From individual to macroscopic models in animal displacements

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Kinetic Description of Multiscale Phenomena
University of Maryland, March 2-5, 2009

Motivation

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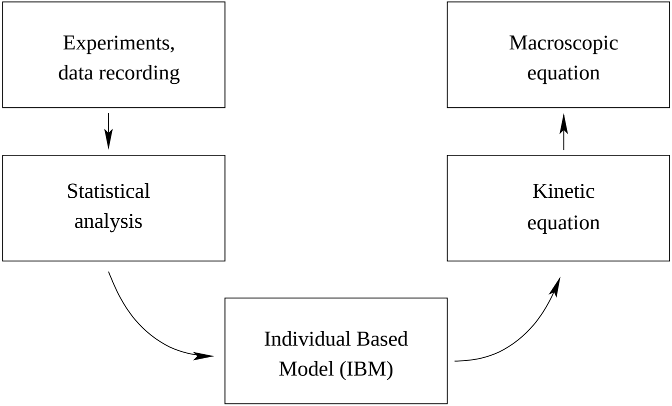
How these structures could emerge from local interactions ?

Work plan



Experiments,
data recording
Statistical
analysis
Individual Based
Model (IBM)
Kinetic
equation
Macroscopic
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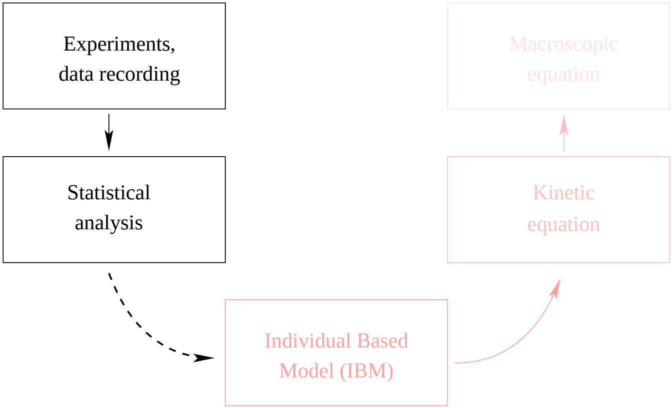
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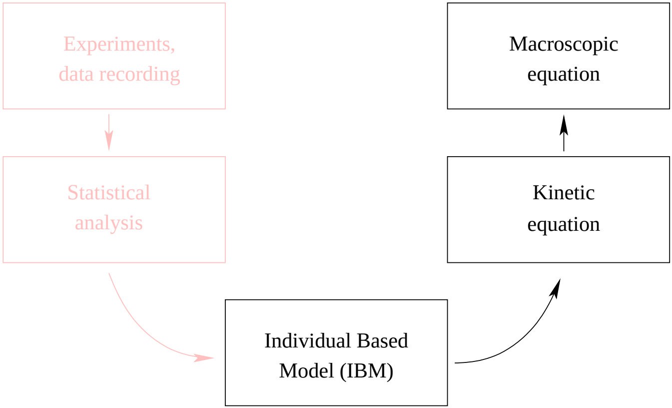
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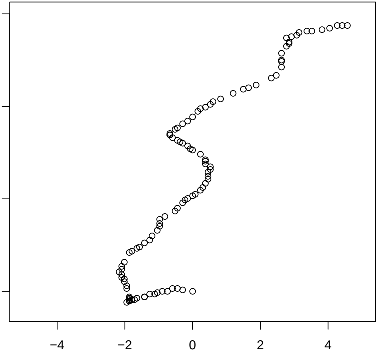

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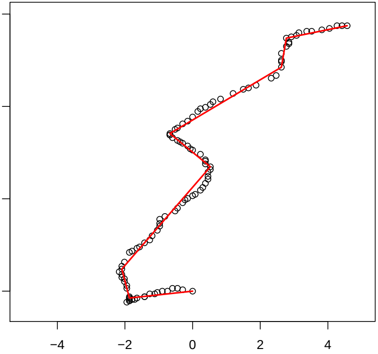

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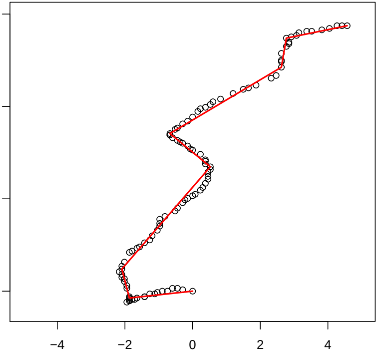

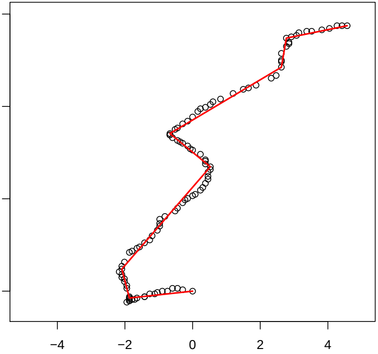

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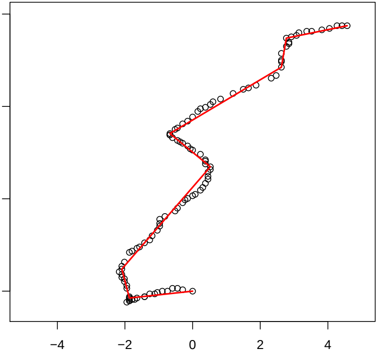

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Kinetic level :

$$\partial_t f + c\vec{\tau} \cdot \nabla_x f = \frac{b^2}{2}\partial_\theta^2 f$$

with $f(x,\theta)$ density in phase space.

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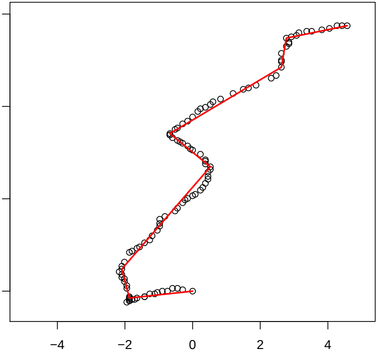

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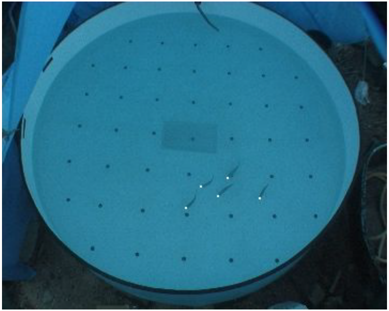
Macroscopic level :

$$\partial_t \rho = D \Delta_x \rho,$$

with ρ mass density, $D = \frac{c^2}{b^2}$.

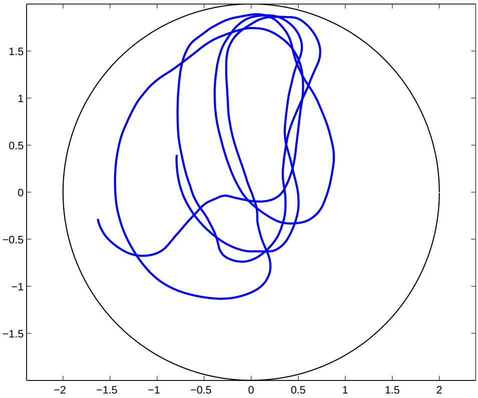
Experiments for fish



- The diameter of the basin is 4 meters
- Species studied : Kuhlia mugil (20-25 cm)

Video , *data recorded*

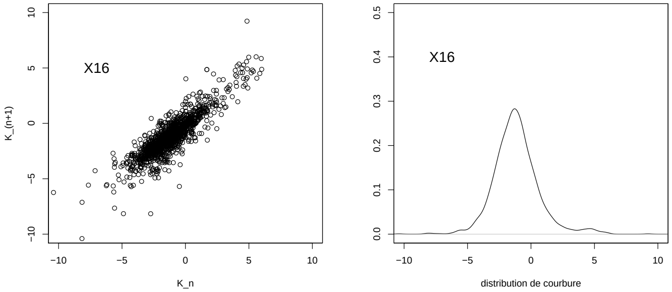
An example of trajectory :



- The norm of the velocity is constant
- The trajectory is smooth, the fish seems to turn constantly

Two key elements in the statistical analysis of the curvature :

- Strong correlation between two time steps
- Gaussian form of the stationary state



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$\Rightarrow$ *Numerical simulation*

Simple computations

The individual model (PTW) in scaled variables :

$$\begin{aligned} \frac{d\vec{x}}{dt} &= \vec{\tau}(\theta) \\ \frac{d\theta}{dt} &= \kappa \\ d\kappa &= -\kappa\, dt + \sqrt{2}\alpha\, dB_t \end{aligned}$$

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- $\kappa(t) = \mathrm{e}^{-t}\kappa_0 + \sqrt{2}\alpha\mathrm{e}^{-t}\int_0^t \mathrm{e}^s\,dB_s.$

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- $\vec{x} \ = ?$

Hyp. : $\vec{x}_0 = (0,0)$, $\theta_0 \sim \mathcal{U}]-\pi,\pi]$, $\kappa_0 \sim \mathcal{N}(0,\alpha^2)$, θ_0, κ_0 and B_t independent.

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Thm. Under above hypothesis, we have :

$$\begin{aligned}
\mathbb{E}\{\vec{x}(t)\} &= (0,0)\,, \qquad \forall\, t \geq 0, \\
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In particular :

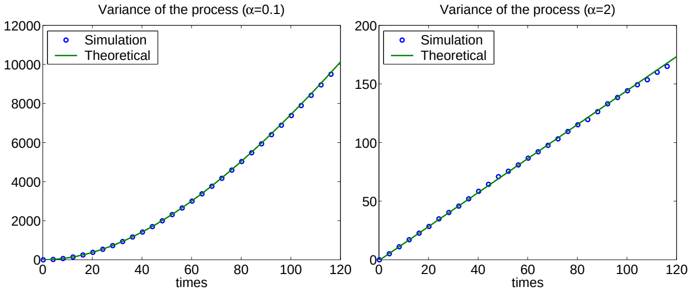
$$\mathrm{Var}\{\vec{x}(t)\} \overset{t\to+\infty}{\sim} 2\mathcal{D}\, t,$$

with :

$$\mathcal{D} = \int_0^{\infty} \exp\left(-\alpha^2(-1+s+\mathrm{e}^{-s})\right)\, ds.$$

Numerical illustration

We use Monte-Carlo method to estimate the mean square displacement :



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To **derive rigorously** a diffusive equation from the PTW model, we can :

- fully characterize the process $x(t)$ using the expression

$$x(t) = x_0 + \int_0^t \vec{\tau}(\theta(s))\, ds$$

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Kinetic method

We start from our PTW model :

$$\frac{d\vec{x}}{dt} = \vec{\tau}(\theta)$$
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$$\partial_t f + \vec{\tau} \cdot \nabla_{\vec{x}} f = Lf.$$

with

$$Lf = -\kappa \partial_\theta f + \partial_\kappa(\kappa f) + \alpha^2 \partial_{\kappa^2} f$$

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We introduce the diffusive rescaling :

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$$f^\varepsilon(t', x', ..) = \frac{1}{\varepsilon^2} f(\frac{t'}{\varepsilon^2}, \frac{x'}{\varepsilon}, ..)$$

which satisfies :

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Question : *What is the limit for f^ε when $\varepsilon \longrightarrow 0$?*

We consider an Hilbert expansion of f^ε :

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  where M Gaussian with zero mean and variance α^2.

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  We introduce an auxiliary function $\vec{\chi}$ satisfying :

$$-L\vec{\chi} = \frac{M(\kappa)}{2\pi}\,\vec{\tau}. \qquad (2)$$

  Then : $f^1 = -\vec{\chi}\cdot\nabla_{\vec{x}}\rho^0 + cM(\kappa).$

Integrating equation (1) in (θ, κ)

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we have the equation of mass conservation :

$$\partial_t \rho^\varepsilon + \nabla_{\vec{x}} \cdot J^\varepsilon = 0,$$

where

$$\rho^\varepsilon(t, \vec{x}) = \int_{\theta,\kappa} f^\varepsilon \, d\kappa \, d\theta, \quad J^\varepsilon(t, \vec{x}) = \int_{\theta,\kappa} \frac{f^\varepsilon}{\varepsilon} \vec{\tau}(\theta) \, d\kappa d\theta.$$

Inserting the Hilbert expansion in the expression of the flux J^ε :

$$\begin{aligned} J^\varepsilon(t,\vec{x}) &= \frac{1}{\varepsilon}\int_{\theta,\kappa} f^\varepsilon \vec{\tau}(\theta)\, d\kappa d\theta \\ &= \int_{\theta,\kappa} f^1\, \vec{\tau}(\theta)\, d\kappa d\theta + O(\varepsilon), \end{aligned}$$

we have at the limit $\varepsilon \longrightarrow 0$:

$$J^0(t,\vec{x}) = -\left(\int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi}\, d\theta\, d\kappa\right) \nabla_{\vec{x}} \rho^0.$$

Diffusive equation

Thm. The distribution f^ε solution of (1) satisfies :

$$f^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} \rho^0 \, \frac{M(\kappa)}{2\pi},$$

with :

$$\partial_t \rho^0 + \nabla_{\vec{x}} \cdot J^0 = 0,$$
$$J^0 = -D \nabla_{\vec{x}} \rho^0,$$

where $D = \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi} \, d\theta \, d\kappa$ and $\vec{\chi}$ solution of (2).

Value of D

$\vec{\chi}$ could be seen as the limit of the parabolic equation :

$$\partial_t \vec{\chi}_t = L\vec{\chi}_t + \vec{\tau}(\theta)\,\frac{M(\kappa)}{2\pi}\,, \quad \vec{\chi}(t=0) = 0.$$

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Therefore :

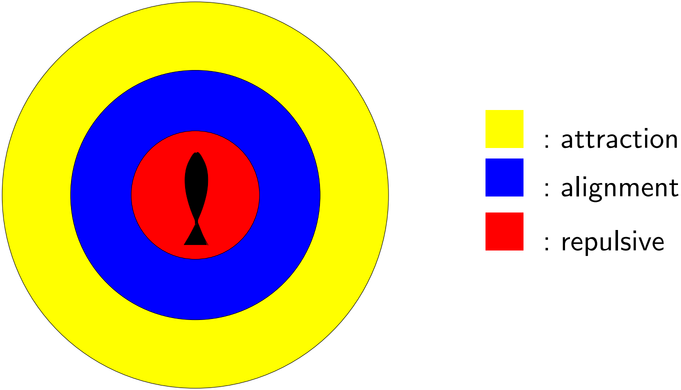
$$\begin{aligned} D &= \lim_{t\to+\infty} \int_{\theta,\kappa} \vec{\tau} \otimes \vec{\chi}_t \, d\theta \, d\kappa \\ &= \dots \\ &= \frac{\mathcal{D}}{2}\mathrm{Id}, \end{aligned}$$

where Id denotes the 2x2 identity tensor and

$$\mathcal{D} = \int_0^\infty \exp\left(-\alpha^2(-1+s+\mathrm{e}^{-s})\right) ds.$$

Fish in interaction

Classical model with 3 zones



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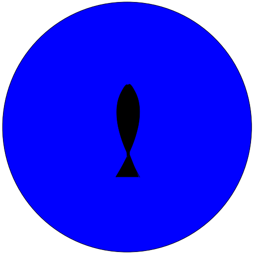
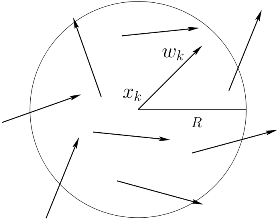


 : alignment

Vicsek model ('95)

Discrete dynamic :



$$\begin{aligned} x_k^{n+1} &= x_k^n + \Delta t\, \omega_k^n \\ \omega_k^{n+1} &= \bar{\omega}_k^n \end{aligned} \qquad (3)$$

$$\text{with } \bar{\omega}_k^n = \frac{\sum_{|x_j - x_k| < R} \omega_j^n}{\left|\sum_{|x_j - x_k| < R} \omega_j^n\right|},$$

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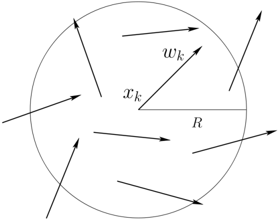

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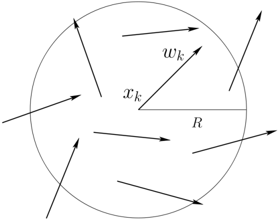

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Continuous dynamic :

$$\begin{aligned} \frac{dx_k}{dt} &= \omega_k \\ d\omega_k &= (\mathrm{Id} - \omega_k \otimes \omega_k)(\nu\, \bar{\omega}_k\, dt + \sqrt{2d}\, dB_t) \end{aligned} \quad (4)$$

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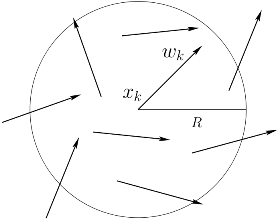

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Remark. (eq. 4) + "$\nu \Delta t = 1$" $\Rightarrow$ (eq. 3)

Kinetic equation

The density of particles $f(t, x, \omega)$ satisfies (formally) :

$$\partial_t f + \omega \cdot \nabla_x f + \nabla_\omega \cdot (F f) = d\Delta_\omega f, \qquad (5)$$

with :

$$F = (\mathrm{Id} - \omega \otimes \omega)\, \nu \Omega(x) \quad , \quad \Omega(x) = \frac{J(x,t)}{|J(x,t)|}$$

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We define :

$$Q(f) = -\nabla_\omega \cdot (Ff) + d\Delta_\omega f.$$

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$$\partial_t f + \omega \cdot \nabla_x f = Q(f) \tag{5}$$

with :

$$F = (\mathrm{Id} - \omega \otimes \omega)\, \nu \Omega(x) \quad , \quad \Omega(x) = \frac{J(x,t)}{|J(x,t)|}$$

$$J(x,t) = \int_{|y-x|<R,\, \omega^* \in \mathbb{S}^1} \omega^* \, f(y, \omega^*, t)\, dy\, d\omega^*$$

We define :

$$Q(f) = -\nabla_\omega \cdot (Ff) + d\Delta_\omega f.$$

Collision operator

- The operator Q can be rewritten :

$$Q(f) = \nabla_\omega \cdot \left(M_\Omega(\omega) \nabla_\omega \left(\frac{f}{M_\Omega(\omega)} \right) \right),$$

with : $M_\Omega(\omega) = C \exp\left(\frac{\omega \cdot \Omega}{T}\right)$ and $T = d/\nu$.

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$$\int_\omega Q(f)\psi \, d\omega = 0 \;\Rightarrow\; \begin{cases} \psi_1(\omega) = 1 \\ \psi_2(\omega) = h_\Omega(\omega) \end{cases}$$

with h solution of an elliptic equation.

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In order to derive a macroscopic equation, we introduce the scaling :

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- Then we integrate the kinetic equation against the collisional invariant :

$$\int_\omega \left[\partial_t f^\varepsilon + \omega \cdot \nabla_{\vec{x}} f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon)\right] \psi \, d\omega$$

Macroscopic equation

Thm. The distribution f^ε solution of (5) satisfies :

$$f^\varepsilon \overset{\varepsilon \to 0}{\rightharpoonup} \rho\, M_{\Omega(t,x)},$$

with :

$$\begin{aligned} \partial_t \rho \quad &+ \nabla_x \cdot (c_1 \rho \Omega) = 0 \\ \partial_t(\rho\Omega) &+ \nabla_x \cdot (c_2 \rho \Omega \otimes \Omega) + \lambda\, (\mathrm{Id} - \Omega \otimes \Omega) \nabla_x \rho = 0, \end{aligned}$$

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- ρ and Ω have different convection speeds ($c_1 \neq c_2$).

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Locally in space, the velocity of particle is distributed according to :

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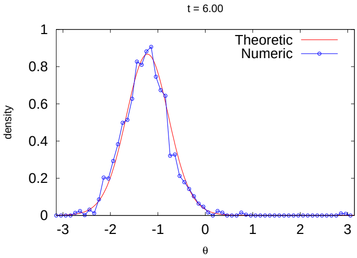
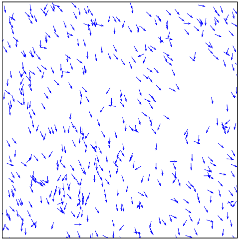
Illustration :





Hyperbolic equation : ρ, Ω

In one direction, the system is written :

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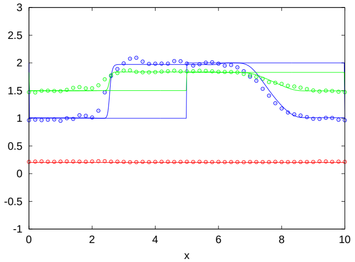

Fig.: Solution of a Riemann problem with : the density ρ, the direction of the velocity θ, the temperature T.

Thank you for your attention.