On a kinetic model of swarms

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Outline

- Swarming: a phenomenon in biological systems
- Some examples
- A particle model
- Kinetic theory and a Vlasov type model
- Particular solutions (steady states)
- Rigorous treatment of the mean field limit
- Conclusions

A natural phenomenon:

Groups of large number of animals (birds, fish, insects) are frequently observed to form aggregations (flocks, shoals, swarms...)

- Localized in space
- Move in an organized way (in the same direction)
- May serve various purposes (migration, transporting heavy objects, protection from predators, mating...)
- Self-organize

Question: How to describe the behavior of the system as a whole based on the principles of interaction of individuums?

A particle system: $(x_i, v_i) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad i = 1 \dots N,$

$$\begin{cases} \dot{x}_i = v_i \\ m_i \dot{v}_i = (\alpha - \beta |v_i|^2)\, v_i - \nabla_{x_i} \sum_{i \neq j} U(|x_i - x_j|) \end{cases}$$

The α-β-term makes the particles to prefer the speed $= \sqrt{\frac{\alpha}{\beta}}$ (self-propulsion balances "friction").

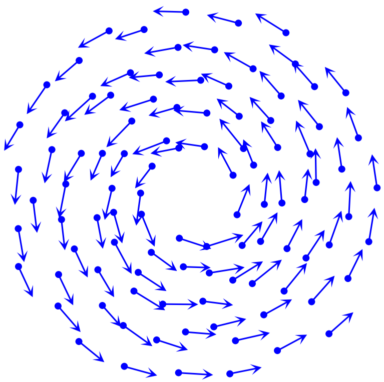
The potential is (Morse)

$$U(r) = C_r e^{-r/\ell_r} - C_a e^{-r/\ell_a}.$$

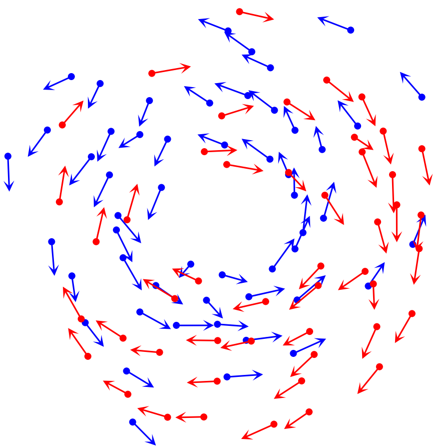
Ref: Levine, Rappel, Cohen; Phys. Rev. E (2000)
D'Orsogna, Chuang, Bertozzi, Chayes; PRL (2006)

Different types of dynamics

- Dispersion
- Rigid body rotation
- Flocks (localized patterns of particles moving with the same velocity)
- Mills (vortex type circular motion)



(single)



(double)

The patterns are observed to be stable with respect to perturbations of the potential and by adding a (small amount of) noise.

Depending on the parameters of the potential, there are different types of behavior with increasing N:

- Catastrophic

  The spacing between particles $\to 0$ with increasing N

- H-stable

  The spacing between particles is preserved as N increases. The system behaves in an extensive way.

Large number of particles $\Rightarrow$ computational cost increases.

In which cases are "fluid-type" models adequate?

A kinetic approach [Carrillo, D'Orsogna, P., to appear in KRM, 2009]

$$f^{(N)}(\{x_i\}, \{v_i\}, t) \quad - \quad N\text{-particle probability density}$$

Conservation of probability:

$$\int_{A_t} f^{(N)}(t)\, d\{x_i\}\, d\{v_i\} = \int_{A_0} f^{(N)}(0)\, d\{x_i\}\, d\{v_i\}, \quad \text{if} \quad A_t = \Phi_t(A_0),$$

where Φ_t is the flow map of the dynamical system.

⇒ Liouville equation

$$\partial_t f^{(N)} + \sum_i \nabla_{x_i} \cdot (v_i\, f^{(N)}) + \sum_i \nabla_{v_i} \cdot (\tfrac{F_i}{m_i} f^{(N)}) = 0.$$

Marginal densities:

$f^{(N-1)}(\dots) = \int f^{(N)}\, dx_N\, dv_N$ – $(N\text{-}1)$-particle

$\vdots$

$f^{(2)}(\dots) = \int \dots \int f^{(N)}\, dx_3\, dv_3 \dots dx_N\, dv_N$ – 2-particle

$$f = f^{(1)}(x_1, v_1, t) = \int \ldots \int f^{(N)}\, dx_2\, dv_2 \ldots dx_N\, dv_N \quad - \quad \text{1-particle}$$

Vlasov's limit (mean field theory)

Assume:

$$m_i = \frac{1}{N}, \quad F_i = \frac{1}{N}(\alpha - \beta|v_i|^2)v_i - \frac{1}{N^2}\nabla_{x_i}\sum_{i \neq j} U(|x_i - x_j|)$$

Molecular chaos:

$$f^{(2)}(x_1, x_2, v_1, v_2, t) = f^{(1)}(x_1, v_1, t)\, f^{(1)}(x_2, v_2, t)$$

Integrate the Liouville equation to obtain

$$f_t + \nabla_x \cdot (vf) + \nabla_v \cdot ((\alpha - \beta|v|^2)vf) - \frac{N-1}{N}\nabla_v \cdot (\nabla_x U * \rho)f) = 0$$

Take the limit $N \to \infty$:

$$f_t + \nabla_x \cdot (vf) + \nabla_v \cdot ((\alpha - \beta|v|^2)vf) - \nabla_v \cdot (\nabla_x U * \rho)f) = 0$$

(Vlasov's equation).

References:

- Bogoliubov (1946), BBGKY...
- Smereka, Russo (1996)
- Neunzert (1977); Braun, Hepp (1977); Maslov (1978); Dobrushin (1979); Golse (JEDP, 2003)
  (Hamiltonian systems)
- Ha, Tadmor (2008), Ha, Liu (2009)
  (Cucker-Smale dynamical system)
- Degond-Motsch (2007)
  Couzin-Vicsek model; continuum limit

Particular solutions at the hydrodynamic level

Equations for the moments

$$\rho = \int f\,dv, \quad \rho u = \int f v\,dv,$$

take the form

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho u) = 0 \\ (\rho u)_t + \nabla_x \cdot (\rho u \otimes u) = (\alpha - \beta |u|^2)\rho u - (\nabla_x U * \rho)\,\rho + \text{[other terms...]} \end{cases}$$

can be closed by the *monokinetic* ansatz

$$f = \rho(x,t)\,\delta(v - u(x,t)),$$

in which case the "other terms" vanish.

In fact, it works both ways:

Lemma. If ρ, u are smooth, then $f = \rho\,\delta(v-u)$ is a solution of

$$f_t + \nabla_x \cdot (vf) + \nabla_v \cdot \big((\alpha - \beta|v|^2)vf - (\nabla_x U * \rho) f\big) = 0 \qquad \text{(K)}$$

$\Leftrightarrow$ ρ, u are solutions of

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho u) = 0 \\ (\rho u)_t + \nabla_x \cdot (\rho u \otimes u) = (\alpha - \beta|u|^2)\rho u - (\nabla_x U * \rho)\,\rho. \end{cases} \qquad \text{(H)}$$

Proof: Write the weak formulation of (K) and (H).

Flocking solutions: $\boxed{u = u_0 = const, \;\; |u| = \sqrt{\frac{\alpha}{\beta}}}$

Then $\rho = \rho_0(x - u_0 t)$, with ρ_0 determined from

$$-(\nabla_x U * \rho_0)\,\rho_0 = 0,$$

$$\text{i. e.} \quad U * \rho_0 = C, \quad \text{whenever } \rho_0 \neq 0.$$

Depending on the potential U there may exist ρ_0 with compact support [Levine, Rappel, Cohen, 2000]

Swarming solutions: $$\rho = \rho(|x|), \quad u(x) = \sqrt{\frac{\alpha}{\beta}}\, \frac{x^\perp}{|x|}$$

(ρ rad. symmetric, u in a rotatory state).

The continuity eqn. is satisfied automatically;
the moment balance eqn. becomes

$$(u \cdot \nabla_x)u = -\nabla_x U * \rho\,,$$

We compute

$$(u \cdot \nabla_x)u = -\tfrac{\alpha}{\beta}\, \frac{x}{|x|^2} = -\tfrac{\alpha}{\beta}\, \nabla_x \log|x|$$

Then ρ is determined from the integral equation

$$U * \rho = D + \tfrac{\alpha}{\beta} \log|x|, \quad \text{whenever} \quad \rho \neq 0,$$

where D is a constant of integration.

Depending on U, there may exist solutions $\rho(r)$ supported in an interval $0 < a \leq r \leq b$. [Levine, Rappel, Cohen, 2000]

Double mill solutions A superposition of two monokinetic solutions

$$f = \rho_1 \, \delta(v - u_1) + \rho_2 \, \delta(v - u_2),$$

with

$$\rho = \rho_1 + \rho_2, \quad \rho u = \rho_1 u_2 + \rho_2 u_2.$$

is a solution of (K) $\Leftrightarrow$

$$\begin{cases} \rho_t + \nabla_x \cdot (\rho u) = 0 \\ (\rho u)_t + \nabla_x \cdot (\rho_1 u_1 \otimes u_1) + \nabla_x \cdot (\rho_2 u_2 \otimes u_2) = -(\nabla_x U * \rho) \, \rho. \end{cases} \tag{H$_2$}$$

In particular, if

$$\rho_1 = \rho_2 = \tfrac{1}{2} \, \rho, \quad u_1 = -u_2$$

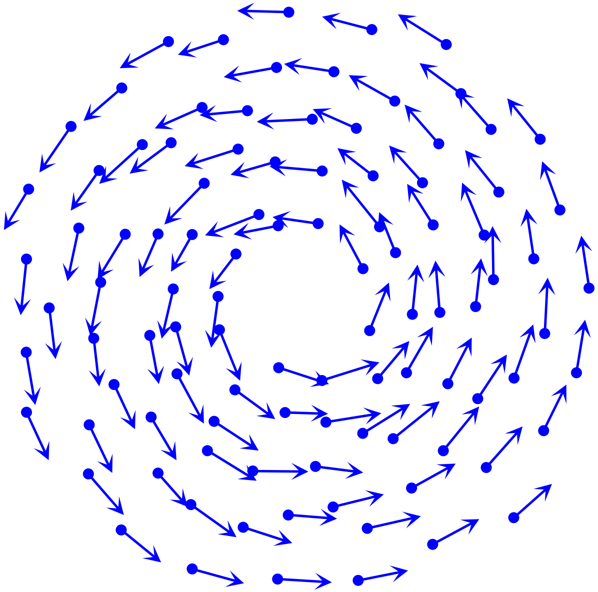
then the steady state is possible precisely when

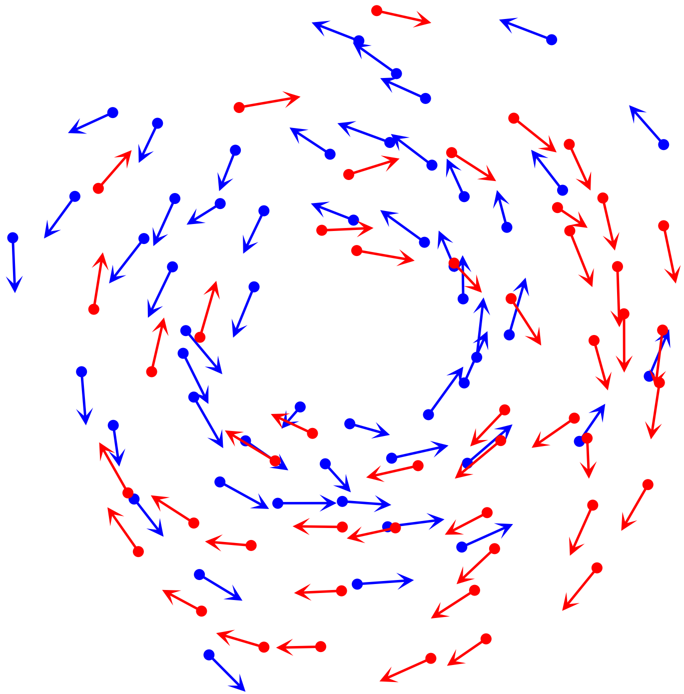
$$\begin{cases} \nabla_x \cdot (\rho u) = 0 \\ \nabla_x \cdot (\rho u \otimes u) = -(\nabla_x U * \rho) \, \rho. \end{cases}$$

Thus, for any single mill solution there is a double mill solution with $\rho_1 = \rho_2 = \frac{1}{2}\rho.$ supported in an interval $0 < a \le r \le b$.

Single and double mills

[snapshot from a simulation with $N = 100$]





Rigorous derivation of the Vlasov dynamics

Assume that $U(r)$ is nice.

(of class C^2, U and U' are bounded, and $U'(0) = 0$)

The dynamical system ($m_i = \frac{1}{N}$; pairwise force $\sim m_i m_j$)

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = (\alpha - \beta |v_i|^2)\, v_i - \frac{1}{N} \nabla_{x_i} \sum\limits_{i \neq j} U(|x_i - x_j|) \end{cases}$$

has global in t solutions $(x_i(t), v_i(t))$ for any initial data (x_i^0, v_i^0), and the dependence on (x_i^0, v_i^0) is smooth.

Consider the "empirical measure" on the phase space (x, v):

$$\mu_N(t) = \sum_{i=1}^{N} m_i\, \delta_{(x_i(t), v_i(t))}$$

Proposition 1. $\mu_N(t)$ is a solution of equation (K), in the sense of distributions.

Introduce the energy

$$\mathcal{E}(t) = \frac{1}{2}\int f\,|v|^2\,dx\,dv + \frac{1}{2}\int U(|x-y|)\,\rho(x,t)\rho(y,t)\,dx\,dy.$$

Then

$$\frac{d\mathcal{E}}{dt} = \int f(\alpha - \beta|v|^2)\,|v|^2\,dx\,dv \le \alpha\int f|v|^2\,dv - \frac{\beta}{M}\Big(\int f|v|^2\,dv\Big)^2,$$

where $M = \int f\,dx\,dv$. Since the potential energy verifies

$$\frac{1}{2}\int U(|x-y|)\,\rho(x,t)\rho(y,t)\,dx\,dy \le CM^2,$$

where $C = \frac{1}{2}\sup|U|$, we have

Proposition 2. The energy $\mathcal{E}(t)$ satisfies

$$\mathcal{E}(t) \le \max\left\{\mathcal{E}(0), CM^2 + \tfrac{\alpha M}{\beta}\right\},$$

where M is the total mass.

Theorem. [Mean-field limit for particle dynamics]

Let $f_0 \geq 0$ be such that $\left|\mathcal{E}[f_0]\right| < +\infty$, and consider a sequence $\{(x_i^0, v_i^0)\}$, such that

$$\mu_N(0) := \sum_{i=1}^{N} m_i \, \delta_{(x_i^0, v_i^0)} \to f_0$$

weakly* as measures and satisfy $\mathcal{E}[\mu_N(0)] = O(1)$. Then $\forall \; T > 0$ the correspoding sequence $\mu_N(t)$ converges in $C([0,T]; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ (weak*) to the unique solution of the Vlasov equation

$$f_t + \nabla_x \cdot (vf) + \nabla_v \cdot \big((\alpha - \beta|v|^2)vf - (\nabla_x U * \rho)f\big) = 0 \qquad \text{(K)}$$

with the initial data f_0.

Proof: [cf. Golse, 2003; Neunzert 1986]

We have

$$\dot{v}_i = (\alpha - \beta|v_i|^2)\, v_i - \frac{1}{N} \sum_{i \neq j} \nabla_{x_i} U(|x_i - x_j|),$$

$\Rightarrow |\dot{v}_i(t)|$ are bounded uniformly in $N \Rightarrow (x_i(t), v_i(t))$ are equicontinuous on $\mathbb{R}_+$.

$\Rightarrow \mu_N(t)$ is rel. compact in $C([0,T]; \mathcal{M}(\mathbb{R}^d \times \mathbb{R}^d))$ (weak*)

Take a subsequence $\mu_N \to \mu$ (weak*), pass to the limit

$$\nabla U * \rho_N \to \nabla U * \rho$$

uniformly on cpt. subsets of $\mathbb{R}_+ \times \mathbb{R}^d$. Then

$$(\nabla U * \rho_N)\, \mu_N \to (\nabla U * \rho_N)\, \mu$$

in the sense of distributions $\Rightarrow \mu$ is a solution of the Vlasov equation.

Conclusions / things to do

- Single and double mills are obtained as exact steady solutions on the level of kinetic equation
- Vlasov's equation is obtained rigorously for a regularized potential in the $1/N$ scaling
- Numerical simulation and comparison of particle/kinetic/fluid solutions
- A more complete description of steady solutions, stability of the steady states.
- Continuum (hydrodynamic) limit of the kinetic model