Stochastic Modeling and Simulation of Highway Traffic

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Outline

- The traffic model: properties and dynamics
- Calibration and parameter estimation
- Simulations and comparisons (one-lane highway)
- Deterministic closures and macroscopic models
- Multi-lane extensions
- Conclusions & References

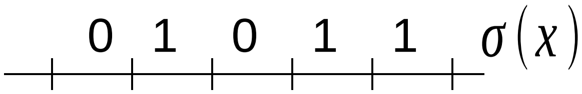
Proposed Traffic Model Properties/Attributes

- Asymmetric Simple Exclusion Process (ASEP)
- Arrhenius microscopic stochastic dynamics
- One directional flow
- Look-ahead interaction potential
- Retarded acceleration
- Timely braking
- Conservation of vehicles (assuming no entrances or exits)
- Numerical simulations via Kinetic Monte Carlo (KMC)
- Extensions to macroscopic traffic flow models and PDEs

Main Statistical Mechanics Concepts

We let Λ denote a lattice of N cells.

We also denote by t (x) the spin configuration at x.



We introduce the microscopic stochastic Ising process $\{\sigma_t\}_{t\geq 0}$

A spin configuration ▪ is an element of the configuration space $\Sigma=\{0,1\}^{\Lambda}$ and we write

$$\sigma=\{\sigma(x): x\in\Lambda\}$$

The stochastic process $\{\sigma_t\}_{t\geq 0}$ is a continuous time jump Markov process on $L^{\infty}(\Sigma, R)$ with generator

$$Mf(\sigma)=\sum_{\substack{x\in\Lambda \\ y\neq x}} c(\sigma)[f(\sigma^{¿})-f(\sigma)]$$

Main Statistical Mechanics Concepts

The corresponding energy Hamiltonian is

$$H(\sigma)=\frac{1}{2}\sum_{x\in\Lambda} U(x)\sigma(x)$$

where the interaction potential is given by

$$U(x)=\sum_{\substack{z\neq x\\ z\in\Lambda}} J(x,z)\sigma(z)-h$$

where J denotes the local interaction potential

$$J(x,y)=\gamma V(\gamma|x-y|)$$

Here $\gamma=1/(2\mathrm{L}+1)$ and L denotes the interaction radius.

Main Statistical Mechanics Concepts

Equilibrium states of the stochastic model are described by the Gibbs measure $\mu_{\beta,N}$ at the prescribed temperature T,

$$\mu_{\beta,N}(d\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)} P_N(d\sigma)$$

where $\beta = \frac{1}{kT}$ and $P_N(d\sigma) = \prod_{x \in \Lambda} \rho(d\sigma(x))$

and $\rho(\sigma(x) = 0) = \frac{1}{2}$, $\rho(\sigma(x) = 1) = \frac{1}{2}$

The Mathematical Model

$$\frac{d}{dx} Ef(\sigma) = EMf(\sigma)$$

where $Mf(\sigma) = \sum_{x \in \Lambda} c(\sigma)[f(\sigma *) - f(\sigma)]$

and $\sigma^{¿}$ denotes a new lattice configuration

Microscopic Arrhenius Spin-Exchange Dynamics

The Arrhenius spin-exchange rate c(x,y,σ)

$$c(x,y,\sigma) = \begin{cases} c_d e^{-U(x)}, & \text{if } \sigma(x)=1, \text{ and } \sigma(y)=0, \\ c_d e^{-U(y)}, & \text{if } \sigma(x)=0, \text{ and } \sigma(y)=1, \\ 0, & \text{otherwise} \end{cases}$$

With exchange rate constant, $c_d = \frac{1}{\tau_0}$

Here τ_0 denotes the characteristic time of the stochastic process.

Again recall that $U(x) = \sum_{\substack{z \neq x \\ z \in \Lambda}} J(x,z)\sigma(z)$

Microscopic Arrhenius Spin-Flip Dynamics

The Arrhenius spin-flip rate c(x,σ) at lattice site x and spin configuration t is given by

$$c_d e^{-U(x)}, \ \text{when } \sigma(x)=0$$
$$c_a \qquad , \ \text{when } \sigma(x)=1$$

¿

$$c(x,\sigma) = ¿\{¿ ¿ ¿$$

¿

With adsorption/desorption constants, $c_a = c_d = \dfrac{1}{\tau_I}$

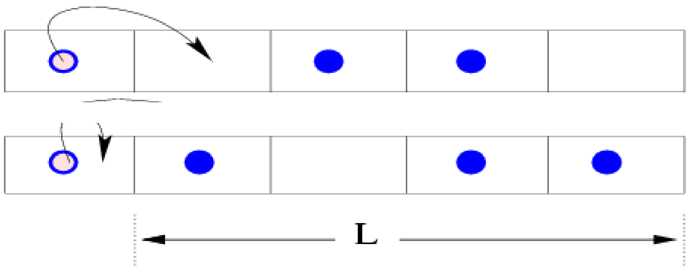
Here τ_I denotes the characteristic time of the stochastic process.

We consider short vehicle potential interactions J,

$$J(x,y)=V(\gamma(x-y)), \qquad x,y\in\Lambda$$

where $\gamma=1/L$ as usual ordains the range of microscopic Interactions.
Here $V:R\rightarrow R$ via,

$$V(r)=\begin{cases} J_0, & \text{if } 0<r<1 \\ 0, & \text{otherwise} \end{cases}$$



Which enforces:

- Exclusion princile
- Vehicles do not go backward in traffic
- Local effect of the interactions

(thus once again, more realistic traffic conditions)

The Traffic Model

$$\frac{d}{dt}Ef(\sigma)=EMf(\sigma)$$

or in more detail,

$$\frac{d}{dt}Ef(\sigma)=E\sum_{\substack{x\in\Lambda \\ y\neq x}} c(x,y,\sigma)[f(\sigma^{¿})-f(\sigma)]$$

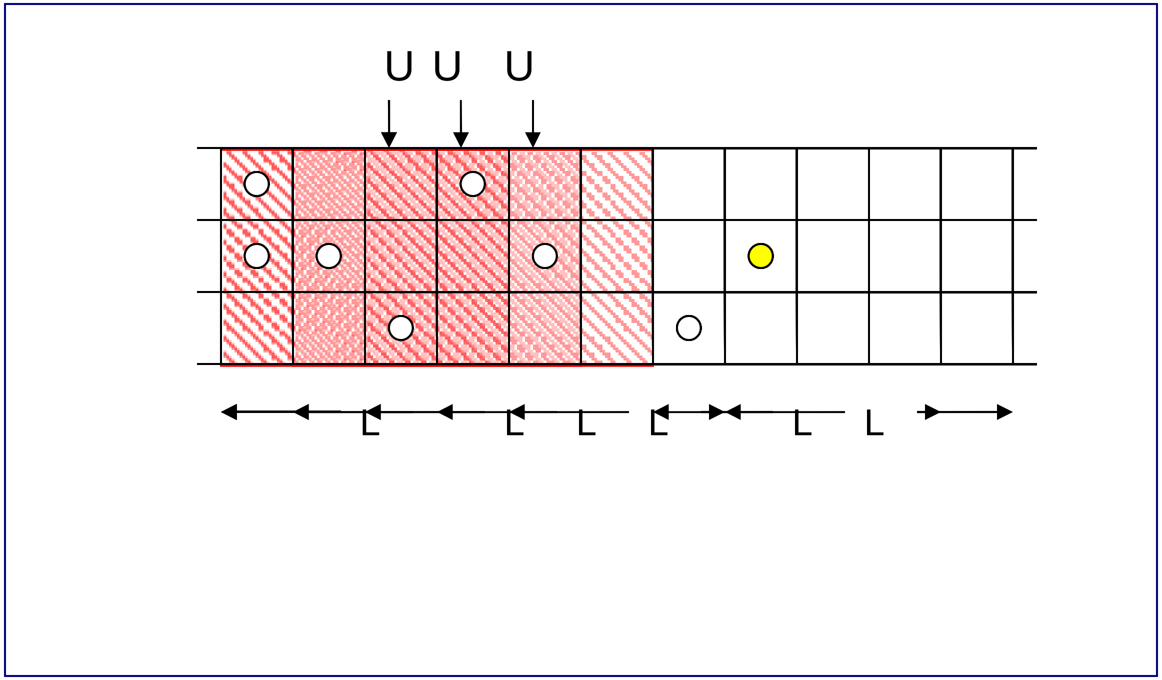
which based on the spin-exchange rate c(x,y,σ) for y=x+1 gives,

$$\frac{d}{dt}Ef(\sigma)=E\sum_{x\in\Lambda} c_0\sigma(x)(1-\sigma(x+1))e^{-U(x,\sigma)}[f(\sigma^{x,x+1})-f(\sigma)]$$

The probability of a spin-exchange between x and y=x+1 during time [t, t+Dt] is

$$c(x,y,\sigma)\Delta t+O(\Delta t^{٢})$$

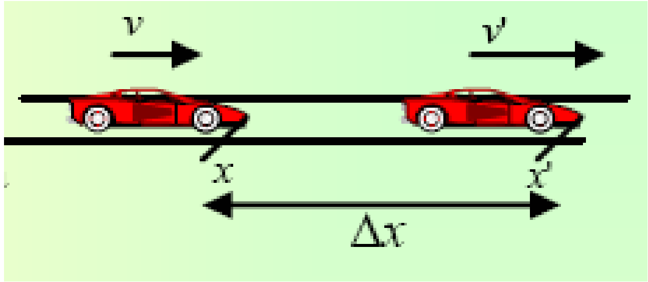
A simple schematic describing the traffic model dynamics



Free Parameters and Calibration

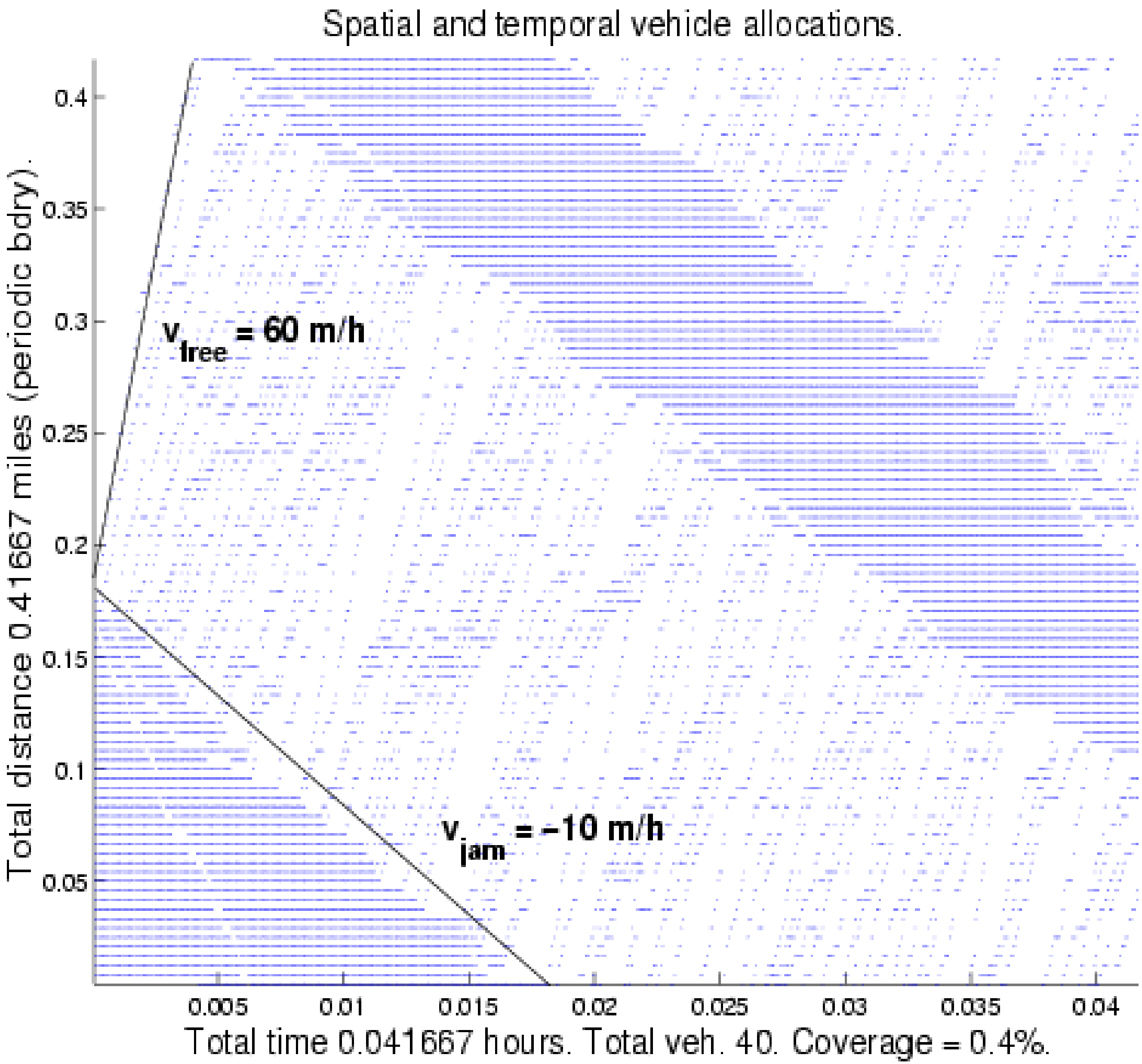
The model is characterized by the following three undetermined parameters:

- τ_0 - the characteristic time of the stochastic process
- J_0 - the strength of the interactions
- L - the interaction potential range



Cell length is assumed to be 22 feet (average vehicle size plus safe distance).

$$\Delta t_{cell} = \frac{22 \text{ feet}}{65 \text{ miles/hour}} \approx \frac{1}{4} \text{ sec}.$$



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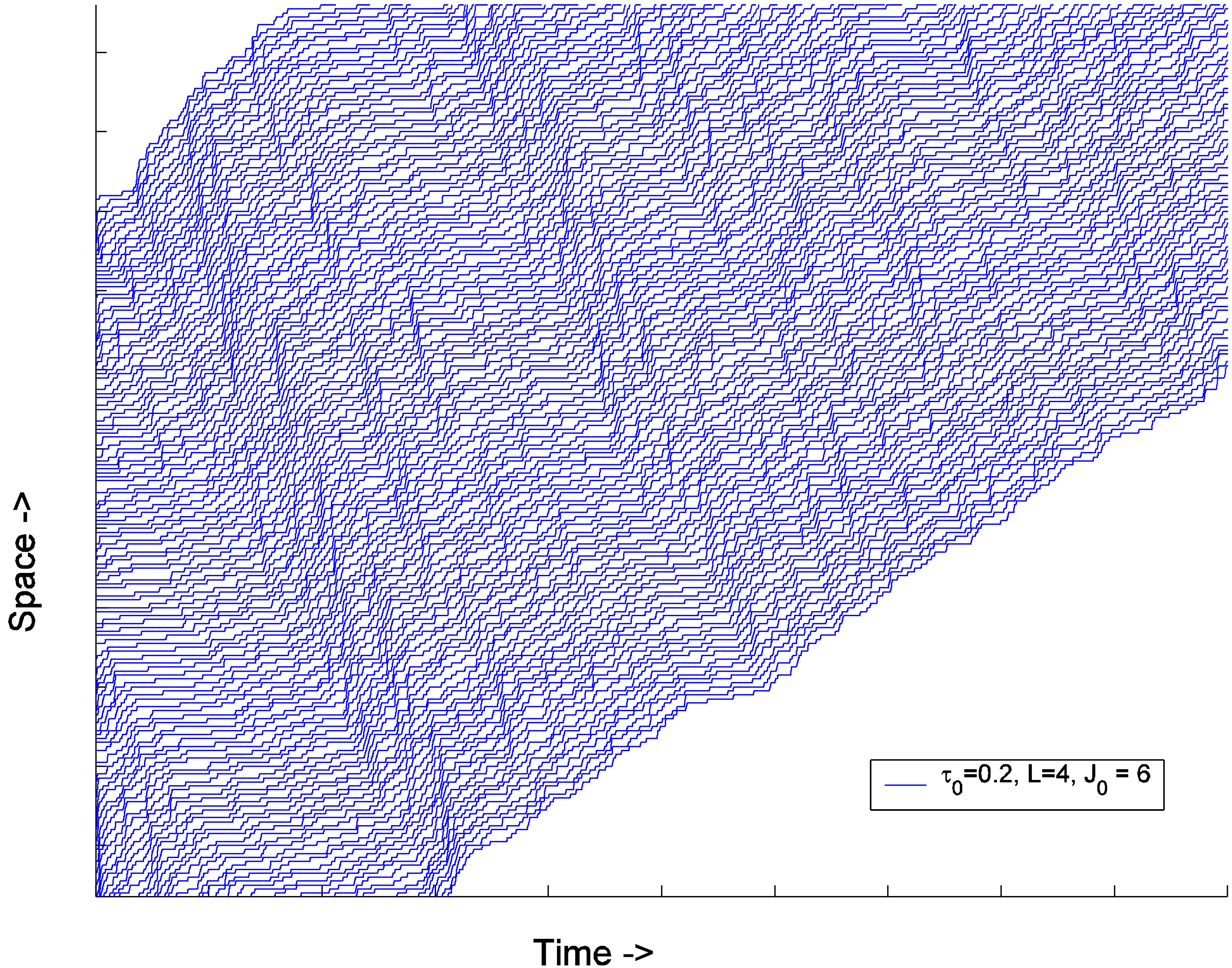
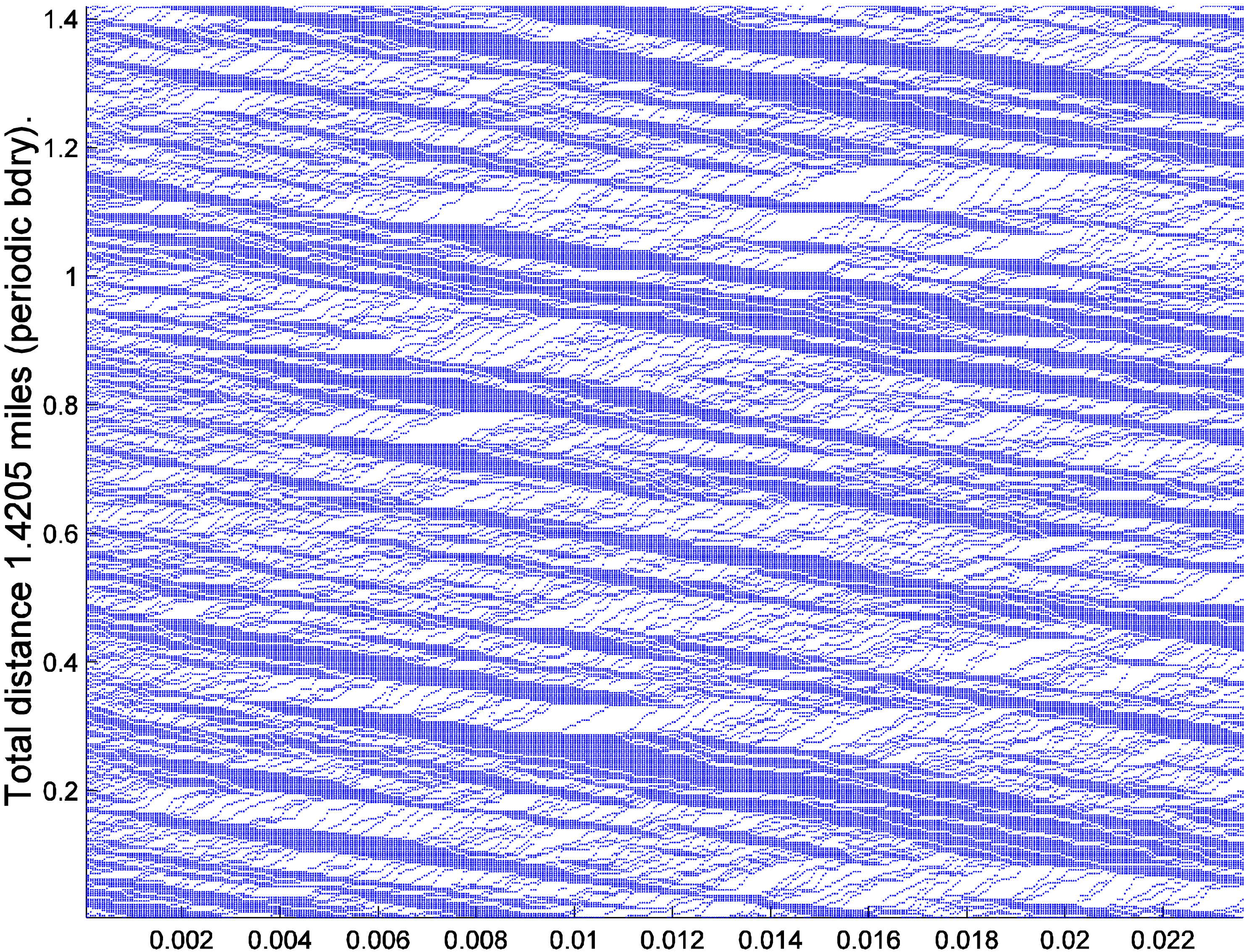
Spatial and temporal vehicle trajectories.

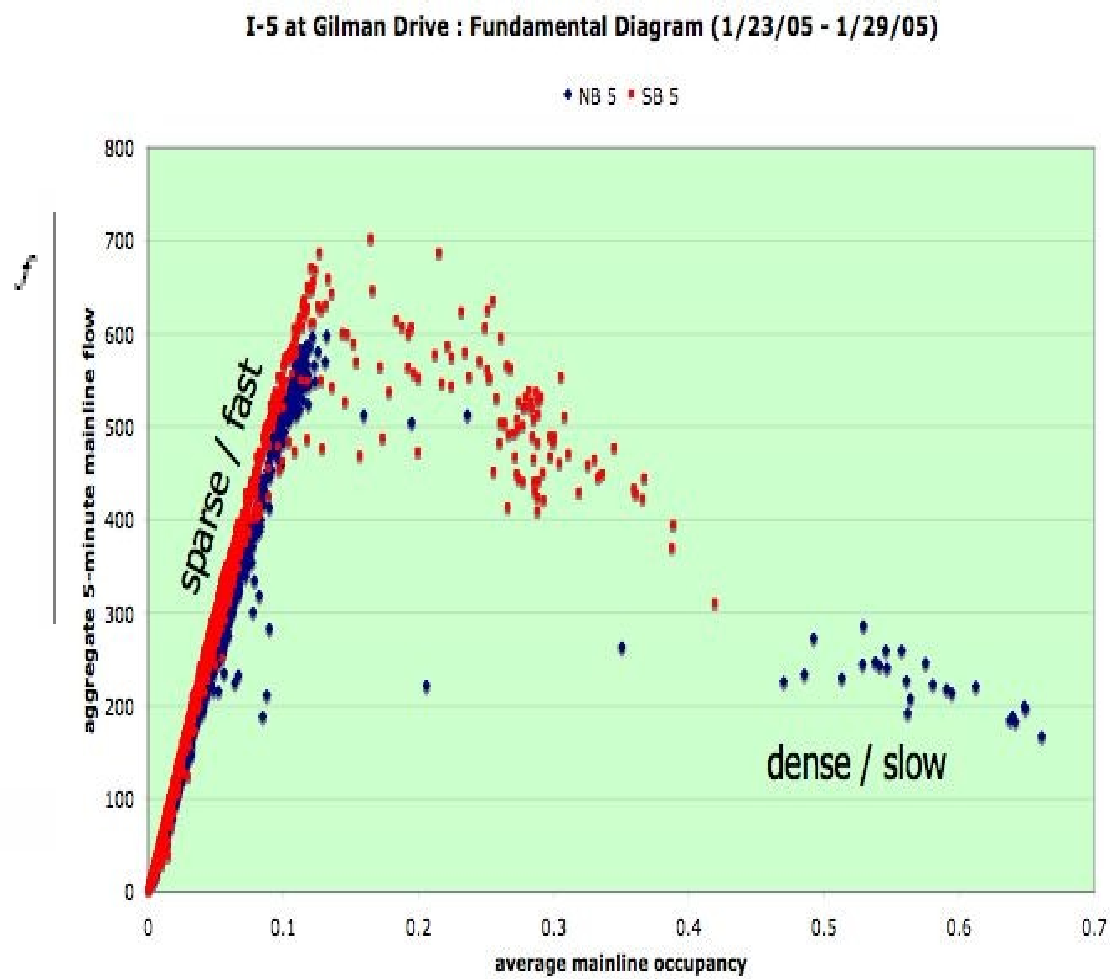


Spatial and temporal vehicle allocations.



Total time 0.023625 hours. Total veh. 256. Coverage = 0.512%.

Flow-Density Diagram
Experimental Data



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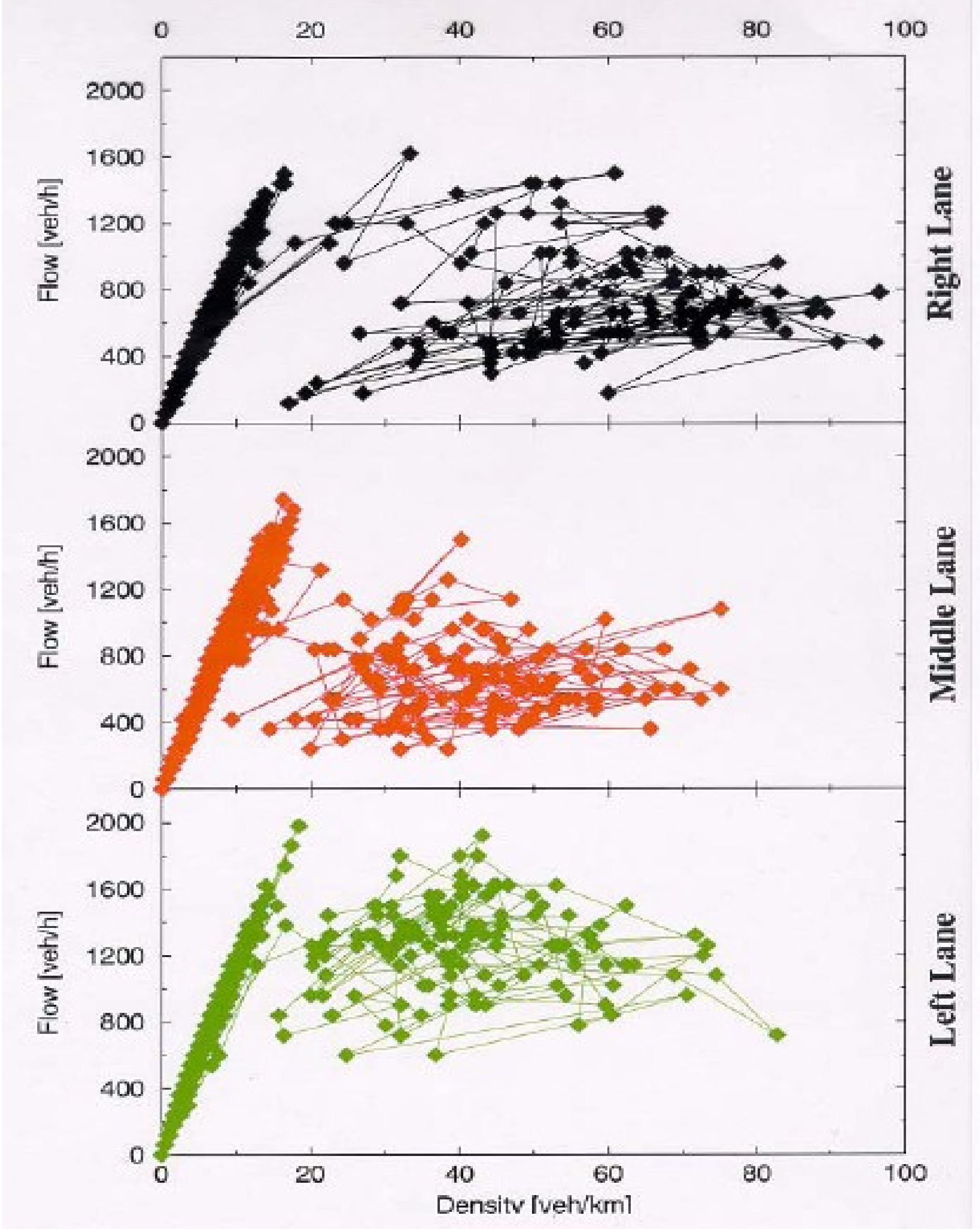

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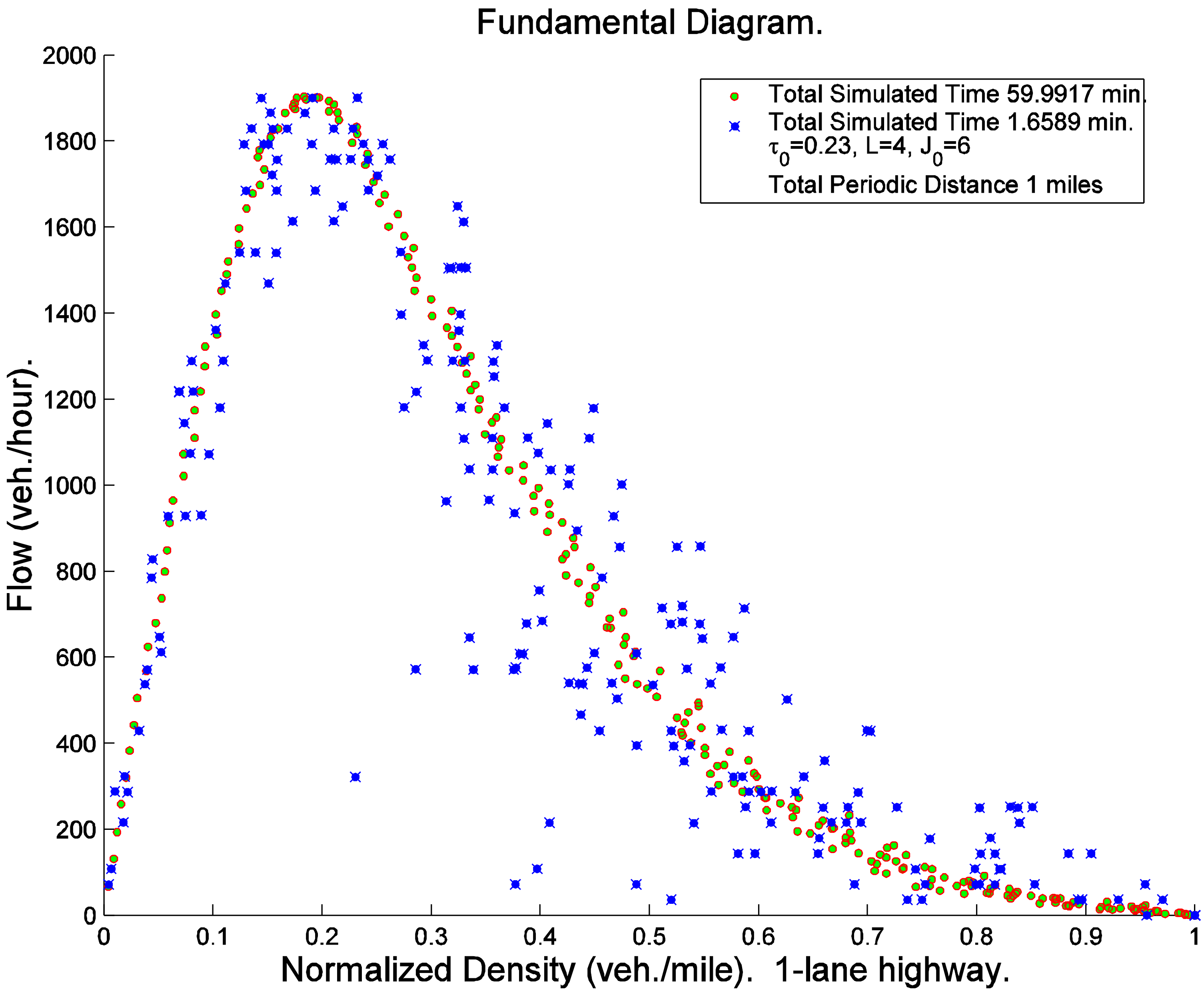

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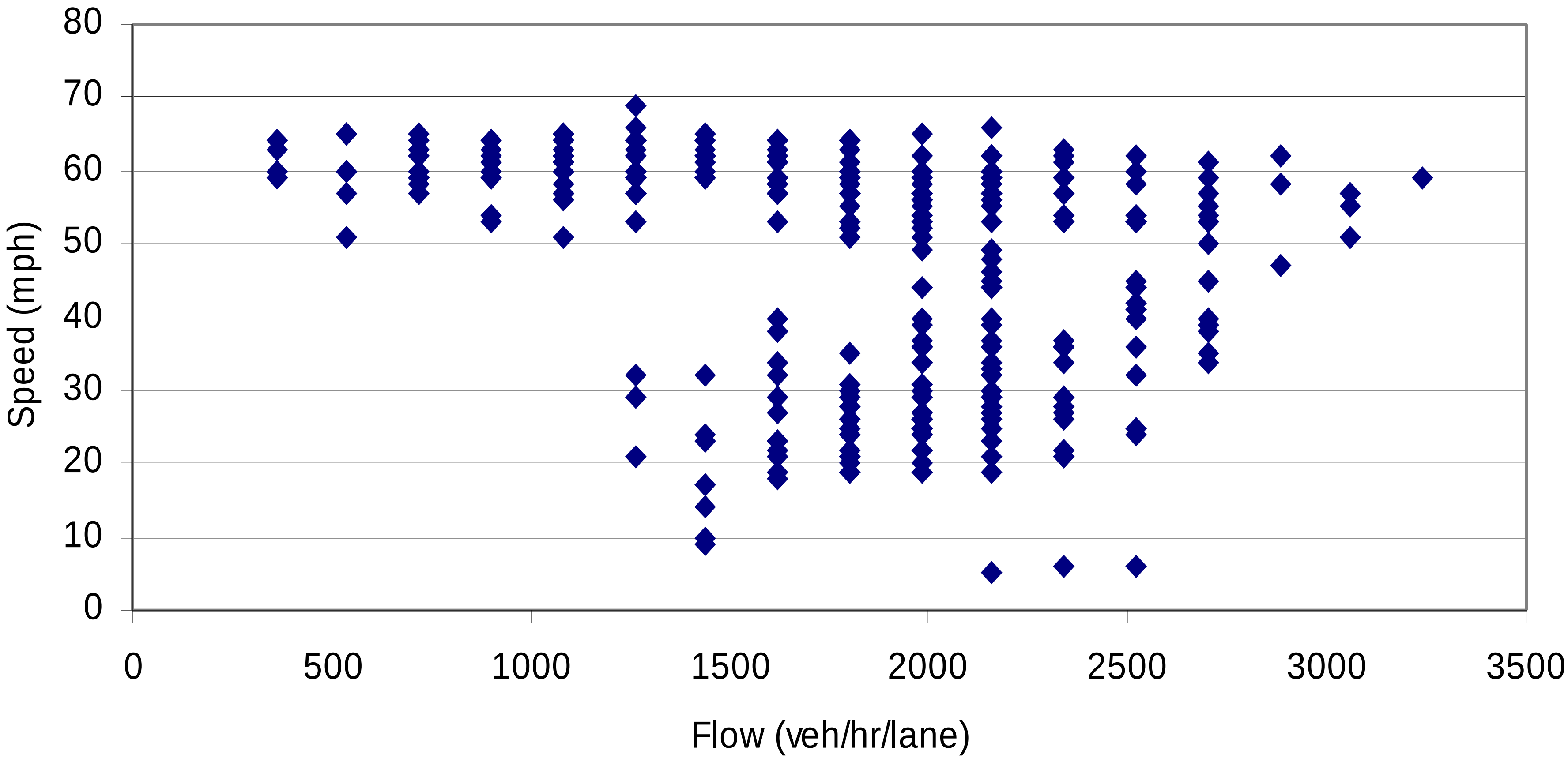
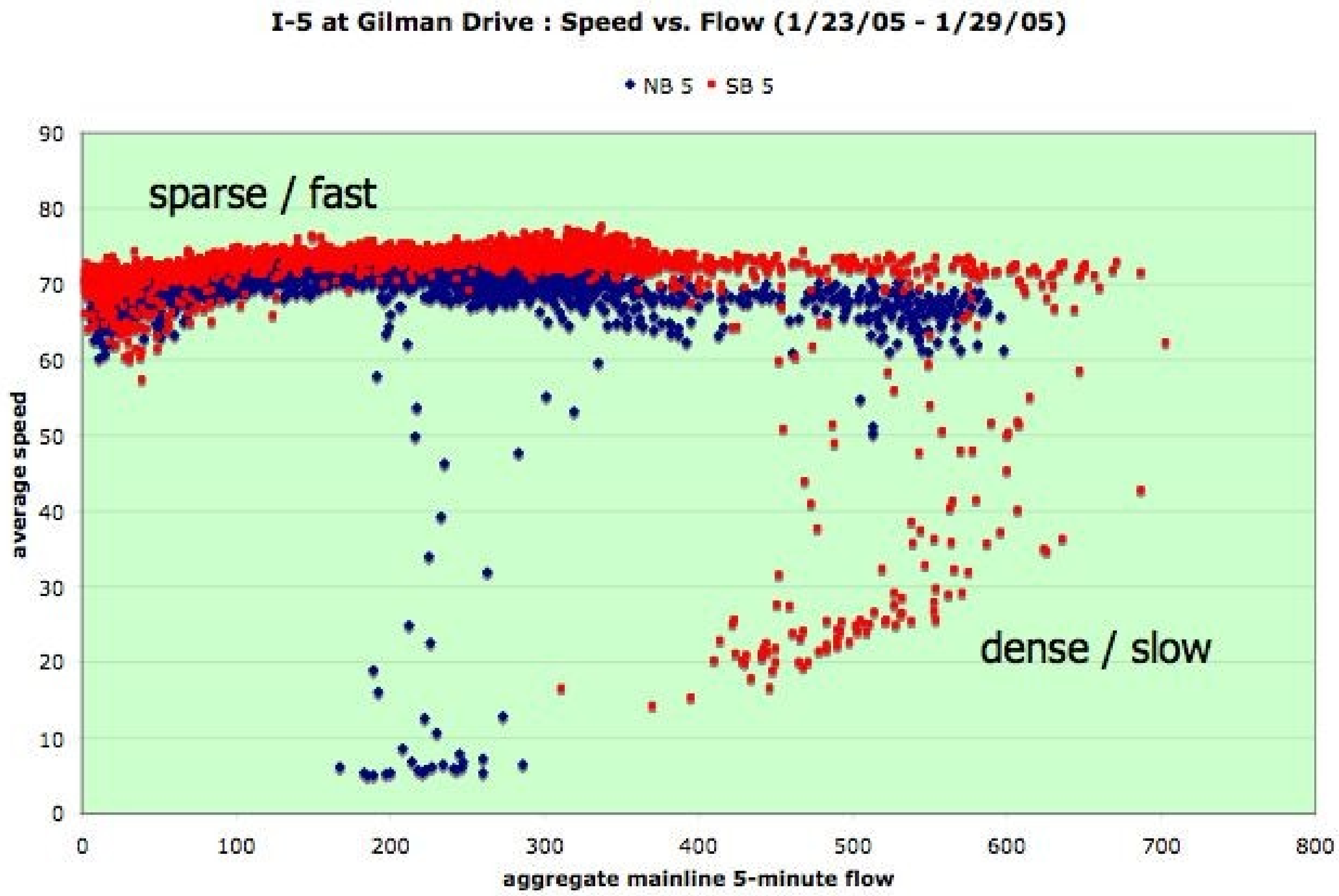


Velocity-Flow Diagram (real freeway data from San Antonio) [Halkias & Mahmassani (MTI 2005)]



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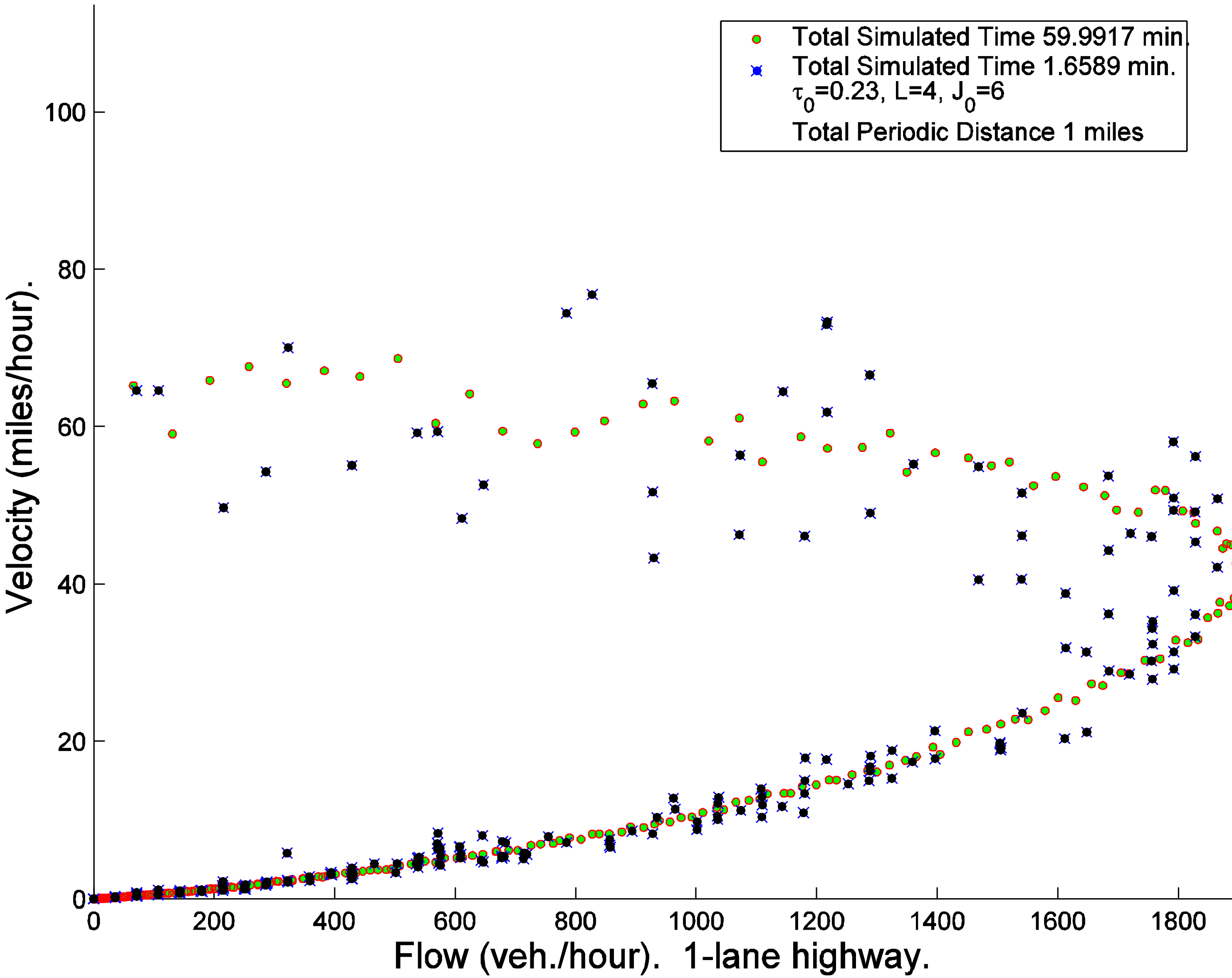

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Deterministic Closures

Recall the traffic flow model is, $\frac{d}{dt}Ef(\sigma)=EMf(\sigma)$ or

$$\frac{d}{dt}Ef(\sigma)=E\sum_{x\in\Lambda}c_0\sigma(x)(1-\sigma(x+1))e^{-U(x,\sigma)}[f(\sigma^{x,x+1})-f(\sigma)]$$

where $U(x,\sigma)=\sum_{\substack{z\neq x\\ z\in\Lambda}}J(x,z)\sigma(z)$. This becomes,

$$\frac{d}{dt}E\sigma_t(z)=-Ec_0\sigma(z)(1-\sigma(z+1))e^{-U(z,\sigma)}+Ec_0\sigma(z-1)(1-\sigma(z))e^{-U(z-1,\sigma)}$$

Exact but not yet closed for $E\sigma_t(z)=\mathrm{Prob}(\sigma_t(z)=1)$

Suppose that J has uniform (J=Jo), weak long interactions.

Finite Difference Scheme

The LLN formally applies and the fluctuations of $\sum_{y \neq x} J(y-x)\sigma(y)$ about their mean will be small.

Then in the long range interaction limit we have,

$$Ee^{-U(x,\sigma)} = Ee^{-\sum J(y-x)\sigma(y)} \overset{N,L\to\infty}{\text{¿}} e^{-\sum J(y-x)E\sigma(y)} + o_N(1)$$

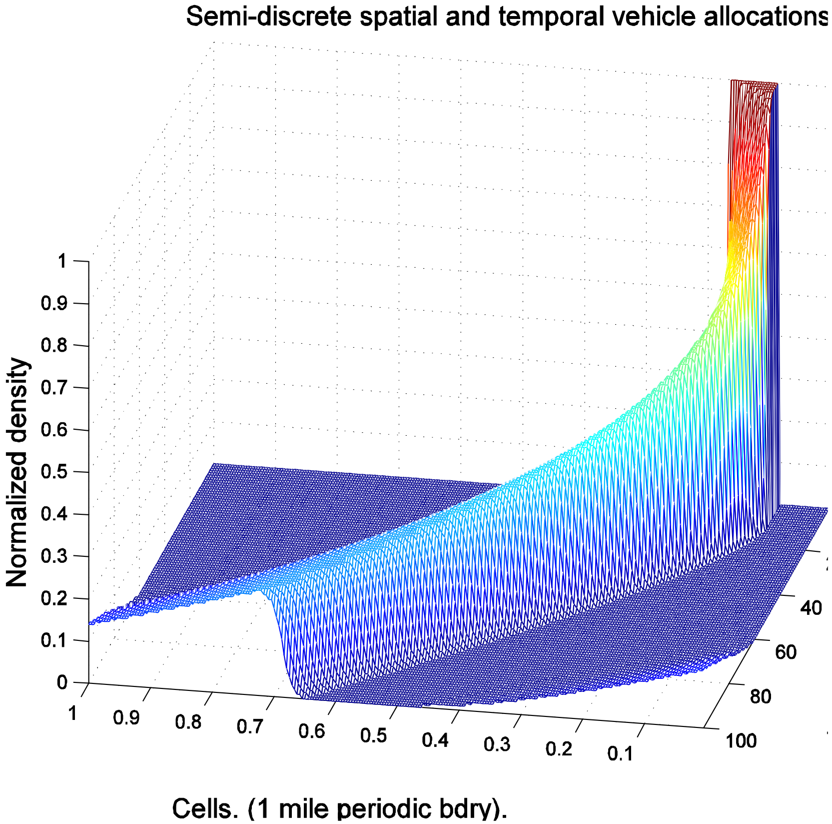
Let $u(z,t) = E\sigma_t(z)$ then we obtain an approximate *semi-discrete finite difference scheme*:

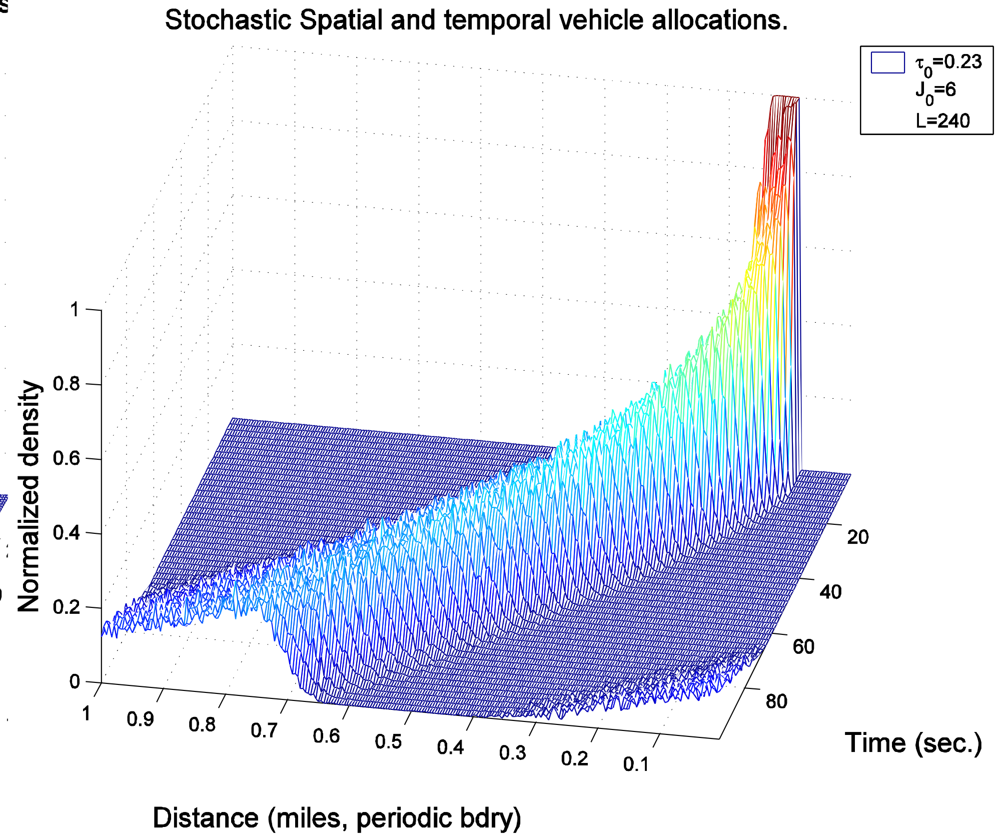
$$\frac{d}{dt}u(z,t) + F(z+1,t) - F(z,t) = 0$$

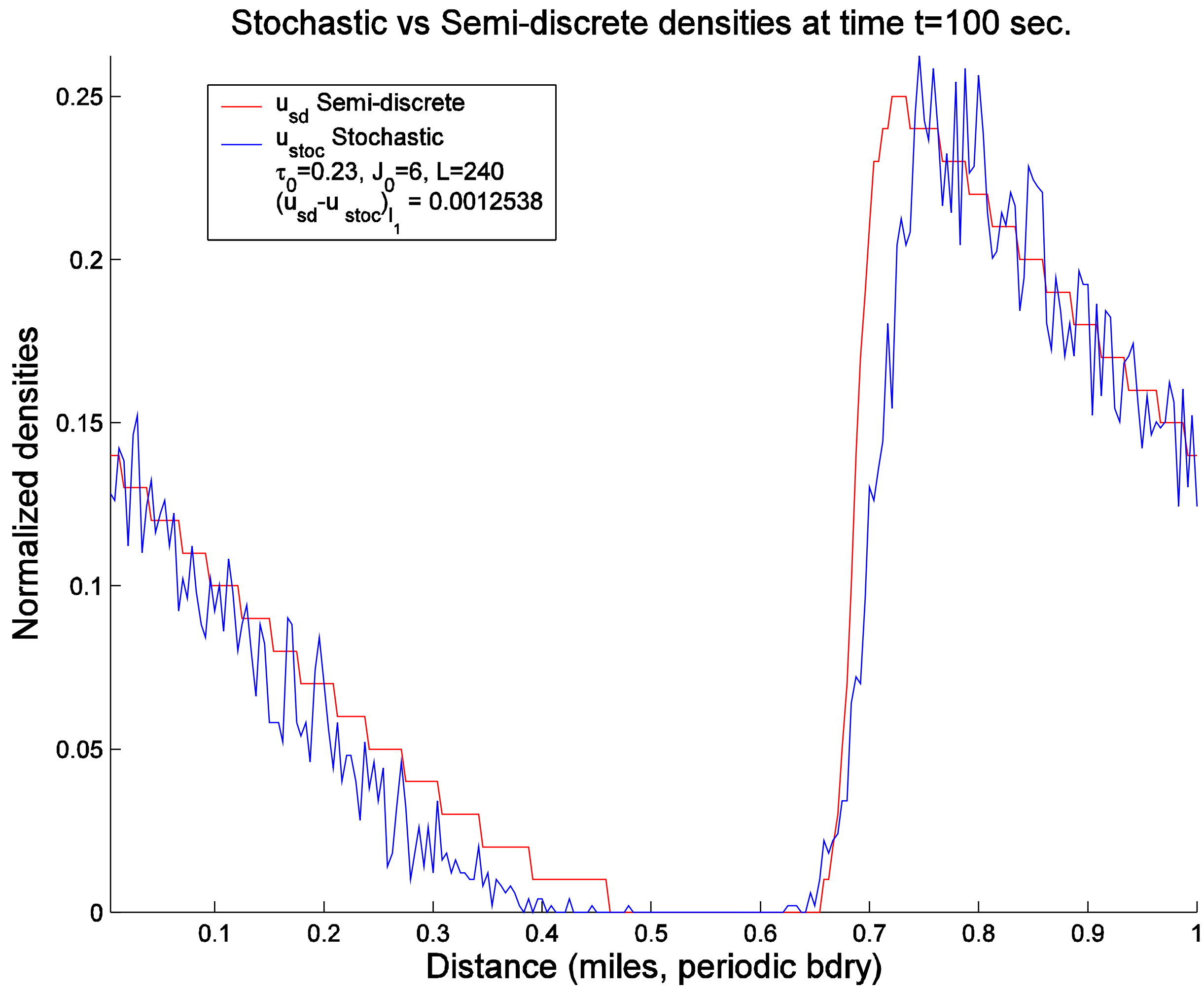
where $F(z,t) = c_0 u(z-1,t)(1-u(z,t))e^{-J \circ u(z-1,t)}$

For a periodic lattice this scheme is conservative.

Comparisons between semi-discrete scheme and microscopic stochastic model







Potential Radius L	240	100	50	10	4	1
l_1 Rel. Error	.0013	.0029	.0051	.0066	.0126	.02

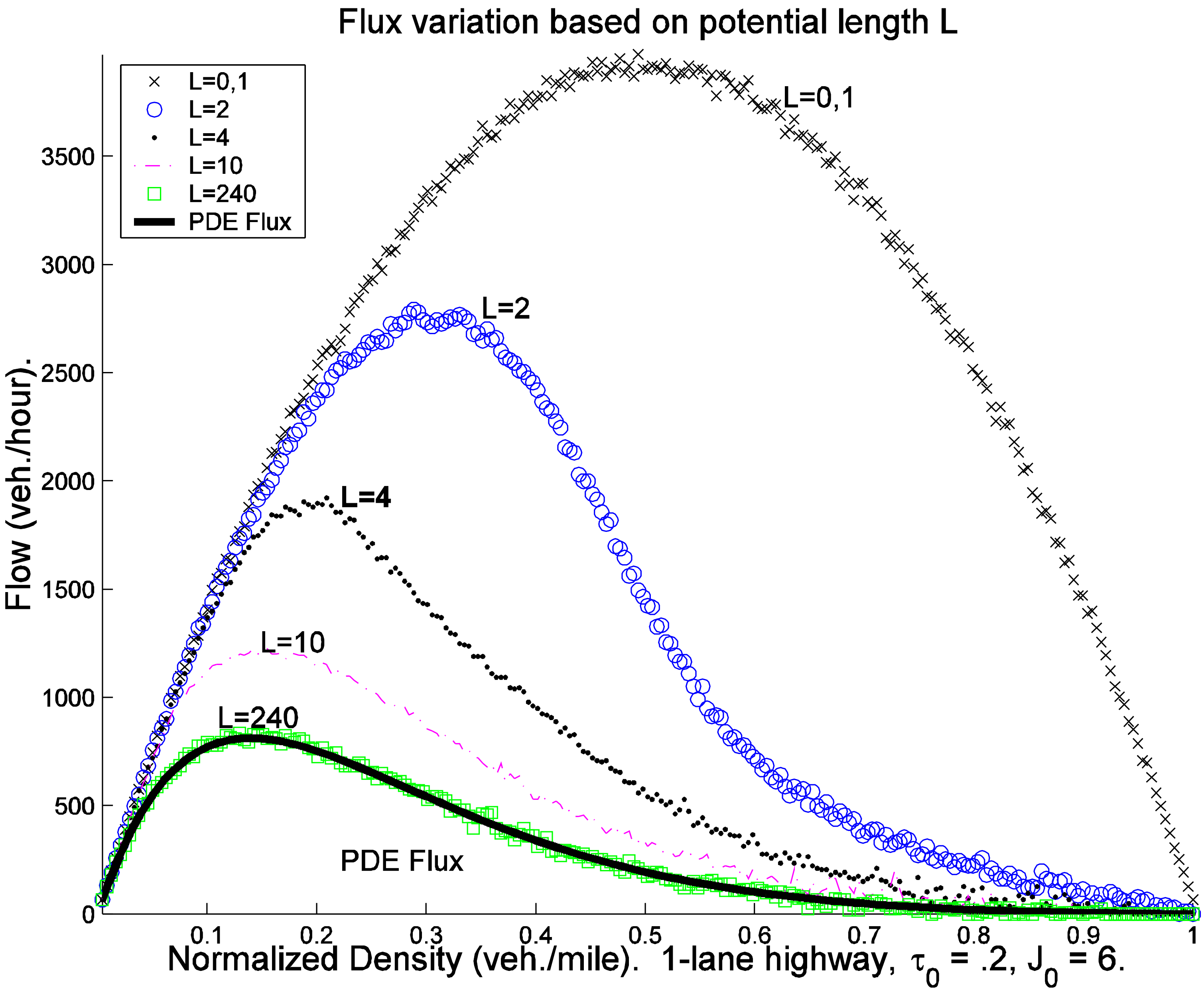
PDE model

Expanding in Taylor we obtain, $\frac{d}{dt}u + [hc_0 u(1-u)e^{-J \circ u}]_z = O(h^2)$

Rescaling time via $t \to th^{-1}$ in order to absorb h and omiting the $O(h^2)$ term, we obtain the following macroscopic transport equation:

$$u_t + F(u)_z = 0$$

where the PDE flux is $F(u) = c_0 u(1-u)e^{-J \circ u}$


Flux variation based on potential length L
L=0,1
L=2
L=4
L=10
L=240
PDE Flux
Flow (veh./hour).
Normalized Density (veh./mile). 1-lane highway, τ_0 = .2, J_0 = 6.
3500
3000
2500
2000
1500
1000
500
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Hierarchical Comparisons

Expanding the convolution,

$$J \circ u = \int_z^\infty V(y-z)u(y)\,dy \overset{x=y-z}{=} \int_0^\infty V(x)u(x+z)\,dx = J_0 u + J_1 u_z + J_2 u_{zz} + \ldots$$

we can approximate the exponential via,

$$e^{-J \circ u} \approx e^{-J_0 u}[1 - J_1 u_z - J_2 u_{zz}]$$

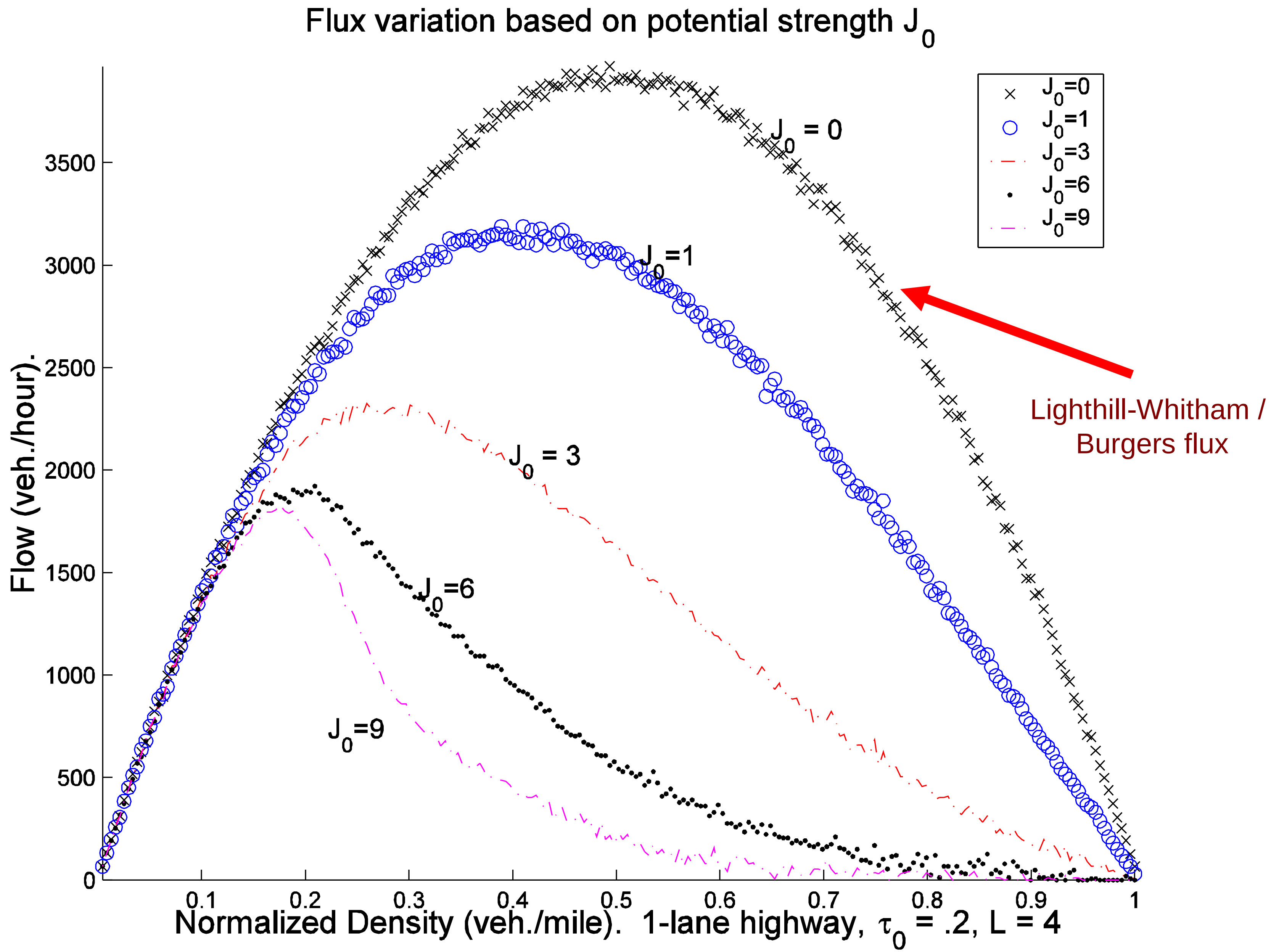
The traffic model PDE $u_t + cu(1-u)e^{-J \circ u} = 0$ therefore becomes…

The traffic model PDE,

$$u_t + c_0\left[u(1-u)e^{-J_0 u}\right]_z = c_0\left[J_1 u(1-u)e^{-J_0 u} u_z\right]_z + c_0\left[J_2 u(1-u)e^{-J_0 u} u_{zz}\right]_z$$

Note:

- No interactions (J=0):
  Lighhill-Whitham/Burger’s eq. $\rightarrow u_t + c_0\left[u(1-u)\right]_z = 0$



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The traffic model PDE,

$$u_t + c_0[u(1-u)e^{-J_0 u}]_z = c_0[J_1 u(1-u)e^{-J_0 u} u_z]_z + c_0[J_2 u(1-u)e^{-J_0 u} u_{zz}]_z$$

Note:

- No interactions (J=0):
  Lighhill-Whitham/Burger's eq. $\rightarrow u_t + c_0 u(1-u) = 0$

- Long range (L=N) uniform (J=Jo) interactions:
  Non-local flux $\rightarrow u_t + c_0[u(1-u)e^{-J_0 \bar{u}}]_z = 0$

- Including terms up to Jo in the convolution,
  Non-convex flux $\rightarrow u_t + c_0[u(1-u)e^{-J_0 u}]_z = 0$

- Terms up to J_1
  Nonlinear diffusive LWR type

- Full model is higher order dispersive (KDV type?)
  with nonlinear coefficients

Multi-lane extensions

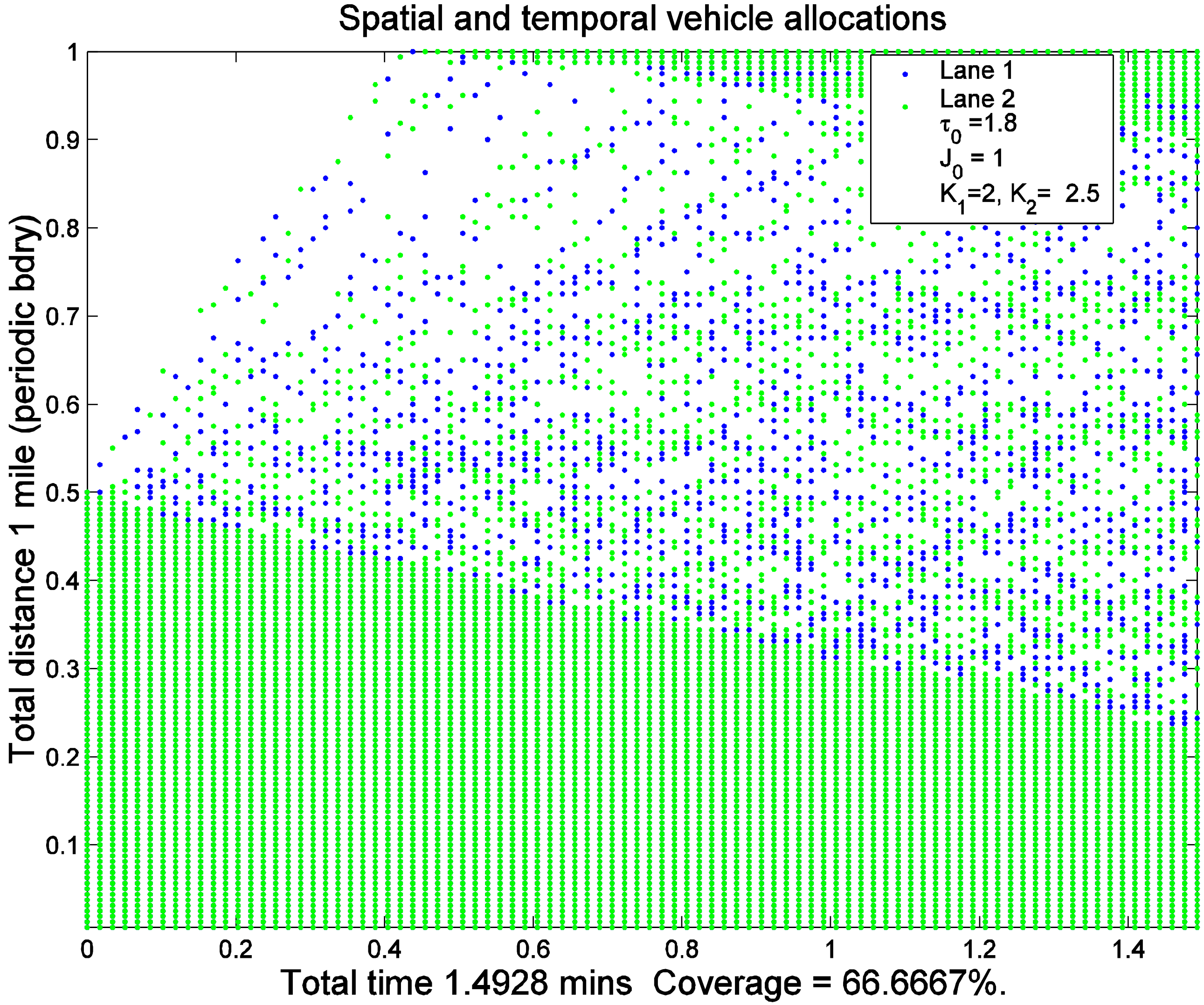
- We assume a two-dimensional domain (multi-lane highway).
- We introducing preferred direction in lane-changing via an anisotropy type potential. Thus our total interaction potential now consists of:

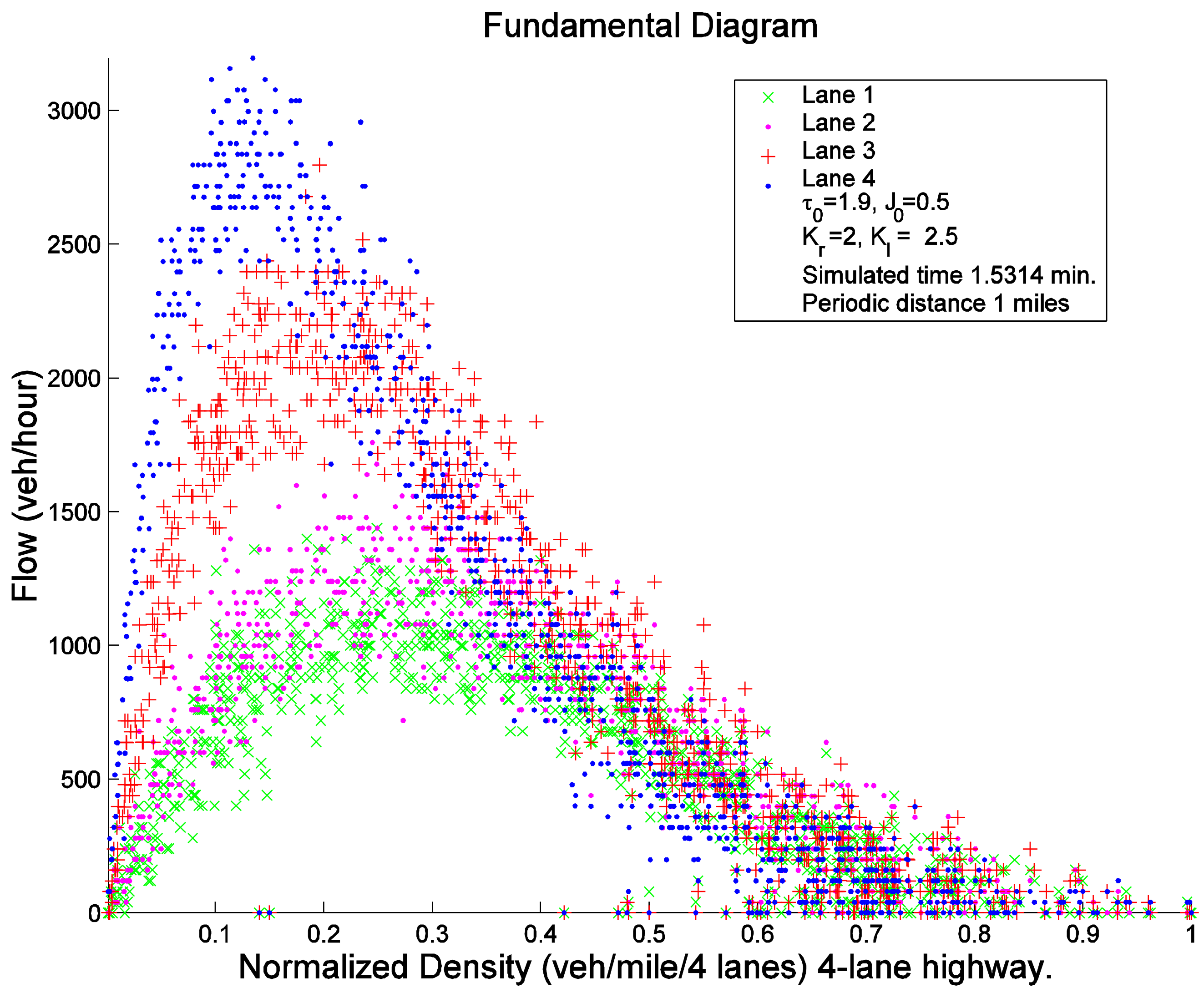
$$U(x) = U_e + U_a + h$$

where $U_a(x) = \sum_{y=nn} \psi(x,y)$ with $\psi(x,y) = \begin{cases} k_l & \text{if } y = x+1 \\ k_r & \text{if } y = x-1 \\ k_f & \text{if } y = x+n \end{cases}$

- Calibrate parameters:

# of Lanes	1	2	3	4
τ_0	0.23	1.1	1.85	1.9
J_0	6	0.7	0.7	0.5
Desired Velocity (mph)	65	62	68	72
Upstream Velocity (mph)	-10	-11	-12	-9.6


Spatial and temporal vehicle allocations
Lane 1
Lane 2
τ_0 =1.8
J_0 = 1
K_1=2, K_2= 2.5
Total distance 1 mile (periodic bdry)
Total time 1.4928 mins Coverage = 66.6667%.
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
1.2
1.4



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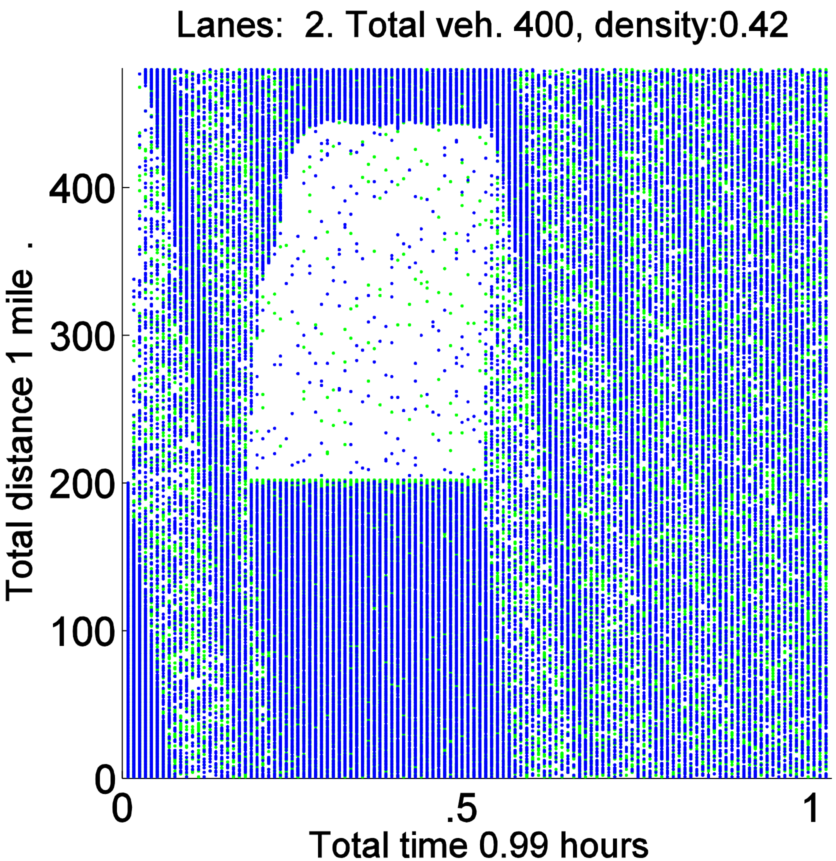
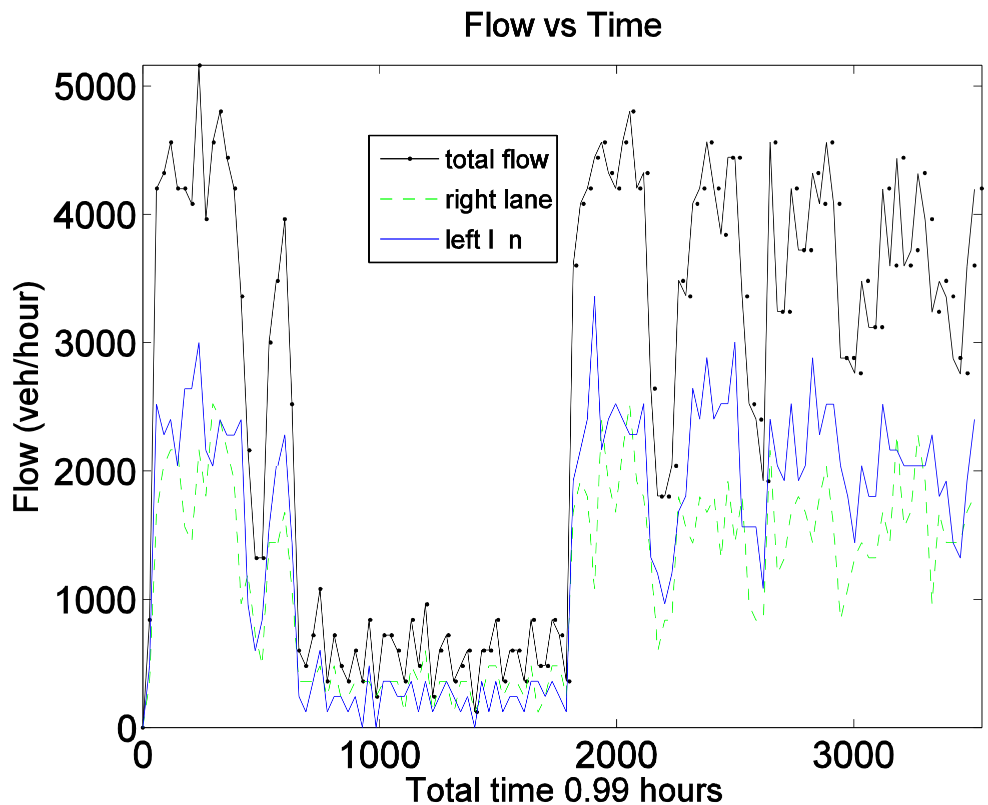
Test case toy problem: an incident

- Assume a 2-lane highway
- Block lane 1 due to an accident





Conclusions

- ✓ Presented a novel modeling approach based on microscopic Arrhenius spin-exchange dynamics
- ✓ Extended method to multi-lane traffic
- ✓ Studied deterministic closures of the microscopic stochastic model at different length scales
- ✓ Obtained formal hierarchical comparisons with other well-known models of traffic
- ✓ Presented Kinetic Monte Carlo simulations which allows for comparisons with actual traffic data as well as other PDE models

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