Shock Reflection-Diffraction and Multidimensional Conservation Laws

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Website: http://www.math.pitt.edu/~dwang/FRG.html

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Bow Shock in Space generated by a Solar Explosion

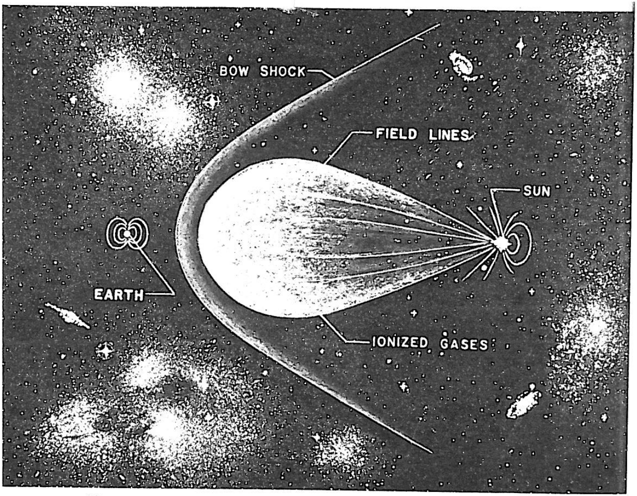


FIG. 50: SOLAR EXPLOSION

A shock wave in space generated by a solar eruption. The sketch shows the fully ionized nucleons attached to the solar magnetic field lines acting as the driving piston for the shock wave. (Courtesy: UTIAS, after Gold, 1962).

Shock Waves generated by Blunt-Nosed and Shape-Nosed Supersonic Aircrafts

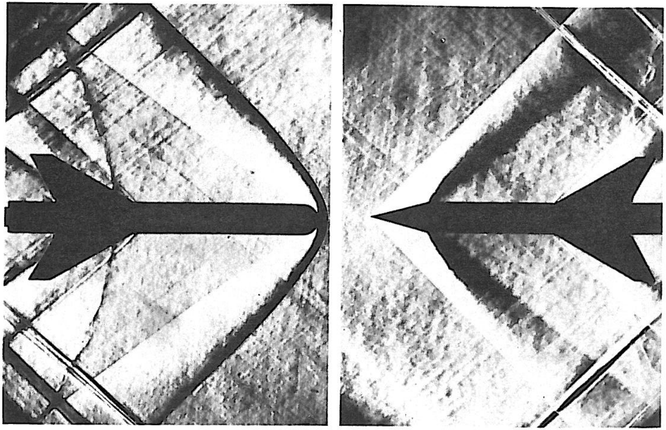


FIG. 41: SHOCK WAVES ABOUT MODEL AEROSPACECRAFT

Schlieren photographs of the wave systems generated about blunt-nosed and sharp-nosed supersonic models at a Mach number $M = 2.5$ in the UTIAS 16 x 16 inch supersonic wind tunnel. (Courtesy: UTIAS).

Blast Wave from a TNT Surface Explosion

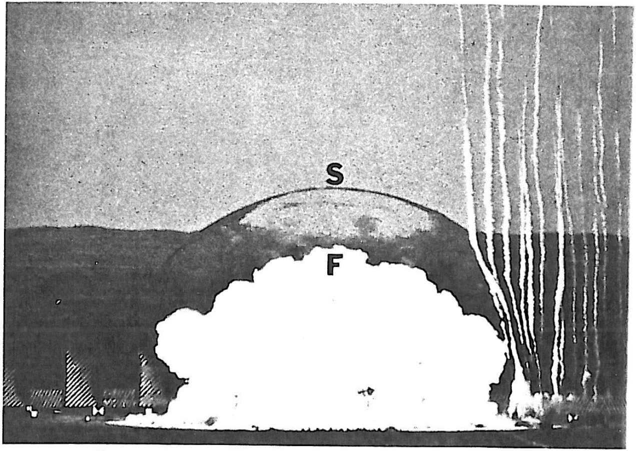


FIG. 22: EXPLOSION FROM A 20-TON HEMISPHERE OF TNT

The blast wave S, and fireball F, from a 20-ton TNT surface explosion are clearly shown. The backdrops are 50 feet by 30 feet and in conjunction with the rocket smoke trails, it is possible to distinguish shock waves and particle paths and to measure their velocities. Owing to unusual daylight conditions, the hemispherical shock wave became visible. (Courtesy: Defence Research Board of Canada).

Shock Wave from an Underwater Nuclear Explosion

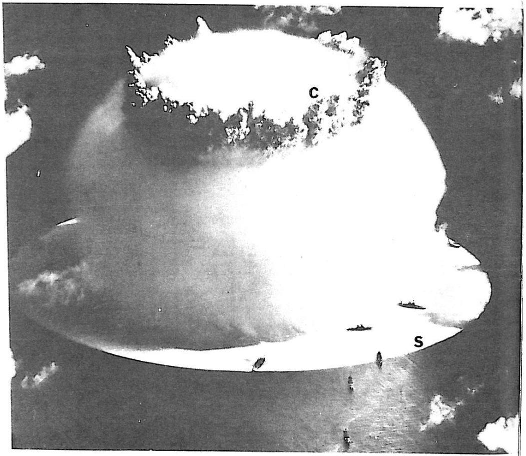
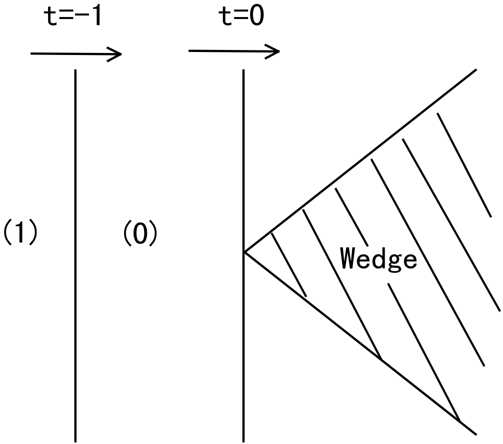


FIG. 33: AN UNDERWATER NUCLEAR EXPLOSION

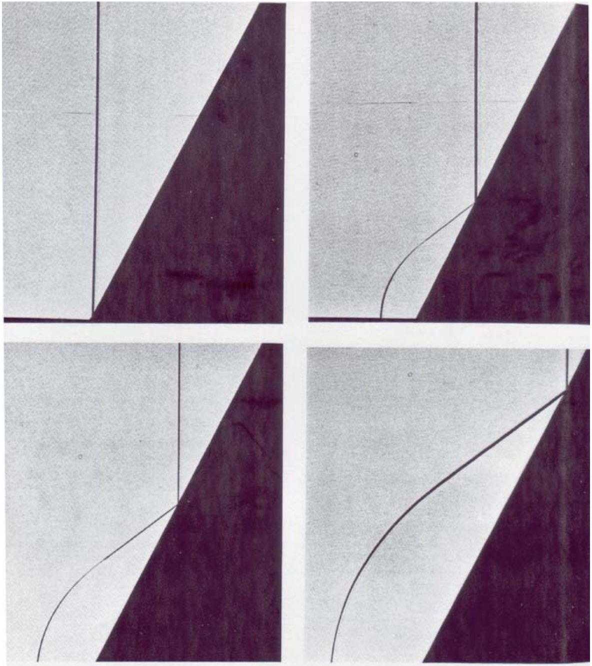
The condensation cloud C, formed just after a shallow underwater nuclear explosion, and the slick S, due to the shock wave on the surface, are clearly illustrated. An appreciation of the tremendous size of the blast zone can be obtained by comparing it with the old destroyers and other naval vessels used in the test. (Courtesy: U.S. Atomic Energy Commission).

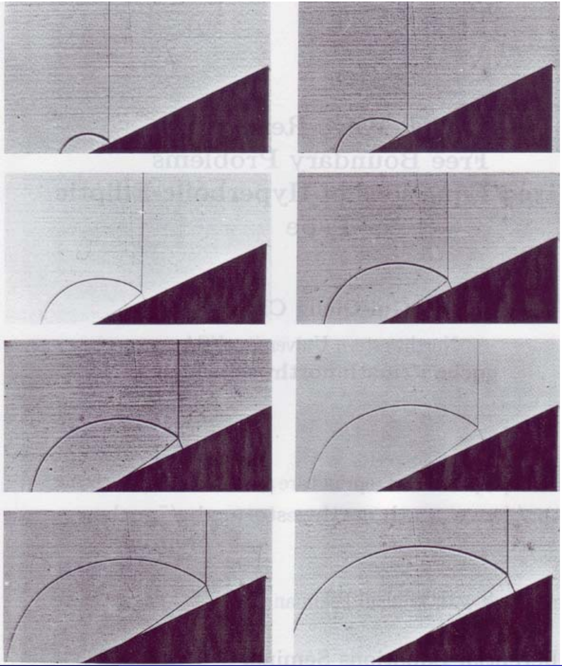


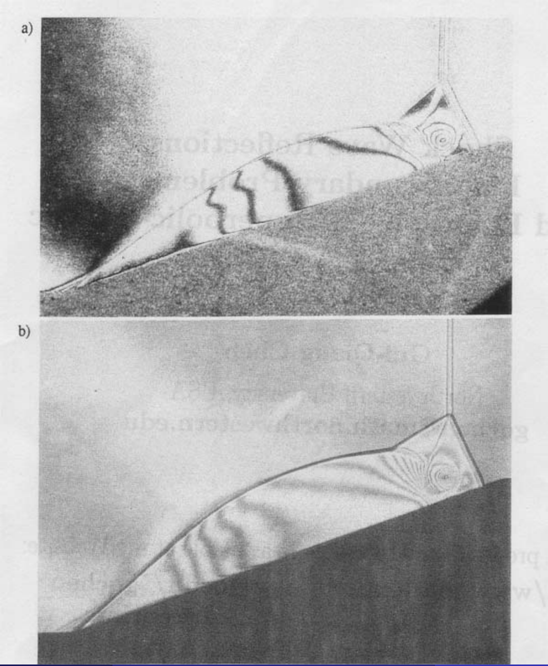
? Shock Wave Patterns Around a Wedge (airfoils, inclined ramps, $\cdots$)

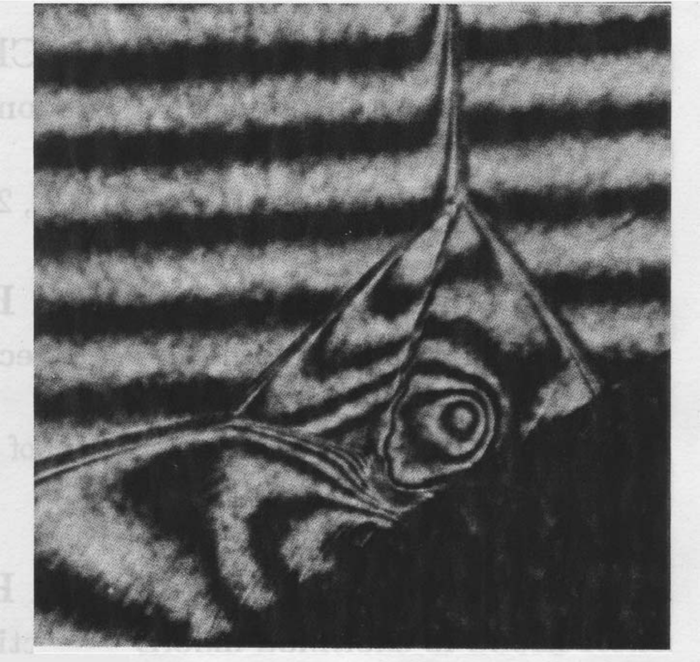
Complexity of Reflection-Diffraction Configurations Was First Identified and Reported by Ernst Mach 1879

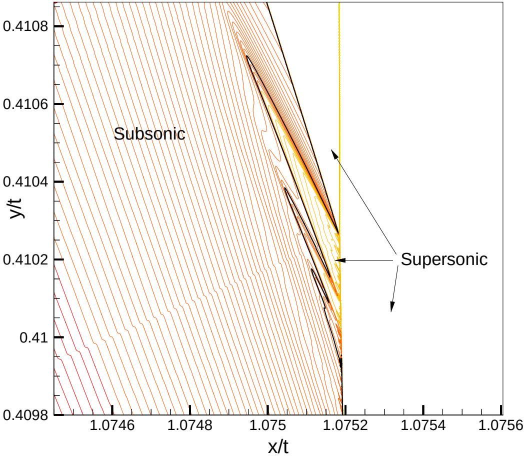
Experimental Analysis: 1940s$\Longrightarrow$: von Neumann, Bleakney, Bazhenova Glass, Takyama, Henderson, $\cdots$






a)
b)





Guderley Mach Reflection:

A. M. Tesdall and J. K. Hunter: TSD, 2002

A. M. Tesdall, R. Sanders, and B. L. Keyfitz: NWE, 2006; Full Euler, 2008

B. Skews and J. Ashworth: J. Fluid Mech. 542 (2005), 105-114

Shock Reflection-Diffraction Patterns

- **Gabi Ben-Dor** Shock Wave Reflection Phenomena
  Springer-Verlag: New York, 307 pages, 1992.

  Experimental results before 1991
  Various proposals for transition criteria

- **Milton Van Dyke** An Album of Fluid Motion
  The parabolic Press: Stanford, 176 pages, 1982.

  Various photographs of shock wave reflection phenomena

- **Richard Courant & Kurt Otto Friedrichs**
  Supersonic Flow and Shock Waves
  Springer-Verlag: New York, 1948.

Scientific Issues

- **Structure of the Shock Reflection-Diffraction Patterns**
- **Transition Criteria among the Patterns**
- **Dependence of the Patterns on the Parameters**

  wedge angle θ_w, adiabatic exponent $\gamma \geq 1$

  incident-shock-wave Mach number M_s
- $\cdots\cdots$

Interdisciplinary Approaches:

- **Experimental Data and Photographs**
- **Large or Small Scale Computing**

  Colella, Berger, Deschambault, Glass, Glaz, ...

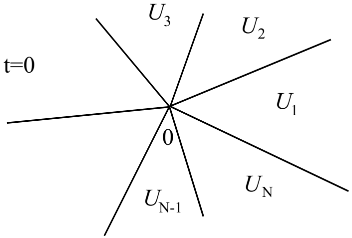
  Anderson, Hindman, Kutler, Schneyer, Shankar, ...

  Yu. Dem'yanov, Panasenko, ...
- **Asymptotic Analysis**: Keller, Lighthill, Hunter, Majda, Rosales, Tabak, Gamba, Harabetian, Morawetz....
- **Rigorous Mathematical Analysis** (Global Analysis?)

  Existence, Stability, Regularity, Bifurcation, $\cdots\cdots$

2-D Riemann Problem for Hyperbolic Conservation Laws

$$\partial_t U + \nabla \cdot F(U) = 0, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$$



Books and Survey Articles

Glimm-Majda 1991, Chang-Hsiao 1989, Li-Zhang-Yang 1998
Zheng 2001, Chen-Wang 2002, Serre 2005, Chen 2005, $\cdots$

Numerical Simulations

Glimm-Klingenberg-McBryan-Plohr-Sharp-Yaniv 1985
Schulz-Rinne-Collins-Glaz 1993, Chang-Chen-Yang 1995, 2000
Lax-Liu 1998, Kurganov-Tadmor 2002, $\cdots$

Riemann Solutions I

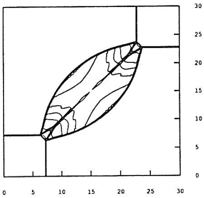


FIG. 5.5A
Density contour curves

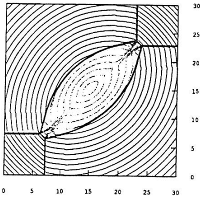


FIG. 5.5B
Self-Mach number contour curves

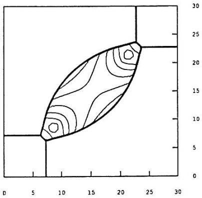


FIG. 5.5C. Pressure contour curves

Riemann Solutions II

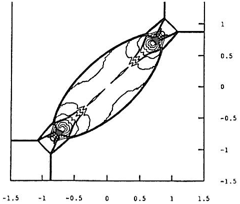


FIG. 5.6A
Density contour curves

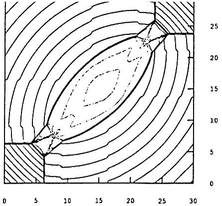


FIG. 5.6B
Self-Mach number contour curves

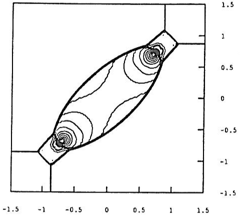


FIG. 5.6C. Pressure contour curves

Asymptotic States and Attractors

Observation (C–Frid 1998):

- Let $R(\frac{x}{t})$ be the unique piecewise Lipschitz continuous Riemann solution with Riemann data: $R|_{t=0} = R_0(\frac{x}{|x|})$
- Let $U(t,x) \in L^\infty$ be an entropy solution with initial data:

$$U|_{t=0} = R_0(\frac{x}{|x|}) + P_0(x), \qquad R_0 \in L^\infty(S^{d-1}), P_0 \in L^1 \cap L^\infty(\mathbb{R}^d)$$

- The corresponding self-similar sequence $U^T(t,x) := U(Tt, Tx)$ is compact in $L^1_{loc}(\mathbb{R}^{d+1}_+)$

$$\Longrightarrow \quad \operatorname{ess}\lim_{t\to\infty} \int_\Omega |U(t,t\xi) - R(\xi)|\, d\xi = 0 \qquad \text{for any } \Omega \subset \mathbb{R}^d$$

Building Blocks and Local Structure

Local structure of entropy solutions

Building blocks for numerical methods

Full Euler Equations (E-1): $(t, \mathbf{x}) \in \mathbb{R}^3_+ := (0, \infty) \times \mathbb{R}^2$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \partial_t(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0 \\ \partial_t(\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e) + \nabla \cdot ((\frac{1}{2}\rho|\mathbf{v}|^2 + \rho e + p)\mathbf{v}) = 0 \end{cases}$$

Constitutive Relations: $p = p(\rho, e)$

- ρ–density, $\mathbf{v} = (v_1, v_2)^\top$–fluid velocity, p–pressure
- e–internal energy, θ–temperature, S–entropy

For a polytropic gas: $p = (\gamma - 1)\rho e, \ \ e = c_v \theta, \qquad \gamma = 1 + \frac{R}{c_v}$

$$p = p(\rho, S) = \kappa \rho^\gamma e^{S/c_v}, \qquad e = e(\rho, S)\frac{\kappa}{\gamma - 1}\rho^{\gamma - 1} e^{S/c_v},$$

- $R > 0$ may be taken to be the universal gas constant divided by the effective molecular weight of the particular gas
- $c_v > 0$ is the specific heat at constant volume
- $\gamma > 1$ is the adiabatic exponent, $\kappa > 0$ is any constant under scaling

Euler Equations: Isentropic or Isothermal (E-2)

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0 \\ \partial_t (\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla p = 0 \end{cases}$$

where the pressure is regarded as a function of density with constant S_0:

$$p = p(\rho, S_0).$$

For a polytropic gas,

$$p(\rho) = \kappa_0 \rho^\gamma, \quad \gamma > 1 \quad (\gamma = 2 \ \text{ also for the shallow water equations})$$

For an isothermal gas,

$$p(\rho) = \kappa_0 \rho \qquad (i.e. \quad \gamma = 1)$$

where $\kappa_0 > 0$ is any constant under scaling

Euler Equations for Potential Flow (E-3): $\mathbf{v} = \nabla\Phi$

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \nabla \Phi) = 0, \\ \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1}; \end{cases}$$

or, equivalently,

$$\partial_t \rho(\nabla\Phi, \partial_t\Phi, \rho_0) + \nabla \cdot \big(\rho(\nabla\Phi, \partial_t\Phi, \rho_0)\nabla\Phi\big) = 0,$$

with

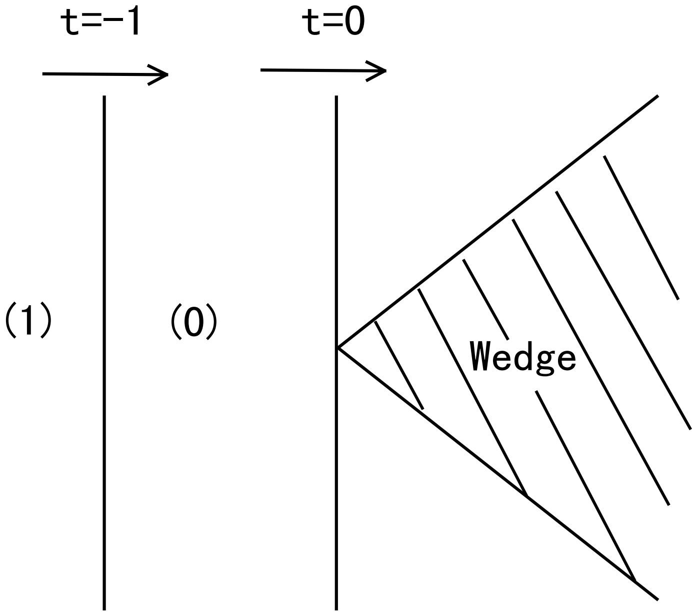
$$\rho(\nabla\Phi, \partial_t\Phi, \rho_0) = \big(\rho_0^{\gamma-1} - (\gamma-1)(\partial_t\Phi + \frac{1}{2}|\nabla\Phi|^2)\big)^{\frac{1}{\gamma-1}}.$$

Celebrated steady potential flow equation of aerodynamics:

$$\nabla \cdot (\rho(\nabla\Phi, \rho_0)\nabla\Phi) = 0.$$

This approximation is well-known in transonic aerodynamics.

We will see the Euler equations for potential flow is actually EXACT in an important region of the solution to the shock reflection problem.

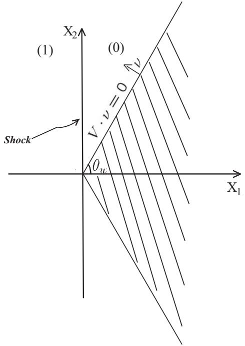

t=-1
t=0
(1)
(0)
Wedge

Initial-Boundary Value Problem: $0 < \rho_0 < \rho_1,\ v_1 > 0$

Initial condition **at** $t = 0$:

$$(\mathbf{v}, p, \rho) = \begin{cases} (0, 0, p_0, \rho_0), & |x_2| > x_1 \tan\theta_w, x_1 > 0, \\ (v_1, 0, p_1, \rho_1), & x_1 < 0; \end{cases}$$

Slip boundary condition **on the wedge bdry:** $\mathbf{v}\cdot\nu = 0$**.**



Invariant under the Self-Similar Scaling: $(t, \mathbf{x}) \longrightarrow (\alpha t, \alpha\mathbf{x}),\ \ \alpha \neq 0$

Seek Self-Similar Solutions

$$(\mathbf{v}, p, \rho)(t, \mathbf{x}) = (\mathbf{v}, p, \rho)(\xi, \eta), \quad (\xi, \eta) = (\tfrac{x_1}{t}, \tfrac{x_2}{t})$$

$$\begin{cases} (\rho U)_\xi + (\rho V)_\eta + 2\rho = 0, \\ (\rho U^2 + p)_\xi + (\rho UV)_\eta + 3\rho U = 0, \\ (\rho UV)_\xi + (\rho V^2 + p)_\eta + 3\rho V = 0, \\ (U(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\xi + (V(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}))_\eta + 2(\frac{1}{2}\rho q^2 + \frac{\gamma p}{\gamma - 1}) = 0, \end{cases}$$

where $q = \sqrt{U^2 + V^2}$ and $(U, V) = (v_1 - \xi, v_2 - \eta)$ is the pseudo-velocity.

Eigenvalues: $\lambda_0 = \frac{V}{U}$ (repeated), $\lambda_\pm = \frac{UV \pm c\sqrt{q^2 - c^2}}{U^2 - c^2}$,
where $c = \sqrt{\gamma p / \rho}$ is the **sonic speed**

When the flow is pseudo-subsonic: $q < c$, the system is

hyperbolic-elliptic composite-mixed

Euler Equations in Self-Similar Coordinates

Entropy: $S = c_v\, ln(p\rho^\gamma)$

Pseudo-velocity Angle: $\lambda_0 = V/U = \tan\Theta$

Pseudo-velocity Magnitude: $q = \sqrt{U^2 + V^2}$

$$\begin{cases} S_\xi + \lambda_0 S_\eta = 0, \\ \rho q(q_\xi + \lambda_0 q_\eta) + p_\xi + \lambda_0 p_\eta = -\rho(U + \lambda_0 V), \\ (U^2 - c^2)p_{\xi\xi} + 2UVp_{\xi\eta} + (V^2 - c^2)p_{\eta\eta} + A_1 p_\xi + A_2 p_\eta + \cdots = 0, \\ (U^2 - c^2)\lambda_{0\xi\xi} + 2UV\lambda_{0\xi\eta} + (V^2 - c^2)\lambda_{0\eta\eta} + A_1\lambda_{0\xi} + A_2\lambda_{0\eta} + \cdots = 0. \end{cases}$$

When the flow is pseudo-subsonic: $q < c$, the system consists of

- 2-transport equations
- 2-nonlinear equations of hyperbolic-elliptic mixed type

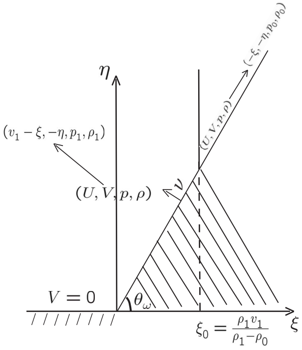
Boundary Value Problem in the Unbounded Domain

Slip boundary condition on the wedge boundary:

$$(U, V) \cdot \nu = 0 \qquad \textbf{on} \quad \partial D$$

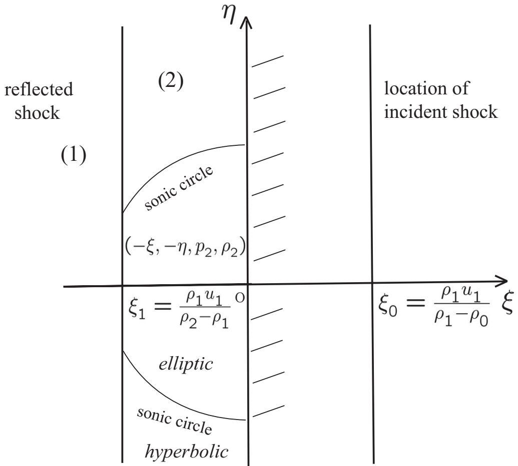
Asymptotic boundary condition as $\xi^2 + \eta^2 \to \infty$:

$$(U+\xi, V+\eta, p, \rho) \to \begin{cases} (0, 0, p_0, \rho_0), & \xi > \xi_0, \eta > \xi \tan\theta_w, \\ (v_1, 0, p_1, \rho_1), & \xi < \xi_0, \eta > 0. \end{cases}$$



Normal Reflection

When $\theta_w = \pi/2$, the reflection becomes the normal reflection, for which the incident shock normally reflects and the reflected shock is also a plane.

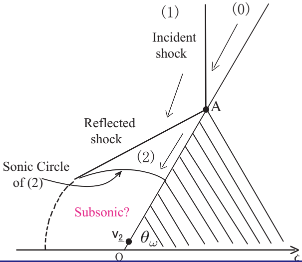


Local Theory for Regular Reflection (cf. Chang-C 1986)

$\exists\, \theta_d = \theta_d(M_s, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_d, \frac{\pi}{2})$, there exist two states $(2) = (U_2^a, V_2^a, p_2^a, \rho_2^a)$ and $(U_2^b, V_2^b, p_2^b, \rho_2^b)$ such that $|(U_2^a, V_2^a)| > |(U_2^b, V_2^b)|$ and $|(U_2^b, V_2^b)| < c_2^b$.

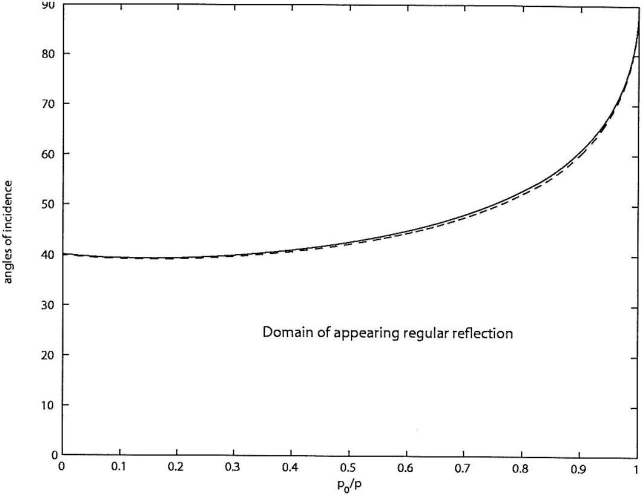
Detachment Conjecture: There exists a Regular Reflection Configuration when the wedge angle $\theta_W \in (\theta_d, \frac{\pi}{2})$.

Sonic Conjecture: There exists a Regular Reflection Configuration when $\theta_W \in (\theta_s, \frac{\pi}{2})$, for $\theta_s > \theta_d$ such that $|(U_2^a, V_2^a)| > c_2^a$ at A.

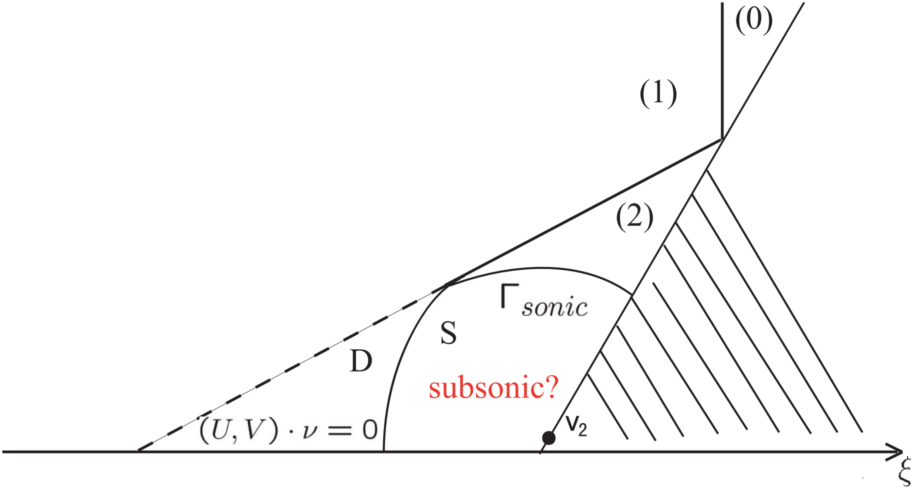


Detachment Criterion vs Sonic Criterion $\theta_c > \theta_s$: $\gamma = 1.4$

Courtesy of W. Sheng and G. Yin: ZAMP, 2008



Global Theory?



Euler Equations under Decomposition $(U, V) = \nabla\varphi + W$

$$\begin{cases} \nabla \cdot (\rho\nabla\varphi) + 2\rho + \nabla \cdot (\rho\nabla W) = 0, \\ \nabla(\frac{1}{2}|\nabla\varphi|^2 + \varphi) + \frac{1}{\rho}\nabla p = \nabla P^*, \\ (\nabla\varphi + W) \cdot \nabla\omega + (1 + \Delta\varphi)\omega = 0, \\ (\nabla\varphi + W) \cdot \nabla S = 0. \end{cases}$$

where

$S = c_v \ln(p\rho^{-\gamma})$–**Entropy**

$\omega = \mathrm{curl}\, W = \mathrm{curl}(U, V)$–**Vorticity**

When $\omega = 0, S = const.$ on a curve transverse to the fluid direction

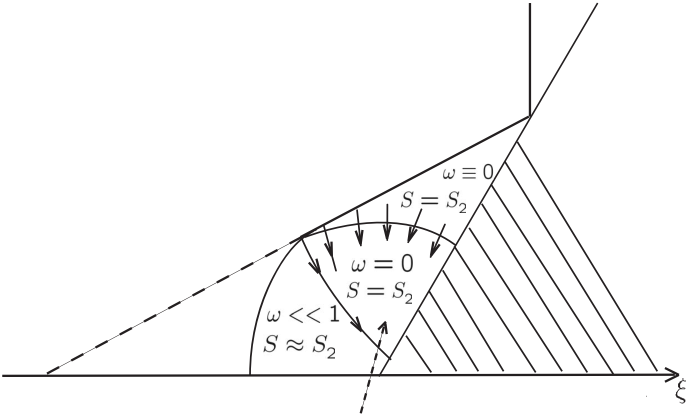
$$\Rightarrow W = 0, \quad \nabla P^* = 0$$

Then we obtain the Potential Flow Equation:

$$\begin{cases} \nabla \cdot (\rho\nabla\varphi) + 2\rho = 0, \\ \frac{1}{2}(|\nabla\varphi|^2 + \varphi) + \frac{\rho^{\gamma-1}}{\gamma - 1} = const. > 0. \end{cases}$$

Potential Flow Equation

$$\begin{cases} \nabla \cdot (\rho \nabla \varphi) + 2\rho = 0, \\ \frac{1}{2}|\nabla \varphi|^2 + \varphi + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\rho_0^{\gamma-1}}{\gamma-1} \end{cases}$$



Potential Flow Equation

$$\nabla \cdot (\rho(\nabla\varphi, \varphi, \rho_0)\nabla\varphi) + 2\rho(\nabla\varphi, \varphi, \rho_0) = 0$$

- Incompressible: $\rho = const. \Longrightarrow \Delta\varphi + 2 = 0$
- Subsonic (Elliptic):

$$|\nabla\varphi| < c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$$

- Supersonic (Hyperbolic):

$$|\nabla\varphi| > c_*(\varphi, \rho_0) := \sqrt{\frac{2}{\gamma+1}(\rho_0^{\gamma-1} - (\gamma-1)\varphi)}$$

Linear and Nonlinear Models

Linear Models

Tricomi Equation: $u_{xx} + xu_{yy} = 0$ (Hyperbolic Degeneracy at $x = 0$);

Keldysh Equation: $xu_{xx} + u_{yy} = 0$ (Parabolic Degeneracy at $x = 0$).

Nonlinear Models

- Transonic Small Disturbance Equation:

$$\left((u - x)u_x + \frac{u}{2}\right)_x + u_{yy} = 0$$

or, for $v = u - x$,

$$v\, v_{xx} + v_{yy} + \text{l.o.t.} = 0.$$

Morawetz, Hunter, Canic-Keyfitz-Lieberman-Kim, $\cdots$

- Pressure-Gradient Equations, Nonlinear Wave Equations

Y. Zheng-K. Song, Canic-Keyfitz-Kim-Katarina, $\cdots$

Steady Potential Flow Equation

- **Pure Elliptic Case: Subsonic Flow past an Obstacle**
  Shiffman, Bers, Finn-Gilbarg, G. Dong, $\cdots$
- **Degenerate Elliptic Case: Subsonic-Sonic Flow past an Obstacle**
  Shiffman, C–Dafermos-Slemrod-Wang, $\cdots$
- **Pure Hyperbolic Case (even Full Euler Eqs.)**:
  Gu, Li, Schaeffer, S. Chen, S. Chen-Xin-Yin, Y. Zheng, $\cdots$
  T.-P. Liu-Lien, S. Chen-Zhang-Wang, C–Zhang-Zhu, $\cdots$
- **Elliptic-Hyperbolic Mixed Case**
  Transonic Nozzles: C–Feldman, S. Chen, J. Chen, Yuan, Xin-Yin,...
  Wedge or Conical Body: S. Chen, B. Fang, C–Fang, $\cdots$
  Transonic Flow past an Obstacle: Morawetz, C-Slemrod-Wang,...

Self-Similar Potential Flow Equation

Glimm-Majda: IMA Volume in Memory of Ronald DiPerna, 1991

Morawetz: CPAM 1994
Shock Reflection Patterns via Asymptotic Analysis

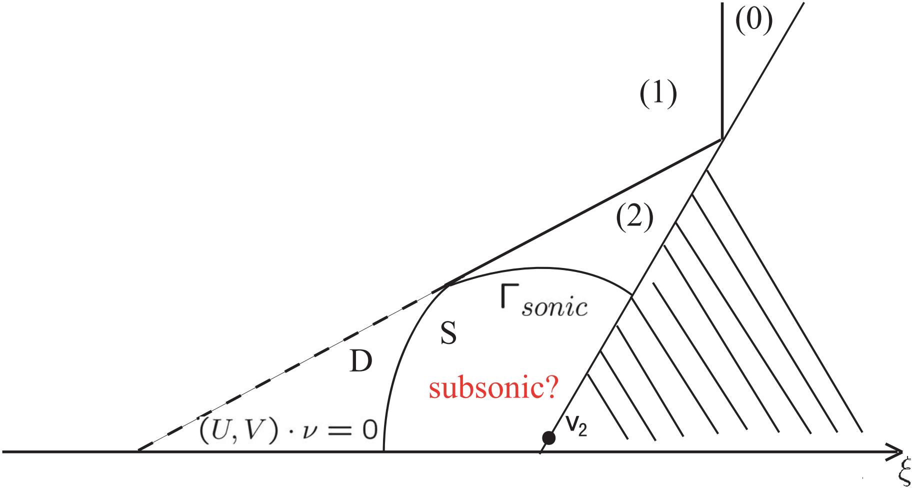
– –

C–Feldman: PNAS 2005, Ann. Math. 2006 (accepted)
Mathematical Existence and Regularity of Global Regular Reflection Configuration for Large-Angle Wedges

Elling-Liu: CPAM 2008 (to appear)
Supersonic Flow onto a Solid Wedge (Prandtl Conjecture)

$\Longrightarrow$ Recent Research Activities $\cdots\cdots$
For example, several talks during this conference

Global Theory?



Setup of the Problem for $\psi := \varphi - \varphi_2$ in Ω

- $\mathrm{div}\left(\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi + \mathbf{v}_2 - (\xi, \eta)\right) + l.o.t. = 0 \qquad (*)$
- $\nabla\psi \cdot \nu|_{wedge} = 0$
- $\psi|_{\Gamma_{sonic}} = 0 \Longrightarrow \psi_\nu|_{\Gamma_{sonic}} = 0$
- Rankine-Hugoniot Conditions on Shock S:
  $[\psi]_S = 0$
  $[\rho(\nabla\psi, \psi, \xi, \eta, \rho_0)(\nabla\psi + \mathbf{v}_2 - (\xi, \eta)) \cdot \nu]_S = 0$

Free Boundary Problem

- $\exists\, S = \{\xi = f(\eta)\}$ such that $f \in C^{1,\alpha}, f'(0) = 0$ and

$$\Omega_+ = \{\xi > f(\eta)\} \cap D = \{\psi < \varphi_1 - \varphi_2\} \cap D$$

- $\psi \in C^{1,\alpha}(\overline{\Omega_+}) \cap C^2(\Omega_+) \begin{cases} \text{solves } (*) \text{ in } \Omega_+, \\ \text{is subsonic in } \Omega_+ \end{cases}$

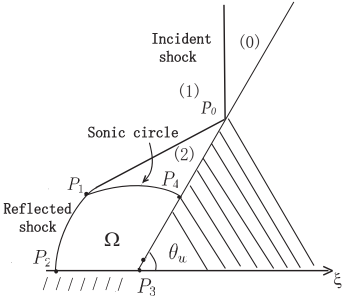
  with $(\psi, \psi_\nu)|_{\Gamma_{sonic}} = 0, \qquad \nabla\psi \cdot \nu|_{wedge} = 0$
- (ψ, f) satisfy the Rankine-Hugoniot Conditions

Theorem (Global Theory for Shock Reflection (C–Feldman 2005))

$\exists\ \theta_c = \theta_c(\rho_0, \rho_1, \gamma) \in (0, \frac{\pi}{2})$ such that, when $\theta_W \in (\theta_c, \frac{\pi}{2})$, there exist (φ, f) satisfying

- $\varphi \in C^\infty(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$ *and* $f \in C^\infty(P_1P_2) \cap C^2(\{P_1\})$;
- $\varphi \in C^{1,1}$ *across the sonic circle* P_1P_4
- $\varphi \longrightarrow \varphi_{NR}$ *in* $W^{1,1}_{loc}$ *as* $\theta_W \to \frac{\pi}{2}$.

$\Rightarrow$ $\Phi(t, \mathbf{x}) = t\varphi(\frac{\mathbf{x}}{t}) + \frac{|\mathbf{x}|^2}{2t}$, $\rho(t, \mathbf{x}) = \left(\rho_0^{\gamma-1} - (\gamma-1)(\Phi_t + \frac{1}{2}|\nabla\Phi|^2)\right)^{\frac{1}{\gamma-1}}$ *form a solution of the IBVP.*



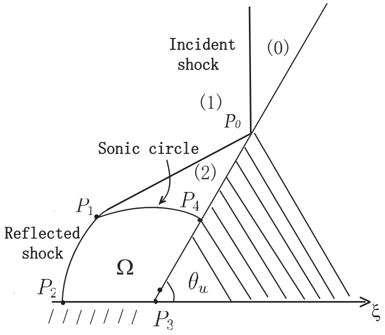
Optimal Regularity and Sonic Conjecture

Theorem (Optimal Regularity; Bae–C–Feldman 2007):

$\varphi \in C^{1,1}$ but NOT in C^2 across P_1P_4;

$\varphi \in C^{1,1}(\{P_1\}) \cap C^{2,\alpha}(\bar{\Omega} \setminus (\{P_1\} \cup \{P_3\})) \cap C^{1,\alpha}(\{P_3\}) \cap C^\infty(\Omega)$;

$f \in C^2(\{P_1\}) \cap C^\infty(P_1P_2)$.



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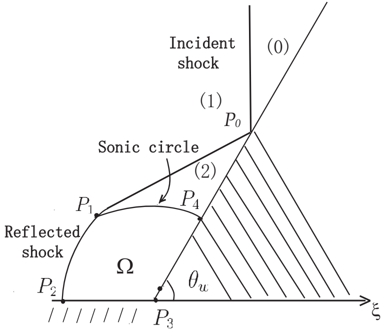
$f \in C^2(\{P_1\}) \cap C^\infty(P_1P_2)$.

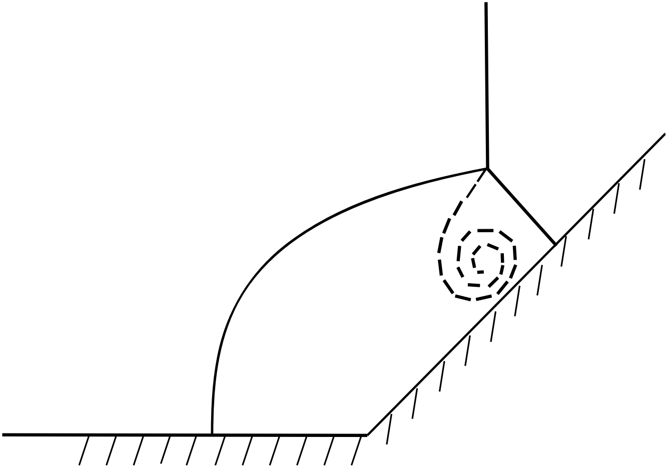
$\Longrightarrow$ The optimal regularity and the global existence hold up to the sonic wedge-angle θ_s for any $\gamma \geq 1$ (the von Neumann's sonic conjecture)



Approach

- **Cutoff Techniques by Shiffmanization**
  $\Rightarrow$ Elliptic Free-Boundary Problem with Elliptic Degeneracy on Γ_{sonic}
- **Domain Decomposition**
  Near Γ_{sonic}
  Away from Γ_{sonic}
- **Iteration Scheme**
  C–Feldman, J. Amer. Math. Soc. 2003
- $C^{1,1}$ **Parabolic Estimates near the Degenerate Elliptic Curve** Γ_{sonic};
- **Corner Singularity Estimates**
  In particular, when the Elliptic Degenerate Curve Γ_{sonic} Meets the Free Boundary, i.e., the Transonic Shock
- **Removal of the Cutoff**
  Require the Elliptic-Parabolic Estimates
- **Topological Argument**
  Extend the Large-Angle to the Sonic-Angle θ_s

Mach Reflection



? Right space for vorticity ω?

? Chord-arc $z(s) = z_0 + \int_0^s e^{ib(s)} ds$, $b \in BMO$?

General Framework for Entropy Solutions to Multidimensional Conservation Laws

Natural Class of Entropy Solutions:

(i) $U(t,x) \in \mathcal{M}$, or $L^p_w, 1 \le p \le \infty$;

(ii) For any convex entropy pair (η, q),

$$\partial_t \eta(U) + \nabla_x \cdot q(U) \le 0 \qquad \mathcal{D}'$$

as long as $(\eta(U(t,x)), q(U(t,x))) \in \mathcal{D}'$

$\Longrightarrow$ $\mathrm{div}_{(t,x)}(\eta(U(t,x)), q(U(t,x))) \in \mathcal{M}$

$\Longrightarrow$ The vector field $(\eta(U(t,x), q(U(t,x)))$
is a Divergence-Measure Field

- Theory of Divergence-Measure Fields for Entropy Solutions

Some of Other Recent Related Developments

D. Serre: Multi-D Shock Interaction for a Chaplygin Gas

S. Canic, B. Keyfitz, J. Katarina, E. H. Kim:
Self-Similar Solutions of 2-D Conservation Laws
Almost Global Solutions for Shock Reflection Problems

V. Elling: Counterexamples to the Sonic and Detachment Criteria

Y. Zheng+al: Solutions to Some 2-D Riemann Problems
Full Euler Equations with Adiabatic Exponent $\gamma \gg 1$

J. Glimm, X. Ji, J. Li, X. Li, P. Zhang, T. Zhang, and Y. Zheng:
Transonic Shock Formation in a Rarefaction Riemann Problem

O. Gues, G. Métivier, M. Williams, and K. Zumbrun;
S. Benzoni-Gavage; $\cdots$: Local Stability of M-D Shock Waves and Phase Boundaries

$\cdots\cdots\cdots$

S.-X. Chen: Stability of Mach Configuration $\cdots$
$\Longrightarrow$ Shuxing Chen's Talk

Shock Reflection-Diffraction vs New Mathematics

- **Free Boundary Techniques**
- **Mixed and Composite Eqns. of Hyperbolic-Elliptic Type**

  Degenerate Elliptic Techniques
  Degenerate Hyperbolic Techniques
  Transport Equations with Rough Coefficients
- **Regularity Estimates when a Free Boundary Meets a Degenerate Curve**
- **Boundary Harnack Inequalities**
- **Spaces for Compressible Vortex Sheets**
- **More Efficient Numerical Methods** $\cdots$
- $\cdots\cdots$

$\Longrightarrow$ Multidimensional Problems in Conservation Laws