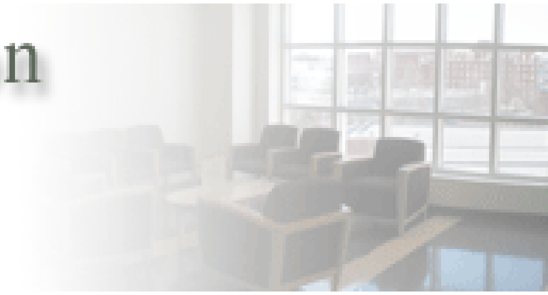

PDEs in Image Processing

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University of Maryland

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Image processing: two-scale decompositions

- Observed images: $\Omega \subset \mathbb{R}^2$

  (grayscale) $f : \Omega \mapsto \mathbb{R}_+$, (colored) $\mathbf{f} = (f_1, f_2, f_3) : \Omega \mapsto \mathbb{R}^3_+$

  measuring "pixel intensity units": 0 (dark)$\mapsto$ 255 (bright)

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- Inverse problems (ill-posed) – from an observed $f \mapsto U$:

Two-scale representation —

variational decompositions of noisy images

- $f = u_\lambda + v_\lambda :$ $\qquad [u_\lambda, v_\lambda] := \arg\inf_{u+v=f} \left\{ \|u\|_X + \lambda \|v\|_Y^2 \right\}$

  — u_λ extracts large scale features: edges, ... $\mapsto U$ (observed)

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$$v_\lambda := f - u_\lambda$$

- ROF model: $\underset{u}{\arginf}\left\{\|u\|_{BV} + \lambda\|f-u\|^2_{L^2}\right\}$,
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- Steepest-descent: $\{U(\cdot,0)=0\} \mapsto u_\lambda := \lim_{t\uparrow\infty} U(\cdot,t)$:

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$$\frac{\partial U}{\partial t} = f - U + \text{div}_x\Big(g(|G_\lambda * \nabla U|)\nabla U\Big), \quad g(s) := \frac{1}{2\lambda s}$$

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- Wavelet shrinkage (DeVore-Lucier): $f = \overbrace{\sum_{|\widehat{f}_{jk}|\geq 1/\lambda} \widehat{f}_{jk}\psi_{jk}}^{u_\lambda} + v_\lambda$

Multiscale representation — (T.-Nezzar-Vese)

- $f = u_\lambda + v_\lambda$: $\quad u_\lambda := \arg\inf_{u+v=f} \left\{ \|u\|_{BV} + 2\lambda \|f - v\|^2_{L^2} \right\}$

- $\begin{cases} u_\lambda : & \text{essential features} \\ v_\lambda : & \text{noise, texture, } \ldots \end{cases}$

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$$v_\lambda = u_{2\lambda} + v_{2\lambda}, \quad u_{2\lambda} := \underset{u+v=v_\lambda}{\arg\inf} \left\{ \|u\|_{BV} + 2\lambda \|v_\lambda - v\|^2_{L^2} \right\}$$

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- A better 2-scale representation: $f = u_\lambda + v_\lambda \approx u_\lambda + u_{2\lambda} + v_{2\lambda}$

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- This process can be continued...

$$v_{2\lambda} = u_{4\lambda} + v_{4\lambda}, \quad u_{4\lambda} := \arg\inf_{u+v=v_{2\lambda}} \left\{ \|u\|_{BV} + 4\lambda \|v_{2\lambda} - v\|_{L^2}^2 \right\},$$

$$\vdots$$

$$v_{2^j\lambda} = u_{2^{j+1}\lambda} + v_{2^{j+1}\lambda}, \quad \text{scale} = 2^{j+1}\lambda, \ldots$$

Hierarchical decompositions — (T.-Nezzar-Vese)

- $u_{2^j\lambda} \mapsto u_j$:

$$v_j = u_{j+1} + v_{j+1}, \quad u_{j+1} := \arg\inf_u \left\{ \|u\|_{BV} + 2^j\lambda \|v_j - u\|_{L^2}^2 \right\}$$

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- After N hierarchical steps:

$$\begin{aligned} f &= u_0 + v_0 = \\ &= u_0 + u_1 + v_1 = \\ &= \cdots\cdots \qquad = \\ &= u_0 + u_1 + \cdots + u_N + v_N \end{aligned}$$

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- Set $\lambda_j = 2^j\lambda_0, \qquad \lambda_0 \geq \dfrac{1}{2\|f\|_*}: \qquad v_j = -\dfrac{1}{2\lambda_j}\text{div}\left(\dfrac{\nabla u_j}{|\nabla u_j|}\right)$:

$$u_{j+1} - \frac{1}{2\lambda_{j+1}}\text{div}_x\left(\frac{\nabla u_{j+1}}{|\nabla u_{j+1}|}\right) = -\frac{1}{2\lambda_j}\text{div}_x\left(\frac{\nabla u_j}{|\nabla u_j|}\right)$$

Multiscale representation of Barbara

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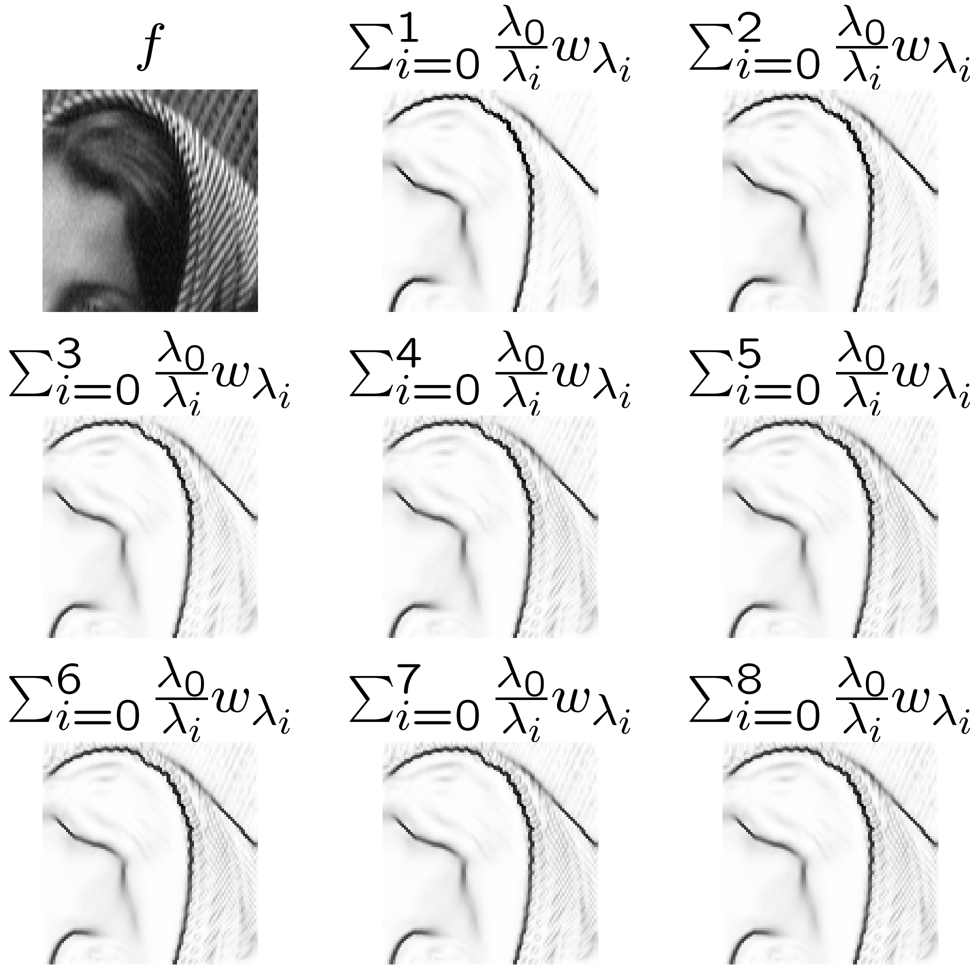
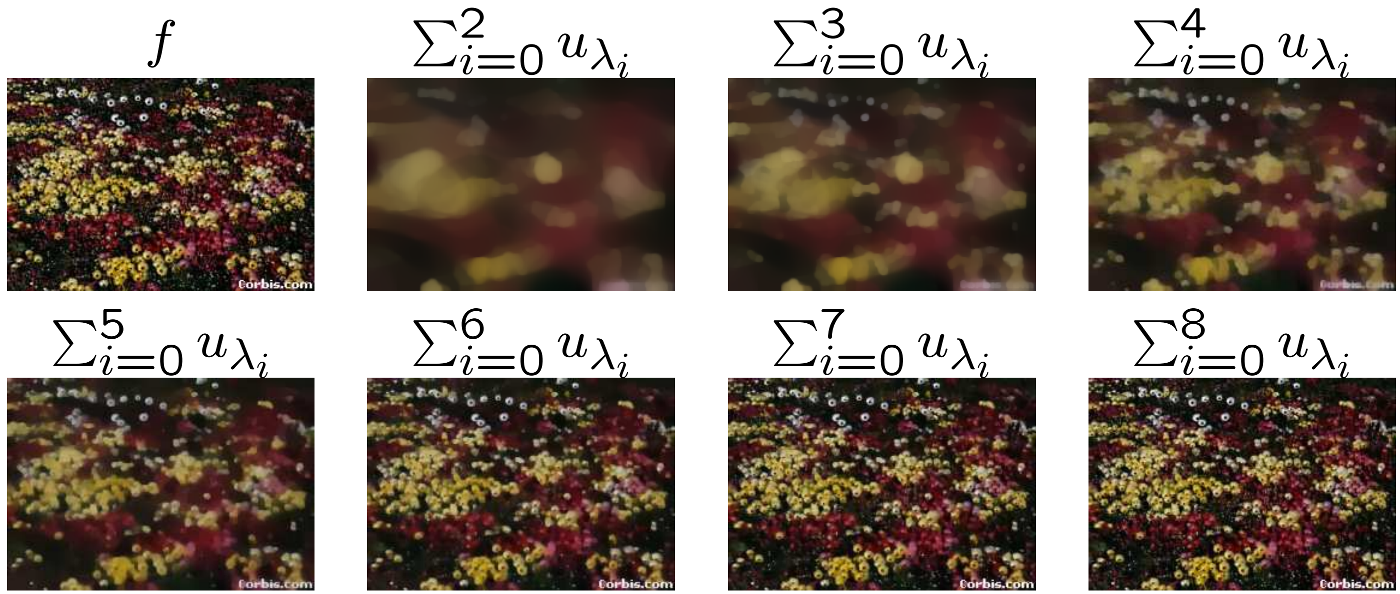
Multiscale representation using MS edge detectors



Mumford-Shah: Ambrosio-Tortorelli $[SBV, L^2](f, \lambda) := \inf_{\{u,v,w \mid u+v=f\}} \{\ldots\}$

$$w_j(x,y) \approx \begin{cases} 0, & \text{in regions of smoothness} \\ 1, & \text{near edges} \end{cases}$$

Multiscale representation of colored images



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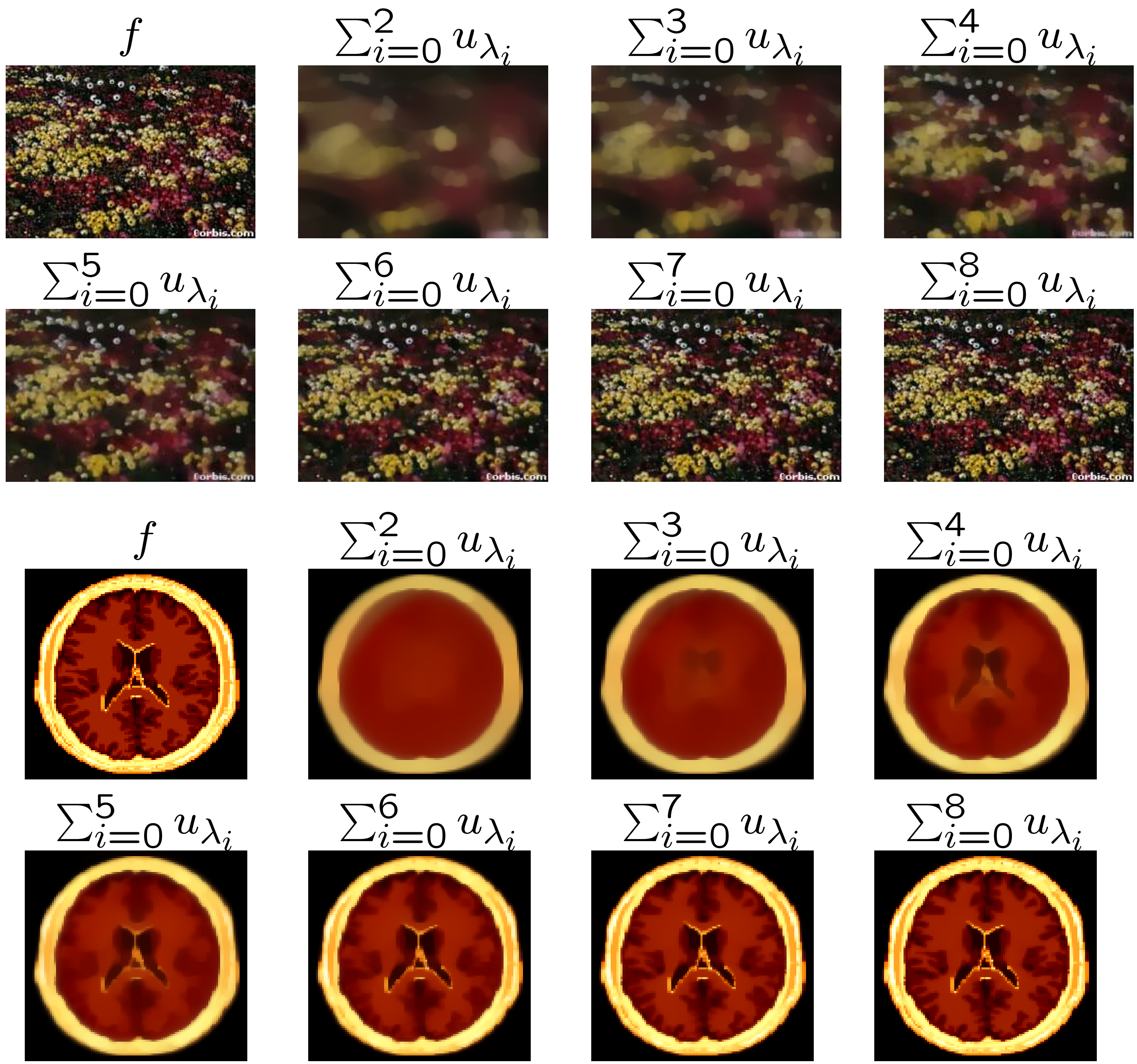

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- Telescoping sum: $\displaystyle\sum_{j=1}^{N} u_j\tau - f = \frac{1}{2\lambda_N}\mathrm{div}_x\left(\frac{\nabla_x u_N}{|\nabla_x u_N|}\right) \mapsto$

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- Integro-Differential Equation (IDE):

$$\boxed{\int_{s=0}^{t} u(s) ds - f = \frac{1}{\lambda(t)} \text{div}_x \left(\frac{\nabla_x u(t)}{|\nabla_x u(t)|} \right), \qquad \frac{\partial u}{\partial \mathbf{n}_{|\partial\Omega}} = 0}$$

Numerical results of IDE

- IDE: $$\int_{s=t_0}^{t} u(s)ds - f = \frac{1}{\lambda(t)}\text{div}_x\left(\frac{\nabla_x u(t)}{|\nabla_x u(t)|}\right)$$

  — inverse scale : $u(t_0) \mapsto U(t) := \int_{s=t_0}^{t} u(s)ds$ (observed)

- Perona-Malik: $$u_t = \frac{1}{\lambda(t)}\text{div}_x\left(\frac{\nabla_x u(t)}{|\nabla_x u(t)|}\right)$$

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$t = 1$ $t = 4$ $t = 6$ $t = 10$

The IDE images, $U(t) := \int_0^t u(\cdot, s)\, ds$ at $t = 1, 4, 6, 10$;

Scaling function: $\lambda(t) := 0.002 \times 2^t$

Parameters: where to start? how to scale?

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- $\frac{1}{\lambda(t)}$ dictates the size of texture: $\|U(\cdot,t) - f\|_{BV^*} = \frac{1}{2\lambda(t)}$

Extended IDE model for image restoration

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- Filtered diffusion — restricted diffusion near edges:

$$\int_0^t u(x,s)\,ds = f(x) + \frac{g(|G_\sigma \star \nabla u(x,t)|)}{2\lambda(t)} \mathrm{div}_x \left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|} \right); \quad \frac{\partial u}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0,$$

subject to $u_0(x) \equiv u(\cdot, 0)$ such that

$$u_0 = f + \frac{1}{2\lambda_0} g(|G_\sigma \star \nabla u_0|) \mathrm{div}_x \left(\frac{\nabla u_0}{|\nabla u_0|} \right), \qquad \lambda_0 > \frac{g(|G_\sigma \star \nabla u_0|)}{2\|f\|_*}.$$

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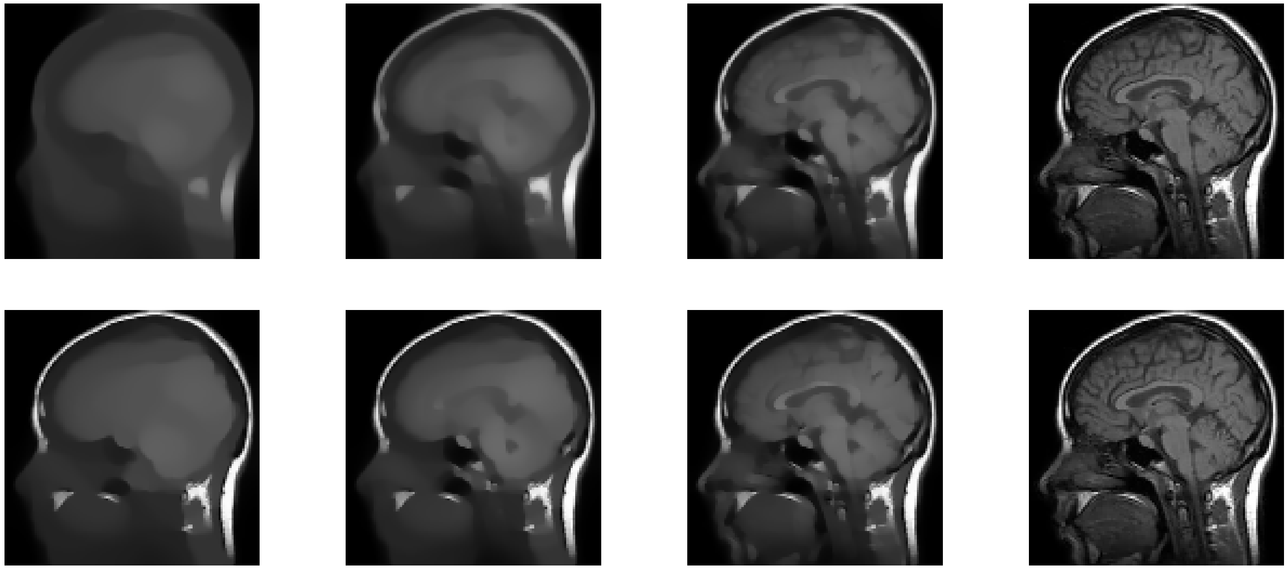
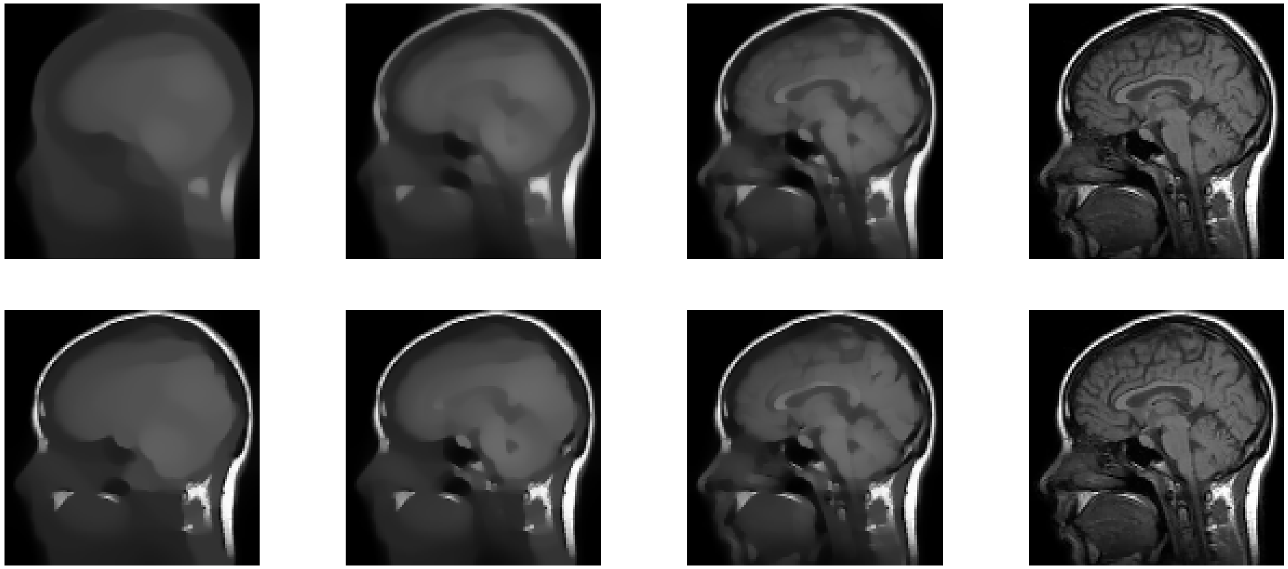
- Tangential smoothing — $u_{\mathbf{tt}} = \Delta u - u_{\mathbf{nn}} = |\nabla u| \text{div}_x \left(\frac{\nabla u}{|\nabla u|} \right)$:

$$\int_{s=0}^t u(x,s)ds - f(x) = \frac{1}{\lambda(t)} |\nabla u(x,t)| \cdot \text{div}_x \left(\frac{\nabla u(x,t)}{|\nabla u(x,t)|} \right)$$



$t = 1$ $t = 4$ $t = 6$ $t = 10$

MRI images, $U(t) = \int_0^t u(\cdot, s)\, ds$ at $t = 1, 4, 6, 10$. Top: unfiltered. Bottom: filtered



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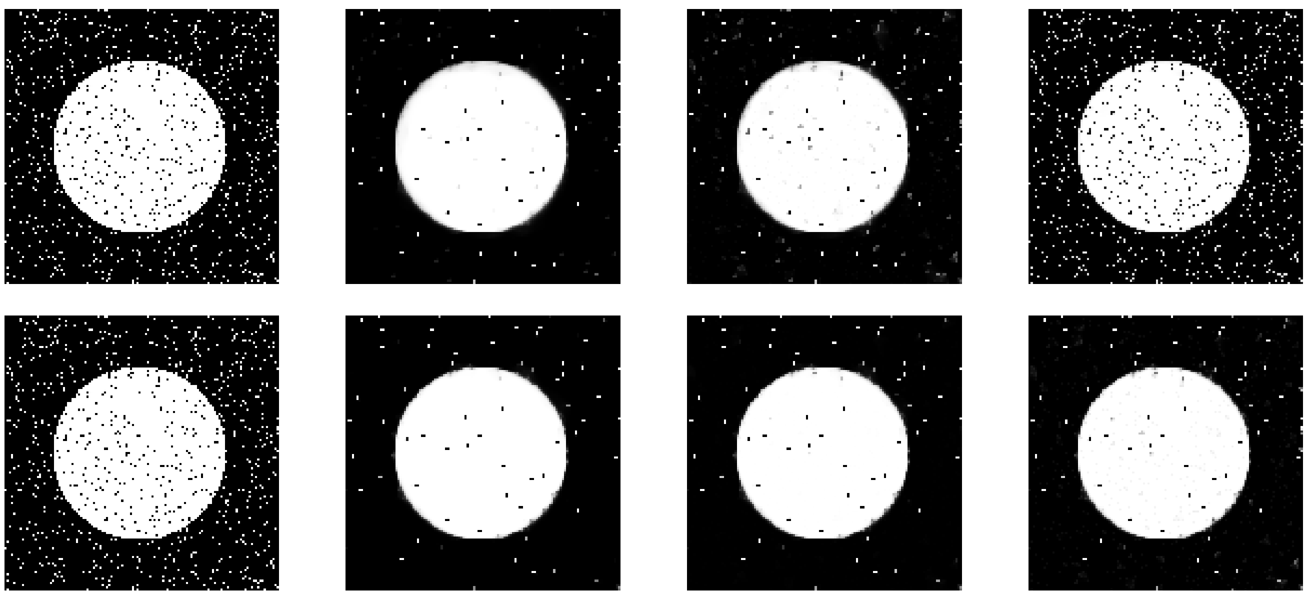

f at $t = 0$ $t = 1$ $t = 4$ $t = 7$

Restoring a noisy image. Top: vanilla IDE. Bottom: IDE with tangential smoothing

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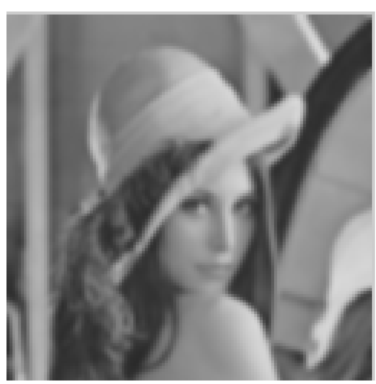


Deblurring Lena using IDE with $G_\sigma, \sigma = 1$

THANK YOU