Acoustic Limit for the Boltzmann equation in Optimal Scaling

Joint work with Yan Guo and Ning Jiang

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$$\mathcal{Q}(F_1, F_2)(v) = \int_{\mathbf{R}^3 \times S^2} b(v-u, \omega)\{F_1(v')F_2(u') - F_1(v)F_2(u)\} d\omega \, du$$

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Here, we take $|b(v-u,\omega)| \leq |v-u|^\gamma$ for $-3 < \gamma \leq 1$

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Collision invariants $\{1, v_k, |v|^2\}$

$$\int_{\mathbf{R}^3} \mathcal{Q}(F,F)dv = \int_{\mathbf{R}^3} \mathcal{Q}(F,F)v_k dv = \int_{\mathbf{R}^3} \mathcal{Q}(F,F)|v|^2 dv = 0$$

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Longer time scale $(G^\varepsilon \to G)$

- $q = 1, m = 1, s = 1$: Incompressible Navier-Stokes-Fourier
- $q = 1, m > 1, s = 1$: Stokes-Fourier
- $q > 1, m = 1, s = 1$: Incompressible Euler-Fourier

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  Justify the convergence to global weak solutions of incompressible Navier-Stokes, Stokes, and Euler flows and compressible acoustic system

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  via Hilbert or Chapman-Enskog expansions
  (Caflisch, De Masi, Esposito, Lebowitz, Guo)

Acoustic Limit

Acoustic scaling

$$\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}(F^\varepsilon, F^\varepsilon), \quad F^\varepsilon = \mu^0 + \delta G^\varepsilon$$

$$\mu^0 \equiv \tfrac{1}{(2\pi)^{3/2}} \exp(-\tfrac{|v|^2}{2}), \quad \delta \to 0 \text{ as } \varepsilon \to 0 \;\; (\delta = \varepsilon^m)$$

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Formal Acoustic limit

$$\varepsilon(G^\varepsilon + v \cdot \nabla_x G^\varepsilon) + LG^\varepsilon = \delta \mathcal{Q}(G^\varepsilon, G^\varepsilon)$$

By letting $\varepsilon \to 0$, $LG = 0 \;\Rightarrow\; G = \left\{\sigma + v \cdot u + \left(\frac{|v|^2-3}{2}\right)\theta\right\}\mu^0$

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G^ε formally satisfy the local conservation laws and thus σ, u, θ satisfy the acoustic system

$$\begin{aligned} \partial_t \sigma + \nabla_x \cdot u &= 0, & \sigma(x,0) &= \sigma^0(x) \\ \partial_t u + \nabla_x(\sigma + \theta) &= 0, & u(x,0) &= u^0(x) \\ \tfrac{3}{2}\partial_t \theta + \nabla_x \cdot u &= 0, & \theta(x,0) &= \theta^0(x) \end{aligned}$$

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The purpose of this work is to establish the acoustic limit in optimal scaling ($0 < m < 1$) via a recent $L^2 - L^\infty$ setting.

Main Theorem

Theorem (Guo-J-Jiang) Let $\tau > 0$ be any given time and let

$$\sigma(0,x) = \sigma^0(x),\ u(0,x) = u^0(x),\ \theta(0,x) = \theta^0(x)\ \in H^s,\ s \geq 4$$

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be any given initial data to the acoustic system. Then, there exist an $\varepsilon_0 > 0$ and a $\delta_0 > 0$ such that for each $0 < \varepsilon \leq \varepsilon_0$ and $0 < \delta \leq \delta_0$, there exists a constant $C > 0$ so that

$$\sup_{0 \leq t \leq \tau} \|G^\varepsilon(t) - G(t)\|_\infty + \sup_{0 \leq t \leq \tau} \|G^\varepsilon(t) - G(t)\|_2 \leq C\{\frac{\varepsilon}{\delta} + \delta\}$$

where $\frac{\varepsilon}{\delta} \to 0$ as $\varepsilon \to 0$, and C depends only on τ and the initial data σ^0, u^0, θ^0.

Idea of Proof

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- The acoustic system is the linearization about the homogeneous state $[1, 0, 1]$ of the compressible Euler system

$$\begin{aligned}
\partial_t \rho + (\mathfrak{u} \cdot \nabla)\rho + \rho \nabla \cdot \mathfrak{u} &= 0 \\
\rho \partial_t \mathfrak{u} + \rho(\mathfrak{u} \cdot \nabla)\mathfrak{u} + \rho \nabla T + T \nabla \rho &= 0 \\
\partial_t T + (\mathfrak{u} \cdot \nabla) T + \tfrac{2}{3} T \nabla \cdot \mathfrak{u} &= 0
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- The first step is to show

$$F^\varepsilon - \mu^\delta = O(\varepsilon)$$

for $\mu^\delta = \frac{\rho^\delta(t,x)}{[2\pi T^\delta(t,x)]^{3/2}} \exp\left\{-\frac{[v-\mathfrak{u}^\delta(t,x)]^2}{2T^\delta(t,x)}\right\}$ local Maxwellians defined from the solutions of (1) constructed from the acoustic initial data $\rho^0 = 1+\delta\sigma^0\,, \quad \mathfrak{u}^0 = \delta u^0\,, \quad T^0 = 1+\delta\theta^0$

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The proof of Theorem :

$$G^\varepsilon(t) - G(t) = \frac{F^\varepsilon(t) - \mu^0}{\delta} - G(t)$$

Rewrite $\frac{F^\varepsilon(t)-\mu^0}{\delta} - G(t) = \frac{F^\varepsilon(t)-\mu^\delta(t)}{\delta} + \frac{\mu^\delta(t)-\mu^0-\delta G(t)}{\delta}$ Therefore,

$$\sup_{0\le t\le\tau} \|G^\varepsilon(t) - G(t)\|_\infty + \sup_{0\le t\le\tau} \|G^\varepsilon(t) - G(t)\|_2 \le C\{\frac{\varepsilon}{\delta} + \delta\}$$

Refined Estimates of Acoustic and Euler solutions

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Introduce difference variables $(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)$ given by the second order perturbation in δ of Euler solutions

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Indeed, $(\sigma_d^\delta\,, u_d^\delta\,, \theta_d^\delta)$ satisfy the *linear* system of equations

$$\begin{aligned}
&\partial_t\sigma_d^\delta + (\mathfrak{u}^\delta\cdot\nabla)\sigma_d^\delta + \rho^\delta\nabla\cdot u_d^\delta + \delta[\nabla\sigma\cdot u_d^\delta + (\nabla\cdot u)\sigma_d^\delta] = -\nabla\cdot(\sigma u)\\
&\rho^\delta\partial_t u_d^\delta + \rho^\delta(\mathfrak{u}^\delta\cdot\nabla)u_d^\delta + \rho^\delta\nabla\theta_d^\delta + T^\delta\nabla\sigma_d^\delta + \delta[(\partial_t u)\sigma_d^\delta + \rho^\delta(u_d^\delta\cdot\nabla)u\\
&\quad + \theta_d^\delta\nabla\sigma + \sigma_d^\delta\nabla\theta] = -\sigma\partial_t u - \rho^\delta(u\cdot\nabla)u - \nabla(\sigma\theta)\\
&\partial_t\theta_d^\delta + (\mathfrak{u}^\delta\cdot\nabla)\theta_d^\delta + \tfrac{2}{3}T^\delta\nabla\cdot u_d^\delta + \delta[\nabla\theta\cdot u_d^\delta + \tfrac{2}{3}(\nabla\cdot u)\theta_d^\delta]\\
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&\quad + \theta_d^\delta\nabla\sigma + \sigma_d^\delta\nabla\theta] = -\sigma\partial_t u - \rho^\delta(u\cdot\nabla)u - \nabla(\sigma\theta)\\
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\end{aligned}$$

The coefficients are smooth and have the uniform bounds up to time τ. The above system can be symmetrized.

Refined Estimates of Acoustic and Euler solutions

Standard energy method of linear symmetric hyperbolic system

$$\|(\sigma_d^\delta, u_d^\delta, \theta_d^\delta)\|_{H^s} \leq C \tag{2}$$

and thus

$$\|(\rho^\delta - 1 - \delta\sigma,\ \mathfrak{u}^\delta - \delta u,\ T^\delta - 1 - \delta\theta)\|_{H^s} \leq C\delta^2$$

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Consider μ^δ as a function of δ and expand it around $\delta = 0$ to derive

$$\mu^\delta(t) = \mu^0 + G\delta + \frac{\mu''(\delta_*)}{2}\delta^2$$

The uniform boundedness of $\|\mu''(\delta_*)\|_2 + \|\mu''(\delta_*)\|_\infty$ follows from the uniform estimate (2)

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Following Caflisch, take the form $F^{\varepsilon} = \sum_{n=0}^{5} \varepsilon^n F_n + \varepsilon^3 F_R^{\varepsilon}$ where $F_0, ..., F_5$ solve the equations ($F_0 = \mu^{\delta} \equiv \mu$)

$$
\begin{aligned}
0 &= \mathcal{Q}(F_0, F_0) \\
\{\partial_t + v \cdot \nabla_x\} F_0 &= \mathcal{Q}(F_0, F_1) + \mathcal{Q}(F_1, F_0) \\
&\dots \\
\{\partial_t + v \cdot \nabla_x\} F_5 &= \mathcal{Q}(F_0, F_6) + \mathcal{Q}(F_6, F_0) + \sum_{\substack{i+j=6 \\ 1 \le i \le 5, 1 \le j \le 5}} \mathcal{Q}(F_i, F_j)
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\end{aligned}
$$

Remainder equation for F_R^{ε}

$$
\partial_t F_R^{\varepsilon} + v \cdot \nabla_x F_R^{\varepsilon} - \frac{1}{\varepsilon}\{\mathcal{Q}(\mu, F_R^{\varepsilon}) + \mathcal{Q}(F_R^{\varepsilon}, \mu)\}
$$

$$
= \varepsilon^2 \mathcal{Q}(F_R^{\varepsilon}, F_R^{\varepsilon}) + \sum_{i=1}^{5} \varepsilon^{i-1}\{\mathcal{Q}(F_i, F_R^{\varepsilon}) + \mathcal{Q}(F_R^{\varepsilon}, F_i)\} + \varepsilon^2 A
$$

Compressible Euler Limit

Theorem Assume that the solution to the Euler equations $[\rho, \mathfrak{u}, T]$ is smooth and $\rho(t,x)$ has a positive lower bound for $0 \leq t \leq \tau$. Furthermore, assume that the temperature $T(t,x)$ satisfies the condition $T_M < \max_{t\in[0,\tau],x\in\Omega} T(t,x) < 2T_M$. Let $F^\varepsilon(0,x,v) = \mu(0,x,v) + \sum_{n=1}^{5} \varepsilon^n F_n(0,x,v) + \varepsilon^3 F_R^\varepsilon(0,x,v) \geq 0$. Then

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Then there is an $\varepsilon_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_0$, and for any $\beta \ge \frac{9-2\gamma}{2}$, there exists a constant $C_\tau(\mu, F_1, ..F_5)$ such that

$$\sup_{0\le t\le\tau} \varepsilon^{\frac{3}{2}} \left\|(1+|v|^2)^\beta \frac{F_R^\varepsilon(t)}{\sqrt{\mu}}\right\|_\infty + \sup_{0\le t\le\tau} \left\|\frac{F_R^\varepsilon(t)}{\sqrt{\mu}}\right\|_2$$
$$\le C_\tau \left\{\varepsilon^{\frac{3}{2}} \left\|(1+|v|^2)^\beta \frac{F_R^\varepsilon(0)}{\sqrt{\mu}(0)}\right\|_\infty + \left\|\frac{F_R^\varepsilon(0)}{\sqrt{\mu}(0)}\right\|_2 + 1\right\}.$$

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$$\leq C_\tau \left\{ \varepsilon^{\frac{3}{2}} \left\|(1+|v|^2)^\beta \frac{F_R^\varepsilon(0)}{\sqrt{\mu}(0)}\right\|_\infty + \left\|\frac{F_R^\varepsilon(0)}{\sqrt{\mu}(0)}\right\|_2 + 1 \right\}.$$

Remark $\quad F^\varepsilon = \sum_{n=0}^{5} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon \implies F^\varepsilon - \mu^\delta = O(\varepsilon)$

Proof of Euler Limit

Introduce

$$f^\varepsilon \equiv \frac{F_R^\varepsilon}{\sqrt{\mu}} \quad \text{and} \quad h^\varepsilon \equiv \{1+|v|^2\}^\beta \frac{F_R^\varepsilon}{\sqrt{\mu_M}}$$

where $\beta \geq \frac{9-2\gamma}{2}$ and $\mu_M = \frac{1}{(2\pi T_M)^{3/2}} \exp\left\{-\frac{|v|^2}{2T_M}\right\}$ so that

$$c_1 \mu_M \leq \mu \leq c_2 \mu_M^\alpha \quad \text{for some} \quad 1/2 < \alpha < 1$$

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L^2 estimates for f^ε

$$\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}f^\varepsilon = \boxed{\frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}} f^\varepsilon} + \varepsilon^2\Gamma(f^\varepsilon, f^\varepsilon)$$

$$+\sum_{i=1}^{5}\varepsilon^{i-1}\{\Gamma(\frac{F_i}{\sqrt{\mu}}, f^\varepsilon) + \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\mu}})\} + \varepsilon^2\bar{A}$$

Difficulty is coming from the boxed term $|v|^3 f^\varepsilon$

For any $\kappa > 0$ and $a = 1/(3-\gamma)$

$$\left\langle \frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}} f^\varepsilon, f^\varepsilon \right\rangle = \int_{|v|\geq \frac{\kappa}{\varepsilon^a}} + \int_{|v|\leq \frac{\kappa}{\varepsilon^a}}$$

$$\leq \{\|\nabla_x\rho\|_2 + \|\nabla_x\mathfrak{u}\|_2 + \|\nabla_x T\|_2\} \times \|\{1+|v|^2\}^{3/2} f^\varepsilon \mathbf{1}_{|v|\geq \frac{\kappa}{\varepsilon^a}}\|_\infty \times \|f^\varepsilon\|_2$$

$$+ \{\|\nabla_x\rho\|_\infty + \|\nabla_x\mathfrak{u}\|_\infty + \|\nabla_x T\|_\infty\} \times \|\{1+|v|^2\}^{3/4} f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2$$

$$\leq C_\kappa \varepsilon^2 \|h^\varepsilon\|_\infty \|f^\varepsilon\|_2$$

$$+ C\|\{1+|v|^2\}^{3/4}\mathbf{P}f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2 + C\|\{1+|v|^2\}^{3/4}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon \mathbf{1}_{|v|\leq \frac{\kappa}{\varepsilon^a}}\|_2^2$$

$$\leq C_\kappa \varepsilon^2 \|h^\varepsilon\|_\infty \|f^\varepsilon\|_2 + C\|f^\varepsilon\|_2^2 + \frac{C\kappa^{3-\gamma}}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2$$

Due to the fact $\{1+|v|^2\}^{3/2} f^\varepsilon \leq \{1+|v|^2\}^{\gamma-3} h^\varepsilon$ for $\beta \geq 3/2 + (3-\gamma)$ and the fact $\mu_M < C\mu$

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$$\frac{d}{dt}\|f^\varepsilon\|_2^2 + \frac{\delta_0}{2\varepsilon}\|\{\mathbf{I}-\mathbf{P}\} f^\varepsilon\|_\nu^2 \leq C\{\sqrt{\varepsilon}\|\varepsilon^{3/2} h^\varepsilon\|_\infty + 1\}(\|f^\varepsilon\|_2^2 + \|f^\varepsilon\|_2)$$

L^∞ estimates for h^ε

$$\sup_{0\leq s\leq\tau}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}\leq C\{\|\varepsilon^{3/2}h_0\|_\infty+\sup_{0\leq s\leq\tau}\|f^\varepsilon(s)\|_2+\varepsilon^{7/2}\}$$

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Following Caflisch, define

$$\mathcal{L}_M g=-\frac{1}{\sqrt{\mu_M}}\{\mathcal{Q}(\mu,\sqrt{\mu_M}g)+\mathcal{Q}(\sqrt{\mu_M}g,\mu)\}=\{\nu(\mu)+K\}g$$

where $\nu(\mu)=\int_{\mathbf{R}^3\times S^2}b(v-v',\omega)\mu(v')dv'd\omega\sim(1+|v|)^\gamma$

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L^∞ estimates for h^ε

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To treat soft potentials, employ a cutoff trick of Guo and Strain
Smooth cutoff function $0\le\chi_m\le 1$ such that for any $m>0$,
$\chi_m(s)\equiv 1$ for $s\le m$ and $\chi_m(s)\equiv 0$ for $s\ge 2m$. Split $K = K^m + K^c$.

$$|K^m g(v)| \le Cm^{3+\gamma}\nu(\mu)||g||_\infty$$

and $K^c g(v) = \int_{\mathbf{R}^3} l(v,v')g(v')dv'$ where the kernel l satisfies

$$l(v,v') \le C_m \frac{\exp\{-c|v-v'|^2\}}{|v-v'|(1+|v|+|v'|)^{1-\gamma}}.$$

Letting $K_w g \equiv wK(\frac{g}{w})$ where $w(v) = \{1 + |v|^2\}^\beta$

$$\boxed{\partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon + \frac{\nu(\mu)}{\varepsilon} h^\varepsilon} + \frac{1}{\varepsilon} K_w h^\varepsilon = \frac{\varepsilon^2 w}{\sqrt{\mu_M}} \mathcal{Q}(\frac{h^\varepsilon \sqrt{\mu_M}}{w}, \frac{h^\varepsilon \sqrt{\mu_M}}{w})$$
$$+ \sum_{i=1}^{5} \varepsilon^{i-1} \frac{w}{\sqrt{\mu_M}} \{\mathcal{Q}(F_i, \frac{h^\varepsilon \sqrt{\mu_M}}{w}) + \mathcal{Q}(\frac{h^\varepsilon \sqrt{\mu_M}}{w}, F_i)\} + \varepsilon^2 \tilde{A},$$

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By Duhamel's principle

$$h^\varepsilon(t,x,v) = \exp\{-\frac{\nu t}{\varepsilon}\} h^\varepsilon(0, x - vt, v)$$

$$- \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\} \left(\frac{1}{\varepsilon} K_w^m h^\varepsilon\right)(s, x - v(t-s), v) ds$$

$$- \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\} \left(\frac{1}{\varepsilon} K_w^c h^\varepsilon\right)(s, x - v(t-s), v) ds$$

$$+ \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\} \left(\frac{\varepsilon^2 w}{\sqrt{\mu_M}} \mathcal{Q}(\frac{h^\varepsilon \sqrt{\mu_M}}{w}, \frac{h^\varepsilon \sqrt{\mu_M}}{w})\right)(s, x - v(t-s), v) ds$$

$$+ \ldots$$

$$
\begin{aligned}
&- \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\} \left(\frac{1}{\varepsilon} K_w^m h^\varepsilon\right)(s, x - v(t-s), v) ds \\
&\leq Cm^{3+\gamma} \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\} \frac{\nu}{\varepsilon} ds \sup_{0\leq t\leq\tau} ||h^\varepsilon(t)||_\infty \leq Cm^{3+\gamma} \sup_{0\leq t\leq\tau} ||h^\varepsilon(t)||_\infty
\end{aligned}
$$

$$
-\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_w^m h^\varepsilon\right)(s,x-v(t-s),v)ds
$$

$$
\leq Cm^{3+\gamma}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\frac{\nu}{\varepsilon}ds \sup_{0\leq t\leq\tau}||h^\varepsilon(t)||_\infty \leq Cm^{3+\gamma}\sup_{0\leq t\leq\tau}||h^\varepsilon(t)||_\infty
$$

And since $|\frac{w}{\sqrt{\mu_M}}Q(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w})| \leq C\nu(\mu)\|h^\varepsilon\|_\infty^2$

$$
\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s,x-v(t-s),v)ds
$$

$$
\leq C\varepsilon^2\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu(\mu)\|h^\varepsilon(s)\|_\infty^2 ds \leq \ C\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2
$$

$$
\begin{aligned}
&-\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_w^m h^\varepsilon\right)(s,x-v(t-s),v)ds \\
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$$
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&\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},\frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s,x-v(t-s),v)ds \\
&\leq C\varepsilon^2\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu(\mu)\|h^\varepsilon(s)\|_\infty^2 ds \leq \; C\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2
\end{aligned}
$$

Let $l_w(v,v')$ be the corresponding kernel associated with K_w^c.

$$
\begin{aligned}
&-\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_w^c h^\varepsilon\right)(s,x-v(t-s),v)ds \\
&\leq \frac{1}{\varepsilon}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')h^\varepsilon(s,x-v(t-s),v')|dv'ds
\end{aligned}
$$

Use the Duhamel expression again

$$\frac{1}{\varepsilon}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')h^\varepsilon(s,x-v(t-s),v')|dv'ds$$

$$\leq \frac{1}{\varepsilon}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\sup_v \int_{\mathbf{R}^3}|l_w(v,v')|dv'$$

$$\times \exp\{-\frac{\nu s}{\varepsilon}\}h^\varepsilon(0,x-v(t-s)-v's,v')ds$$

$$+\frac{1}{\varepsilon^2}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_w(v,v')|$$

$$\times \int_0^s \exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}|\{K^m h^\varepsilon\}(s_1,x-v(t-s)-v'(s-s_1),v')|dv'ds_1 ds$$

$$+\frac{1}{\varepsilon^2}\int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3\times\mathbf{R}^3}|l_w(v,v')l_w(v',v'')$$

$$\times \int_0^s \exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}h^\varepsilon(s_1,x-v(t-s)-v'(s-s_1),v'')|dv'dv''ds_1 ds$$

$$+\ldots$$

The estimate for double I term : Divide into
Case 1 : $|v| \geq N$
Case 2 : $|v| \leq N, |v'| \geq 2N$ or $|v'| \leq 2N, |v''| \geq 3N$
Case 3a : $|v| \leq N$, $|v'| \leq 2N, |v''| \leq 3N$, and $s - s_1 \leq \kappa\varepsilon$
Case 3b : $|v| \leq N$, $|v'| \leq 2N, |v''| \leq 3N$, and $s - s_1 \geq \kappa\varepsilon$

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Case 3b : $|v| \leq N$, $|v'| \leq 2N, |v''| \leq 3N$, and $s - s_1 \geq \kappa\varepsilon$

To see how to get $\|f^\varepsilon\|_2$, in Case 3b, integrate over v' to get

$$
\begin{aligned}
& C_N \int_{|v'|\leq 2N} |h^\varepsilon(s_1, x-(s-s_1)v', v'')|dv' \\
& \leq C_N \left\{ \int_{|v'|\leq 2N} \mathbf{1}_\Omega(x-(s-s_1)v')|h^\varepsilon(s_1, x-(s-s_1)v', v'')|^2 dv' \right\}^{1/2} \\
& \leq \frac{C_N}{\kappa^{3/2}\varepsilon^{3/2}} \left\{ \int_{|y-x_1|\leq (s-s_1)3N} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2} \\
& \leq \frac{C_N\{(s-s_1)^{3/2}+1\}}{\kappa^{3/2}\varepsilon^{3/2}} \left\{ \int_{\mathbf{\Omega}} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2}
\end{aligned}
$$

Here we have made a change of variable $y = x - (s-s_1)v'$, and for $s - s_1 \geq \kappa\varepsilon$, $|\frac{dy}{dv'}| \geq \frac{1}{\kappa^3\varepsilon^3}$

Thus approximate version of Case 3b

$$
\begin{aligned}
& C \int_0^t \int_B \int_0^{s-\varepsilon\kappa} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} |l_N(v,v') l_N(v',v'') \\
& \qquad h^\varepsilon(s_1, x_1 - (s-s_1)v', v'')| ds_1 dv' dv'' ds \\
& \leq \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\} \\
& \qquad \int_{|v''|\leq 3N} \left\{ \int_{\boldsymbol{\Omega}} |h^\varepsilon(s_1, y, v'')|^2 dy \right\}^{1/2} dv'' ds_1 ds \\
& \leq \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\} \\
& \qquad \left\{ \int_{|v''|\leq 3N} \int_{\Omega} |f^\varepsilon(s_1, y, v'')|^2 dy dv'' \right\}^{1/2} ds_1 ds \\
& \leq \frac{C_{N,\kappa}}{\varepsilon^{3/2}} \sup_{0\leq s\leq t} \|f^\varepsilon(s)\|_2
\end{aligned}
$$

Thus approximate version of Case 3b

$$C\int_0^t \int_B \int_0^{s-\varepsilon\kappa} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} |l_N(v,v')l_N(v',v'')$$
$$h^\varepsilon(s_1, x_1-(s-s_1)v', v'')|ds_1 dv' dv'' ds$$
$$\leq \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\}$$
$$\int_{|v''|\leq 3N} \left\{ \int_{\Omega} |h^\varepsilon(s_1,y,v'')|^2 dy \right\}^{1/2} dv'' ds_1 ds$$
$$\leq \frac{C_{N,\kappa}}{\varepsilon^{7/2}} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}} e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}} \{(s-s_1)^{3/2}+1\}$$
$$\left\{ \int_{|v''|\leq 3N} \int_{\Omega} |f^\varepsilon(s_1,y,v'')|^2 dy dv'' \right\}^{1/2} ds_1 ds$$
$$\leq \frac{C_{N,\kappa}}{\varepsilon^{3/2}} \sup_{0\leq s\leq t} \|f^\varepsilon(s)\|_2$$

In summary, one can obtain for any $\kappa > 0$ and large $N > 0$,

$$\sup_{0\leq s\leq t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \leq \{Cm^{\gamma+3} + C_{N,m}\kappa + \frac{C_m}{N}\}\sup_{0\leq s\leq t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}$$
$$+C\varepsilon^{7/2} + C_{\varepsilon,N}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}C\sup_{0\leq s\leq t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}^2$$
$$+C_{m,N,\kappa}\sup_{0\leq s\leq t}\|f^\varepsilon(s)\|_2$$

$$\sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \le \{Cm^{\gamma+3} + C_{N,m}\kappa + \frac{C_m}{N}\} \sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}$$
$$+C\varepsilon^{7/2} + C_{\varepsilon,N}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}C \sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}^2$$
$$+C_{m,N,\kappa}\sup_{0\le s\le t}\|f^\varepsilon(s)\|_2$$

For sufficiently small $\varepsilon > 0$, first choosing m small, then N sufficiently large, and finally letting κ small so that $\{Cm^{\gamma+3} + C_{N,m}\kappa + \frac{C_m}{N}\} < \frac{1}{2}$, one gets

$$\sup_{0\le s\le \tau}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \le C\{\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\le s\le \tau}\|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\}$$

and this completes the proof.