Nonlocal aggregation models.

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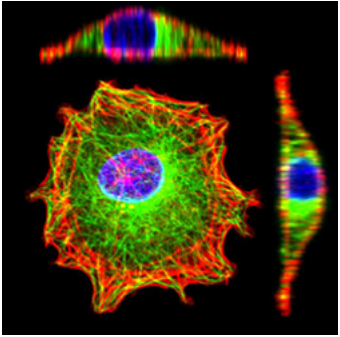
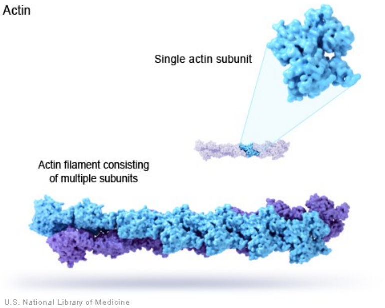
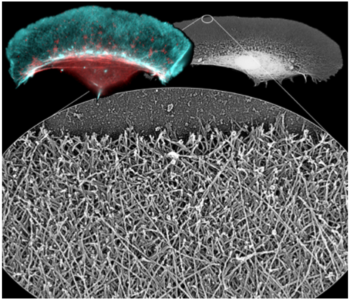
The model.

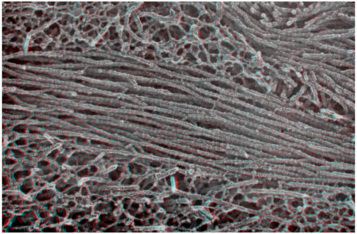
Actin filaments.



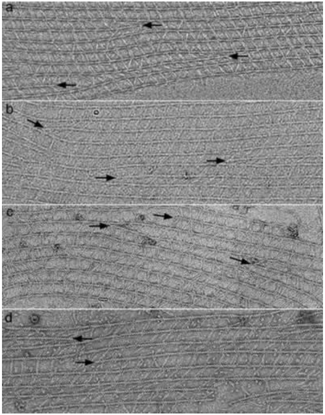


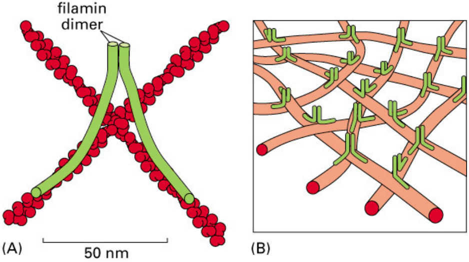
Mesh of actin filaments.





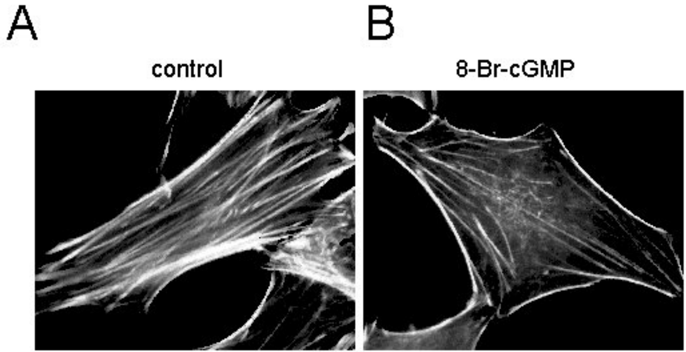
Cross-linking proteins.



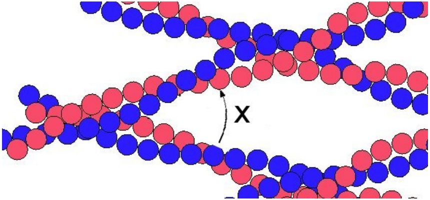


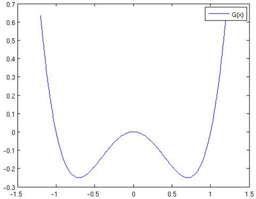
filamin, ABP-50, fibrillin, villin, fascin...

Actin filaments with or without cross-linking proteins.



Action of Cross-linking proteins.





$G'(x)$: moment force between two filaments.

Notations.

We the case where the density of proteins is homogeneous in space.

- $\rho(t,\cdot) \in M^1(S^1 \text{ or } \mathbb{R})$. $\rho(t,x)$ is the density of filaments of orientation x. We normalise it by :

$$\int_{\mathbb{R}^n} d\rho(t,\cdot) = 1,$$

- $G(x)$ is the interaction potential between two filaments. We assume that this potential is symetric.

The model.

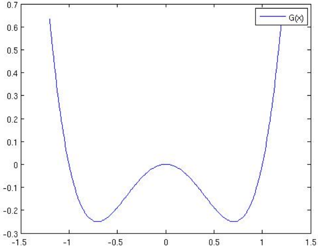
The force applied to one filament is :

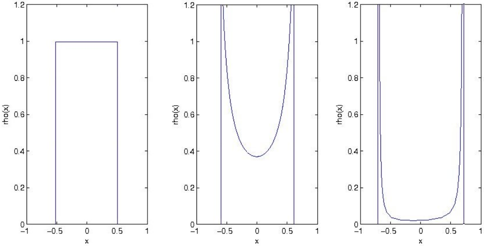
$$\partial_x \left(G * \rho \right).$$

We assume that the rotating speed of a particule is proportionnal to the moment applied. Then, ρ evolves as :

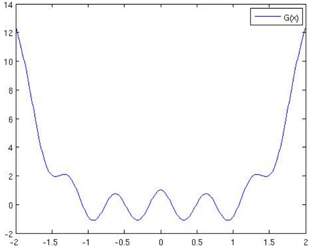
$$\begin{cases} \rho(0,\cdot) = \rho_0, \\ \partial_t \rho = \partial_x \left(\rho \partial_x \left(G *_x \rho \right) \right). \end{cases} \tag{1}$$

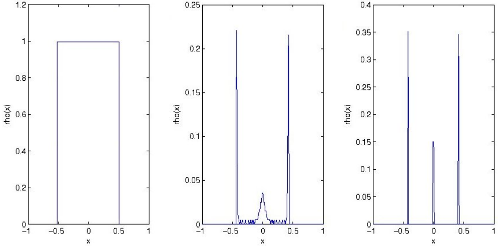
simulations



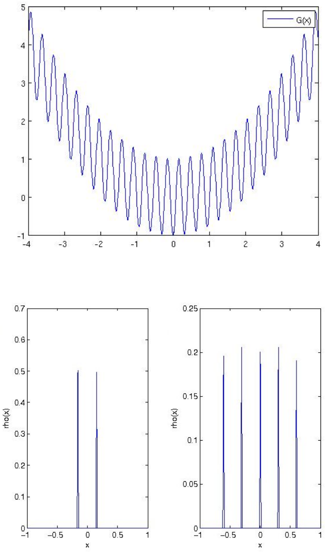


simulations

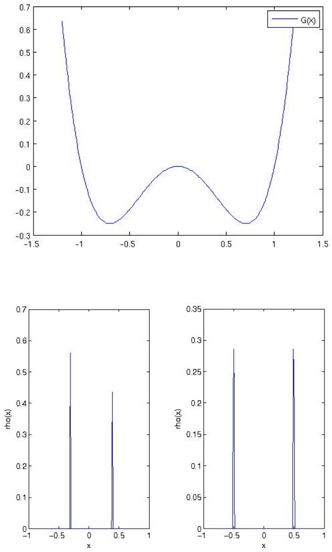




non-unicity of steady-states



non-unicity of steady-states



Local stability analysis

Pseudo-inverse

Let consider a measure $\rho \in M^1(\mathbb{R})$, of total mass 1 : $\int_{\mathbb{R}} d\rho = 1$.
Then,

$$x \mapsto \int_{-\infty}^{x} d\rho,$$

is a increasing function $\mathbb{R} \to [0,1]$. One can then define its pseudo-inverse $u : [0,1] \to \mathbb{R}$ as :

$$u(z) = \inf \left\{ x \in \mathbb{R};\ \int_{-\infty}^{x} d\rho \geq z \right\}.$$

The Pseudo-inverse equation

u then satisfies the following equation :

$$\partial_t u(t,z) = \partial_x \left(G *_x \rho(u(t,z)) \right), \qquad (2)$$

or :

$$\partial_t u(t,z) = \int_0^1 G' \left(u(t,\xi) - u(t,z) \right) \, d\xi.$$

steady-states are sums of Dirac masses

Proposition

If G is analytical, $G \in L^1(\mathbb{R})$ and $\left|\int_{\mathbb{R}} G\right| < \infty$, $\int_{\mathbb{R}} G \neq 0$, then every steady solution of equation (1) is a sum of Dirac masses $\bar{\rho} = \sum_{i\in\mathbb{N}} \rho_i \delta_{x_i}$, where the sequence $(x_i)_i$ has no accumulation point (this sum is finite if the support of $\bar{\rho}$ is bounded).

Proof :

$$\forall z \in [0,1], \quad 0 = \partial_x \left(G * \bar{\rho}(\bar{u}(z))\right).$$

Then, if $u([0,1])$ has an accumulation point, $0 = \partial_x \left(G * \bar{\rho}\right)$. Then $Cte = G * \bar{\rho}$, and if we apply a Fourier transform and evaluate it in 0, we get :

$$Cte\, \delta_0(0) = \left(\int_{\mathbb{R}} G\right)\left(\int_{\mathbb{R}} d\rho\right),$$

which is absurd.

condition to be a steady-state

$$\partial_t u(t,z) = \int_0^1 G'\left(u(t,\xi) - u(t,z)\right)\, d\xi.$$

Proposition

$\bar{\rho} = \sum_{i=1}^n \bar{\rho}_i \delta_{\bar{u}_i}$, $\bar{\rho}_i \neq 0$ *is steady state of eq. (1) if and only if :*

$$(\bar{\rho}_i)_i \in Ker\left(\left(G'(\bar{u}_i - \bar{u}_j)\right)_{i,j}\right),$$

that is $\forall i = 1, \ldots, n, \quad \partial_x (G *_x \rho)(u_i) = 0.$

Proof :

$$\begin{aligned}\partial_t \bar{u} &= \int_0^1 G'\left(\bar{u}(t,\xi) - \bar{u}(t,z)\right)\, d\xi \\ &= \sum_i \bar{\rho}_j G'\left(\bar{u}_j - \bar{u}_i\right)\end{aligned}$$

Necessary conditions for linear stability 1

Proposition

For a steady solution $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{x_i}$, $\rho_i \neq 0$ of eq. (1) to be linearly stable under small dislocations, it is necessary that :

$$\forall i = 1, \ldots, n, \quad \partial_{xx}^2 (G * \rho)(u_i) > 0.$$

Necessary conditions for linear stability 2

Proposition

For a steady solution $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{u_i}$, $\rho_i \neq 0$ *of eq. (1) to be linearly stable under small perturbations of the* u_i, *it is necessary that the linear application* $\mathcal{L}_M$ *defined by the matrix*

$$M = \left(\rho_i G''(u_i - u_j)\right)_{i,j} - diag\left(\left(G''(u_i - u_j)\right)_{i,j} (\rho_j)_j\right),$$

has a spectrum included in $\mathbb{R}^*_- \times i\mathbb{R}$ *when restricted to the hyperspace* $\{(w_i)_{i=1,\ldots,n}; \sum_{i=1}^{n} w_i = 0\}$.

Local stability with support conditions

Proposition

A steady-state $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{u}_i}$ *is locally stable under a support condition, that is :*

$$\|u(t) - \bar{u}\|_\infty \leq Ce^{-\kappa t}, \ \kappa = \kappa(\|G'''\|_\infty, n) > 0,$$

$$\textit{as soon as } \|u(0) - \bar{u}\|_\infty \textit{ is small enough,}$$

if and only if it is satisfies the linear stability conditions 1 and 2.

Proof1

We consider a perturbation $u = \bar{u} + v$ of the steady-state $\bar{u}$. We first estimate $\int_{I_i} v$:

$$\begin{aligned} \partial_t v &= \int_0^1 G'\left(u(\xi) - u(z)\right) \, d\xi \\ &= \sum_{j=1}^{n} G''\left(\bar{u}_j - \bar{u}_i\right) \int_{I_j} v(\xi) \, d\xi - v(z) \sum_{j=1}^{n} G''\left(\bar{u}_j - \bar{u}_i\right) \bar{\rho}_j \\ &\quad + O(\|v\|_\infty^2). \end{aligned}$$

Then, if we integrate the equation over I_i,

$$\begin{aligned} \frac{d}{dt} \int_{I_i} v &= \left[\left(\rho_i G''(u_i - u_j)\right)_{i,j} - diag\left(\left(G''(u_i - u_j)\right)_{i,j} (\rho_j)_j \right) \right] \left(\int_{I_i} v \right)_i \\ &\quad + O(\|v\|_\infty^2). \end{aligned}$$

Proof2

We now get estimates on $|v|$:

$$
\begin{aligned}
\partial_t |v| &= sgn(v(z)) \int_0^1 G'\left(u(\xi) - u(z)\right) d\xi \\
&= -G'' * \bar{\rho}(x_i)|v| + O\left(\left\| (\int_{I_i} v)_i \right\|_\infty \right) \\
&\quad + O(\|v\|_\infty^2)
\end{aligned}
$$

Proof3

So, if we consider the vector $w := \begin{pmatrix} \|v\|_\infty \\ \int_{I_1} v \\ \vdots \\ \int_{I_n} v \end{pmatrix}$, we get thanks to previous estimates :

$$\frac{d}{dt} w = \begin{pmatrix} -G'' * \bar{\rho}(x_{i_0}) & O(1) \\ 0 & (\rho_i G''(u_i - u_j))_{i,j} - diag\left((G''(u_i - u_j))_{i,j} (\rho_j)_j \right) \end{pmatrix} + O(\|w\|^2),$$

and a Gronwall lemma shows the proposition.

A Lemma

Lemma

If the steady state $\bar{\rho}$ satisfies the linear stability condition 2, there exist $\eta > 0$, such that if $\|(\tilde{\rho}_i)_{i=1,\dots,n} - (\bar{\rho}_i)_{i=1,\dots,n}\| < \eta$, then there exist a unique $(\tilde{u}_i)_{i=1,\dots,n}$ close to $(\bar{u}_i)_{i=1,\dots,n}$ such that $\tilde{\rho} := \sum_{i=1}^{n} \tilde{\rho}_i \delta_{\tilde{u}_i}$ is a stable steady solution of (1), and such that $\tilde{\rho}$ has the same center of mass as $\bar{\rho}$.

Notice that the steady state $\tilde{\rho}$ obtained this way is also stable.

Proof

We consider the function

$$F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i) := (G'(\tilde{u}_i - \tilde{u}_j))_{i,j}(\tilde{\rho}_i)_i.$$

We want to apply the implicit function theorem to find to describe the set $\{F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i) = 0\}$. The derivative to consider then is :

$$D_{(\tilde{u}_i)_i} F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i),$$

which is equal to the matrix of the linear stability assumption 2.

Necessary conditions for linear stability 3.

Proposition

For a steady state $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{x}_i}$ to be linearly stable under distant invasions, it is necessary that there exist $(y_i)_{i=1,\dots,n-1}$ such that :

$$G' * \rho < 0 \ \ on \ (-\infty, x_1), \ G' * \rho > 0 \ \ on \ (x_n, +\infty),$$

$$\big(G' * \rho(x) = 0, \ x \in (x_i, x_{i+1})\big) \Leftrightarrow x = y_i, \tag{3}$$

and $G'' * \rho(y_i) < 0$.

Local stability without support conditions

Proposition

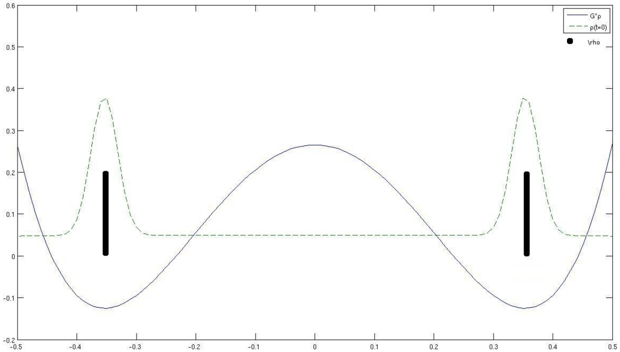
if a steady-state $\bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{x_i}$ *satisfy the three stability conditions, then it is locally stable, that is :*

if $\|u(0) - \bar{u}\|_{L^1}$ *is small enough, and is not flat on neighbourhoods of the points* $(y_i)_i$ *defined in (3), then there exist* $\tilde{\rho} = \sum_{i=1}^{n} \tilde{\rho}_i \delta_{\tilde{x}_i}$ *close to* $\bar{\rho}$*, such that :*

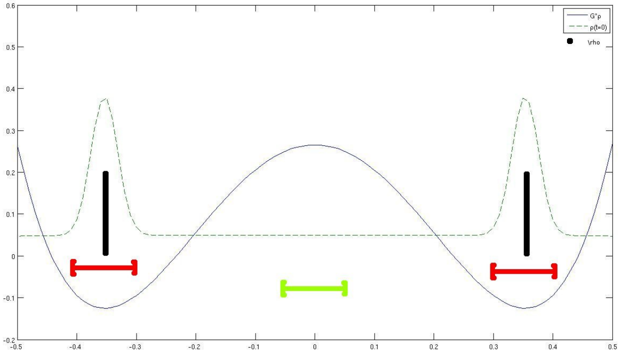
$$\|u(t) - \tilde{u}\|_{L^1} \leq Ce^{-ct},$$

where $c > 0$.

Idea of the proof 1



Idea of the proof 2



Idea of the proof 3

We want to estimate ρ over the green interval I of size ε. At all times, there exist $y(t) \in I$ such that $\partial_x (G *_x \rho)(y(t)) = 0$. Then, for $z \in I$,

$$\partial_t u(t,z) = \left(\partial^2_{xx}(G *_x \rho)(y(t)) + O(\varepsilon)\right)\left(u(t,z) - u(t,y(t))\right).$$

Then,

$$u(t,z) \;=\; \left[u(0,z) - \int_0^t \left(y(s) e^{\int_0^s G''*\rho(y(\tau))+O(\varepsilon)\,d\tau}\right)\,ds\right] e^{\int_0^t G''*\rho(y(\tau))+O(\varepsilon)\,d\tau},$$

that is $u(t) = c(t)u(0) + d(t)$, where :

$$c(t) = e^{\int_0^t G''*\rho(y(\tau))+O(\varepsilon)\,d\tau} \sim e^{ct} \to \infty.$$

And all the mass escapes from the interval I.

Remarks

refined model.

A more complex model has been introduced by Kang, Perthame, Primi, Stevens and Velazquez :

$$\partial_t \rho(t,x) = \int T[\rho](y,x)\rho(t,y)\,dy - \int T[\rho](x,y)\rho(x,t)\,dx,$$

where :

$$T[\rho](x,y) = \int \Gamma_\sigma\left(y - x - G'(z-x)\right)\rho(t,z)\,dz.$$

A refined kinetic model.

They say that for σ small, an approximation of their kinetic model is :

$$\partial_t \rho(t,x) = \frac{\sigma^2}{2}\partial_{xx}^2 \rho(t,x) + \partial_x \left(\rho(t,x) \int G'(x-y)\rho(t,y)\, dy \right)$$

The local stability result doesn't pass to this generalised model : the diffusion can transport mass from one Dirac to another. My guess would be that the result is true under the additional assumption that $\forall i,j, \quad G *_x \bar{\rho}(\bar{u}_i) = G *_x \bar{\rho}(\bar{u}_j)$...

Attractive singular kernels

If G is repulsive and singular at 0, ρ explodes in finite time :

Theorem

(Bertozzi, Carrillo, Laurent) Let ρ be a solution of (1) in $\mathbb{R}^N$, with a nonnegative compactly supported initial data in L^∞. Let G satisfy $\int_0^1 \frac{1}{G'(x)}\,dx < +\infty$ and $\frac{G'(r)}{r}$ monotone decreasing. Then there exists a maximal time $T^ast < \infty$ and a unique solution ρ on the interval $[0, T^*)$. Moreover,

$$\lim_{t \to T^*} \|\rho(\cdot, t)\|_{L^q} = \infty,$$

for $q \in [2, \infty]$ if $N > 2$ and $q \in (2, \infty]$ if $N = 2$.

And if $\int_0^1 \frac{1}{G'(x)}\,dx = +\infty$, the solution is global in time.

Attractive singular kernels

If we add a diffusion to the model and if $G = -\frac{1}{2\pi} log|x|$ (in dimension 2), then (1) becomes the Keller-Segel model :

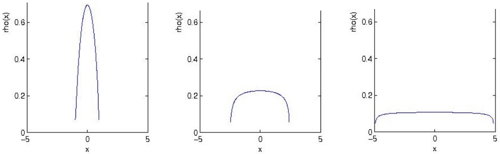
$$\begin{cases} \partial_t n = \Delta n - \xi \nabla \cdot (n \nabla c), \\ -\Delta c = n. \end{cases}$$

repulsive singular kernels

If $G := \delta_0$, equation (1) becomes the porous medium equation :

$$\partial_t \rho = \partial_x(\rho \partial_x \rho).$$

If G is singular and repulsive, it also behaves numerically as a slow diffusion :



repulsive singular kernels

We only have very few results on this up to now :

Proposition

if G is Lipschitz continuous and C^2 on $\mathbb{R} \setminus \{0\}$, then (1) has a unique solution.

Proposition

Let $G_\varepsilon \to (x \mapsto -|x|)$ a sequence of regular kernels, $\bar{\rho}_\varepsilon$ and $\bar{\rho}$ steady-states of :

$$\partial_t \rho = \partial_x \left(\rho \partial_x \left(G *_x \rho \right) + V \right).$$

Then,

$$\bar{\rho}_\varepsilon \to \bar{\rho} \text{ in } M^1.$$