any F_0 sufficiently close to M_∞.

(b) If $\mathcal{M}^\infty > -1$, such a solution exists only for F_0 close to M_∞ and satisfying certain admissible conditions. The set of admissible F_0 forms a smooth manifold whose co-dimension is 1 for the case $0 < \mathcal{M}^\infty < -1$, 4 for $0 < \mathcal{M}^\infty < 1$ and 5 for $\mathcal{M}^\infty > 1$, respectively.

The problem (1) arises in the theory of the kinetic boundary layer, the analysis of the condensation-evaporation and so on. The corresponding linearized problem has been studied by many authors, e.g. [2],[3],[4],[5], mainly in the context of the classical Milne and Kramers problems and hence with auxiliary conditions on boundary fluxes. In [6], an existence theorem was established for the nonlinear case with the specular boundary condition but the method of proof does not apply to other boundary conditions, especially the Dirichlet condition. Recently, nonlinear existence and stability theorems have been established for the discrete velocity model of the Boltzmann equation [7], [9]. Our result is the first existence theorem on the full nonlinear problem. Furthermore, it provides a new aspect of the linearized problem and also a partial proof of the numerical results established in [1], [8], on (1) with F_0 fixed to be the standard Maxwellian. The talk will include the details on these points as well as the idea of proof and the stability of our stationary solution.

References

[1] K. Aoki, K. Nishino, Y. Sone, & H. Sugimoto, Phys. Fluids A **3** (1991), 2260–2275.

[2] C. Bardos & R. E. Caflish & B. Nicolaenko, Comm. Pure Appl. Math. **49**(1986), 323-352.

[3] C. Cercignani, in" Trends in Applications of Pure Mathematics to Mechanics" (eds. E. Kröner and K. Kirchgässner), Springer-Verlag, Berlin, 1986, pp35–50.

[4] F. Coron, F. Golse, & C. Sulem , Commun. Pure Appl. Math. **41**(1988), 409–435.

[5] F. Golse & F. Poupaud, Math. Methods Appl. Sci., **11**(1989), 483-502.

[6] F. Golse, B. Perthame, & C. Sulem, Arch. Rational Mech. Anal. **103**(1988), 81-96.

[7] S. Kawashima, S. Nishibata, Commun. Math. Phys. **207** (1999), 385-409 & Commun. Math. Phys., **211** (2000), 183-206.

[8] Y. Sone, *Kinetic Theory and Fluid Dynamics*, to appear.

[9] S. Ukai, in "Advances in Nonlinear Partial Differential Equations and Stochastic" (eds. S. Kawashima and T. Yangisawa), Series on Advances in Mathematics for Applied Sciences, Vol. 48, World Scientific, Singapore-New York, 1998, pp. 160-174.