Weak solution and critical threshold for the one-dimensional Vlasov-Poisson equation

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Vlasov-Poisson Equations

- One-dimensional Vlasov-Poisson equations

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - E(x,t)\frac{\partial f}{\partial v} = 0$$

$$\frac{\partial^2}{\partial x^2}\phi = b(x) - \int_{-\infty}^{\infty} f(x,v,t)\,dv, \qquad E = \frac{\partial \phi}{\partial x}$$

$f(x,v,t)$ — density of electrons at location x traveling with velocity v at time t

$E(x,t)$ — electric field

$\phi(x,t)$ — electric potential

$b(x)$ — fixed charged background

2D Euler equations with non-negative vorticity

- Incompressible 2D Euler equations

$$\frac{D\mathbf{v}}{Dt} = -\nabla p$$
$$div\mathbf{v} = 0$$

- Vorticity-stream form

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla w = 0$$
$$div\mathbf{v} = 0$$
$$curl\mathbf{v} = \omega$$

- Analogy between the 2D Euler and 1D Vlasov-Poisson equaiton

$$\frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f = 0, \quad \mathbf{u} = (v, -E(x,t))$$
$$div\mathbf{u} = 0$$
$$curl\mathbf{u} = \int_{-\infty}^{\infty} f(x,v,t)dv - 2$$

Electron Sheet Initial Data

- Smooth electron sheet initial data

$$C(\alpha, 0) = \big(x(\alpha, 0), v(\alpha, 0)\big) = (\alpha, g(\alpha))$$

$$f(x, v, 0) = h(\alpha)\left|\frac{dC(\alpha, 0)}{d\alpha}\right|^{-1}\delta\big((x, v) - C(\alpha, 0)\big)$$

- $\delta\big((x, v) - C(\alpha)\big)$ — a surface measure supported on the curve

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi(x, v)\delta\big((x, v) - C(\alpha)\big)dvdx = \int_a^b \psi(\alpha, g(\alpha))\left|\frac{dC(\alpha)}{d\alpha}\right|d\alpha$$

- local density $\rho(x, t) = \int_{-\infty}^{\infty} f(x, v, t)dv$

$$\rho(\alpha, 0) = h(\alpha)$$

Weak solution of the Vlasov-Poisson equation (MMZ94)

- (E,f): f non-negative measure $f(x,v,t)$, (E,f) is 1-periodic in x

1. $f(x,v,t)$ is a probability measure for each $t>0$, i.e.,

$$\int_0^1 \int_{-\infty}^{\infty} f(x,v,t)dvdx = 1$$

2. (E,f) satisfies the Poisson equation in the distributional sense, $E_x = 1 - \int_{-\infty}^{\infty} f dv$, and the normalizing condition is compatible with Condition 1, $\int_0^1 E_x dx = 0$

3. (E,f) satisfies the Vlasov equation in the weak form: $\bar{E} = \frac{1}{2}(E_l + E_r)$

$$\int_0^T \int_0^1 \int_{-\infty}^{\infty} (\psi_t f + \psi_x v f) dvdxdt - \int_0^T \int_0^1 \bar{E}\left(\int_{-\infty}^{\infty} \psi_v f dv\right) dxdt = 0$$

Plan of this talk

- Exact weak solution of the Vlasov-Poisson equaiton

  — critical threshold

- Weak solution afther the critical time

  — multi-valued solution to the Euler-Poisson equation

- A novel algorithm and numerical examples

Exact Weak Solution

An electron sheet defined by

$$C(\alpha, t) = (x(\alpha, t), v(\alpha, t)),$$

$$x(\alpha, t) = g(\alpha)\sin t + (\alpha - H(\alpha))\cos t + H(\alpha),$$

$$v(\alpha, t) = g(\alpha)\cos t - (\alpha - H(\alpha))\sin t,$$

$$f(x, v, t) = h(\alpha)\left|\frac{dC(\alpha, t)}{d\alpha}\right|^{-1}\delta\big((x, v) - C(\alpha, t)\big),$$

$$E(\alpha, t) := E(x(\alpha, t), t) = x(\alpha, t) - H(\alpha).$$

Dziurzynski 87 ($x = \alpha,\ h(\alpha) \equiv 1$)

Construction of Weaksolution

- Method of Characteristics

$$\frac{dx(\alpha,t)}{dt} = v(\alpha,t)$$
$$\frac{dv(\alpha,t)}{dt} = -E(x(\alpha,t),t)$$
$$E(x,t) = \int_0^x \Big(1 - \int_{-\infty}^{\infty} f(y,v,t)dv\Big)dy$$

- $M(x,t) := \int_0^x \int_{-\infty}^{\infty} f(y,v,t)dvdy, \quad M(x(\alpha,t),t) = M(x(\alpha,0),0)$

  — Constant, as long as $C(\alpha,t)$ is a graph

Method of Characteristics

- Equation for $x(\alpha, t), v(\alpha, t)$

$$\frac{dx(\alpha, t)}{dt} = v(\alpha, t)$$

$$\frac{dv(\alpha, t)}{dt} = -M(\alpha, t) + x(\alpha, t)$$

- Equation of the local density ρ

$$\rho(\alpha, t) = \frac{\partial M(x(\alpha, t), t)}{\partial x}$$

$$\frac{dM}{dt} = \frac{\partial M}{\partial t} + v\frac{\partial M}{\partial x} = 0$$

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial x} + \frac{\partial v}{\partial x}\rho = 0$$

Weak solution — Method of Characteristics

$$\frac{d\rho}{dt} + \frac{\partial v}{\partial x}\rho = 0$$

$$\frac{\partial v(\alpha,t)}{\partial x} = \frac{\partial v(\alpha,t)}{\partial \alpha}\Big/\frac{\partial x}{\partial \alpha}$$

$$\frac{\partial v}{\partial \alpha} = \frac{\partial \Gamma(\alpha,t)}{\partial t}, \quad \Gamma(\alpha,t) := \frac{\partial x}{\partial \alpha}$$

$$\rho(\alpha,t) = \frac{\rho(\alpha,0)}{\Gamma(\alpha,t)}\Gamma(\alpha,0) = \frac{h(\alpha)}{\Gamma(\alpha,t)}$$

$$f(x,v,t) = h(\alpha)\left|\frac{dC(\alpha,t)}{d\alpha}\right|^{-1}\delta\big((x,v) - C(\alpha,t)\big)$$

Engelberg, Liu, Tadmor, 2001 (Euler-Poisson)

Critical Threshold

- Will the momentum of this weak solution blow up in finite time?

  — $\rho(\alpha, t) = \dfrac{h(\alpha)}{\Gamma(\alpha, t)}$

  — $\Gamma(\alpha, t) := \dfrac{\partial x}{\partial \alpha} = g'(\alpha)\sin t + (1 - h(\alpha))\cos t + h(\alpha)$

  — if $\Gamma(\alpha, t)$ remains positive

- Critical threshold

  — $\big(g'(\alpha)\big)^2 < 2h(\alpha) - 1, \qquad \forall \alpha \in [0, 1]$

- Agrees with Engelberg, Liu, Tadmor, 2001 (Euler-Poisson Critical Threshold)

Example: concentration in charge

Vlasov-Poisson with electron sheet initial data:

$$h(\alpha) = 1,$$

$$g(\alpha) = \begin{cases} \alpha & 0 \le \alpha \le \frac{1}{4} \\ \frac{1}{2} - \alpha & \frac{1}{4} \le \alpha \le \frac{1}{2} \\ 0 & \alpha \ge \frac{1}{2} \end{cases}$$

Weak solution

$$x(\alpha, t) = g(\alpha)\sin t + \alpha$$

$$v(\alpha, t) = g(\alpha)\cos t$$

Valid for $0 \le t < \frac{\pi}{2}$, at the critical time $t^* = \frac{\pi}{2}$,

$$x(\alpha, t) = \frac{1}{2}, \quad v(\alpha, t) = 0, \quad for \quad t \in [\frac{1}{4}, \frac{1}{2}].$$

Two-component Vlasov-Poisson equation

$$
\begin{cases}
\dfrac{\partial f_-}{\partial t} + v\dfrac{\partial f_-}{\partial x} - E(x,t)\dfrac{\partial f-}{\partial v} = 0 \\
\dfrac{\partial f_+}{\partial t} + v\dfrac{\partial f_+}{\partial x} + E(x,t)\dfrac{\partial f+}{\partial v} = 0 \\
\dfrac{\partial^2}{\partial x^2}\phi = \displaystyle\int_{-\infty}^{\infty}(f_+ - f_-)(x,v,t)dv, \qquad E = \dfrac{\partial \phi}{\partial x}
\end{cases}
$$

— f_+ and f_-, the density of positively charged ions and electrons

— ϕ is the electric potential

— E is the electric field

Two-component Vlasov-Poisson equation

- Simplest stationary weak solution

  — a uniform electron sheet and a uniform ion sheet

$$f_{-}(x,v,0) = \delta((x,v) - (x,0))$$
$$f_{+}(x,v,0) = \delta((x,v) - (x,0))$$

- Impose small density and velocity perturbations

  — Weak solution by method of characteristics

  — Weak solution is valid at least for a short time $T > 0$

Two-component Vlasov-Poisson equation

- Moments of the two-component Vlasov-Poisson equations

$$\frac{\partial \rho_-}{\partial t} + \frac{\partial}{\partial x}(\rho_- u_-) = 0$$
$$\frac{\partial}{\partial t}(\rho_- u_-) + \frac{\partial}{\partial x}(\rho_- u_-^2) = -\rho_- E = -\rho_- \phi_x$$
$$\frac{\partial \rho_+}{\partial t} + \frac{\partial}{\partial x}(\rho_+ u_+) = 0$$
$$\frac{\partial}{\partial t}(\rho_+ u_+) + \frac{\partial}{\partial x}(\rho_+ u_+^2) = \rho_+ E = \rho_+ \phi_x$$
$$\phi_{xx} = \rho_+ - \rho_-$$

- Finite time blow up – an example

— $f_-(x,v,0) = f_+(x,v,0), \quad u_-(x,t) = u_+(x,t), \quad \rho_-(x,t) = \rho_+(x,t)$

Solution after critical time

- Moments of the weak solution

$$\rho(\alpha,t)=\lim_{\epsilon\longrightarrow 0}\int_{v(\alpha,t)-\epsilon}^{v(\alpha,t)+\epsilon} f(x,v,t)dv, \qquad u(\alpha,t)=v(\alpha,t)$$

- Multi-valued solution to Euler-Poisson

- Method of characteristics

- Choose the parameter α

$$\forall\alpha\in[0,1], \quad \text{find z such that} \quad \int_0^z \rho_0(x)dx=\alpha, \quad \text{then} \quad x(\alpha,0)=z$$

$$M(\alpha,0)=M(\alpha,t)=\alpha$$

Electric Field

Without concentration

$$E(\alpha,t) = x(\alpha,t) - \int_0^1 \chi_{\left[0,x(\alpha,t)\right)}\big(x(\beta,t)\big)d\beta$$

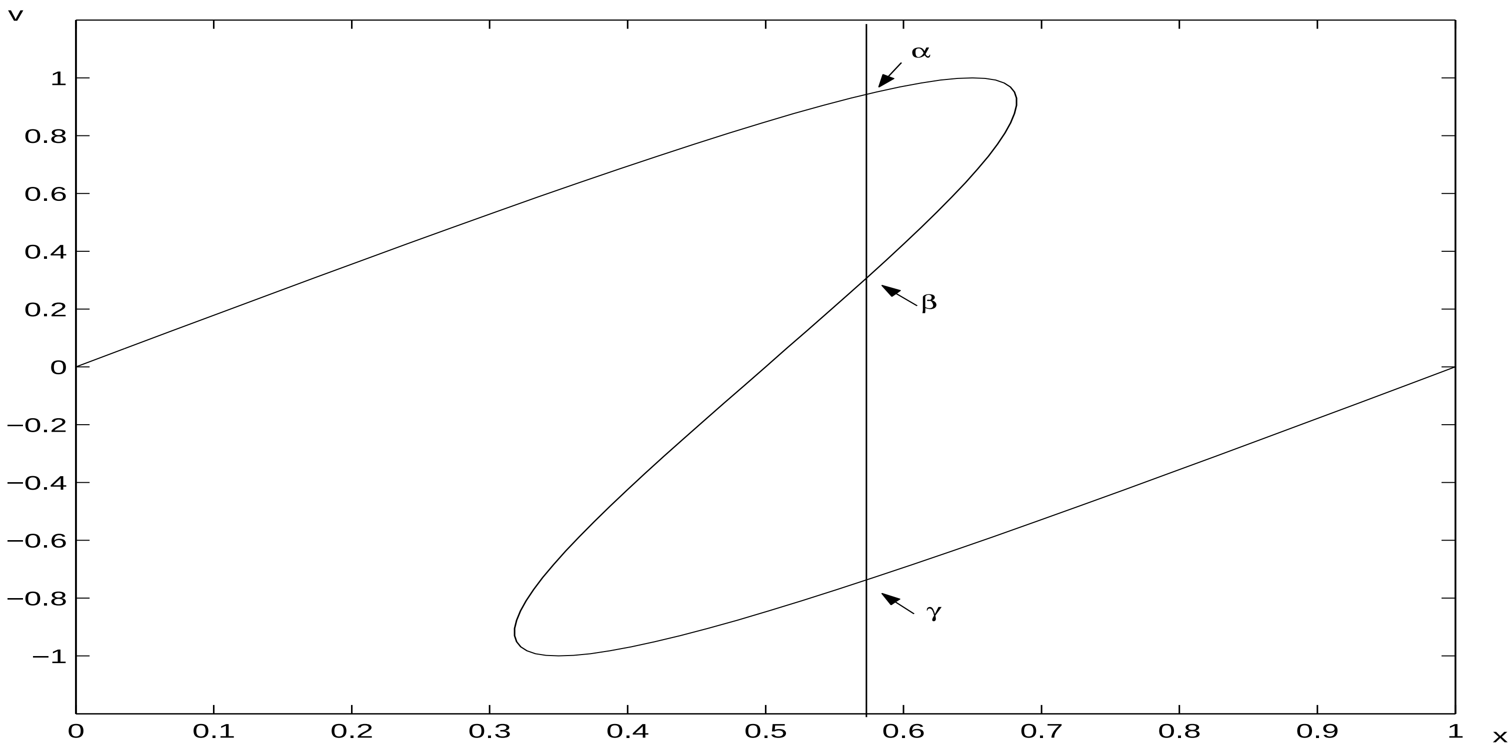


Fig1. x(α,t)=x(β,t)=x(γ,t), then by (4.4), E(α,t)=x(α,t)−(α+γ−β).

Electric Field

With concentration:

$$E(\alpha,t) = x(\alpha,t) - \int_0^1 \chi_{\left[0,x(\alpha,t)\right)}\big(x(\beta,t)\big)d\beta - \big|\Omega \cap \{\theta \leq \alpha\}\big|$$

$$x(\theta,t) = \bar{x} \quad \text{if and only if} \quad \theta \in \Omega$$

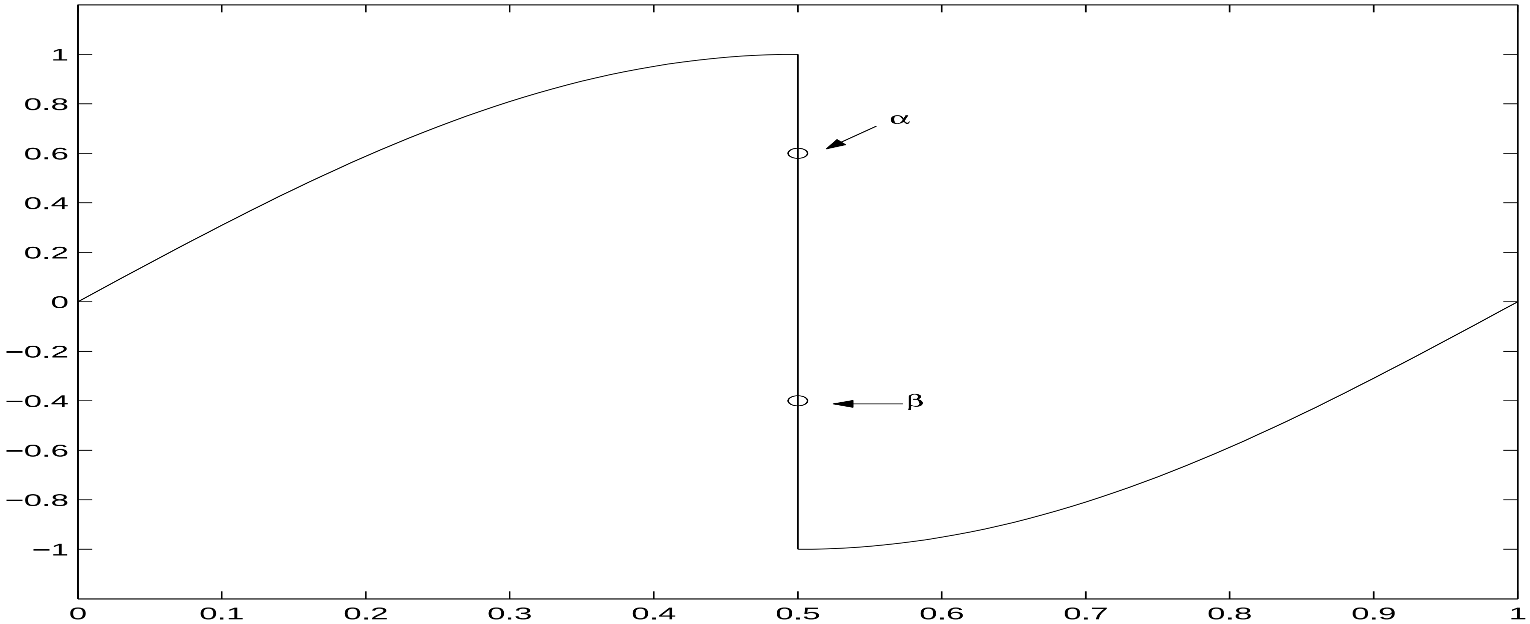


Fig.2 x(α,t)=x(β,t), E(α,t)=x(α,t)−α, E(β,t)=x(β,t)−β

Multi-valued solution

- Euler-Poisson

$$\rho(\alpha,t) = |\frac{\partial x}{\partial \alpha}|^{-1}$$

- Vlasov-Poisson

— initial data: $f(x,v,0) = h(\alpha)\Big|\frac{dC(\alpha,0)}{d\alpha}\Big|^{-1}\delta\big((x,v)-C(\alpha,0)\big)$

$$h(\alpha) \equiv 1$$

— weak solution

$$f(x,v,t) = \Big|\frac{dC(\alpha,t)}{d\alpha}\Big|^{-1}\delta\big((x,v)-C(\alpha,t)\big)$$

Charge Index

- Initial data

$$C(s,0) = \big(x(s,0), v(s,0)\big),$$

$$f(x,v,0) = h(s)\Big|\frac{dC(s,0)}{ds}\Big|^{-1}\delta\big((x,v) - C(s,0)\big)$$

— $h(s)$ could be 0 or a delta function

- Change index

$$s(\alpha) = \min_p\{p \,|\, \int_0^p h(y)dy \geq \alpha\}$$

- Example: charge concentration initial data

Zero diffusion limit of the Fokker-Planck

- Fokker-Planck equation as a regularization to the Vlasov-Poisson equation

$$\frac{\partial f^\eta}{\partial t} + v\frac{\partial f^\eta}{\partial x} - E^\eta(x,t)\frac{\partial f^\eta}{\partial v} = \eta\frac{\partial^2 f^\eta}{\partial v^2},$$

$$\frac{\partial^2}{\partial x^2}\phi = b(x) - \int_{-\infty}^{\infty} f^\eta(x,v,t)dv, \quad E^\eta = \frac{\partial \phi^\eta}{\partial x},$$

- $\eta \to 0$, weak solution of the Vlasov-Poisson?

Example — Continuous fission solution

- Vlasov-Poisson equation with concentrated electron sheet initial data

$$f(x, v, 0) = \delta(x - \frac{1}{2})\delta(v),$$

i.e., all the charge concentrates on $x = \frac{1}{2}$ with velocity 0 at t=0

- With charge index α

$$x(\alpha, 0) = \frac{1}{2}, \quad v(\alpha, 0) = 0, \quad E(\alpha, t) = \alpha.$$

- Solving equations

$$\frac{dx(\alpha, t)}{dt} = v(\alpha, t), \quad \frac{dv(\alpha, t)}{dt} = -E(x(\alpha, t), t),$$

Example — Continuous fission solution

- Obtain the weak solution

$$x(\alpha,t) = \alpha + (\frac{1}{2} - \alpha)\cos t,$$

$$v(x(\alpha,t),t) = (\alpha - \frac{1}{2})\sin t,$$
$$f(x(\alpha,t), v(\alpha,t), t) = \frac{1}{1-\cos t}.$$

- Continuous fission weak solution

- Zero diffusion limit of the Fokker-Planck equation

  Majda, Majda, & Zheng 1994

A novel algorithm for 1D Euler-Poisson

Step 1. Establish the "charge index" according to the initial data.

Step 2. Input space step size $\Delta h = 1/N$ and time step size Δt.

Step 3. Input initial values $(x(\alpha_k, 0), u(\alpha_k, 0))$, here $\alpha_k = k/N$.

Step 4. **for** i from 1 to $T/\Delta t$ **do**

Step 5. Update $x(\alpha, t + \Delta t)$ and $u(\alpha, t + \Delta t)$

$$\frac{dx}{dt} = v, \qquad \frac{dv}{dt} = -E.$$

Step 6. Update $E(\alpha, t + \Delta t)$

Step 7. Construct $\rho(\alpha, t + \Delta t)$ by the following formula

$$\rho\Big(\alpha \in (\frac{k}{N}, \frac{k+1}{N}], t + \Delta t\Big) = \frac{1}{N|x(\frac{k}{N}, t + \Delta t) - x(\frac{k+1}{N}, t + \Delta t)|}$$

Step 8. **end**

Moment system for multi-valued solutions

- Moments of the Vlasov-Poisson equations

$$m_l = \int_{\mathbb{R}} f(x,v,t) v^l dv, \quad l = 0, 1, \cdots, 2K.$$

- Moment equations in the physical space

$$\begin{aligned}
\frac{\partial}{\partial t} m_0 + \frac{\partial}{\partial x} m_1 &= 0\,, \\
\frac{\partial}{\partial t} m_1 + \frac{\partial}{\partial x} m_2 &= -m_0 \partial_x \phi\,, \\
&\cdots \\
\frac{\partial}{\partial t} m_{2K-1} + \frac{\partial}{\partial x} m_{2K} &= -(2K-1) m_{2K-2} \partial_x \phi\,, \\
\partial_{xx} \phi &= 1 - \sum_{k=1}^{K} \rho_k\,.
\end{aligned}$$

- Second order kinetic scheme

  Jin & Li, 2003

Numerical Examples

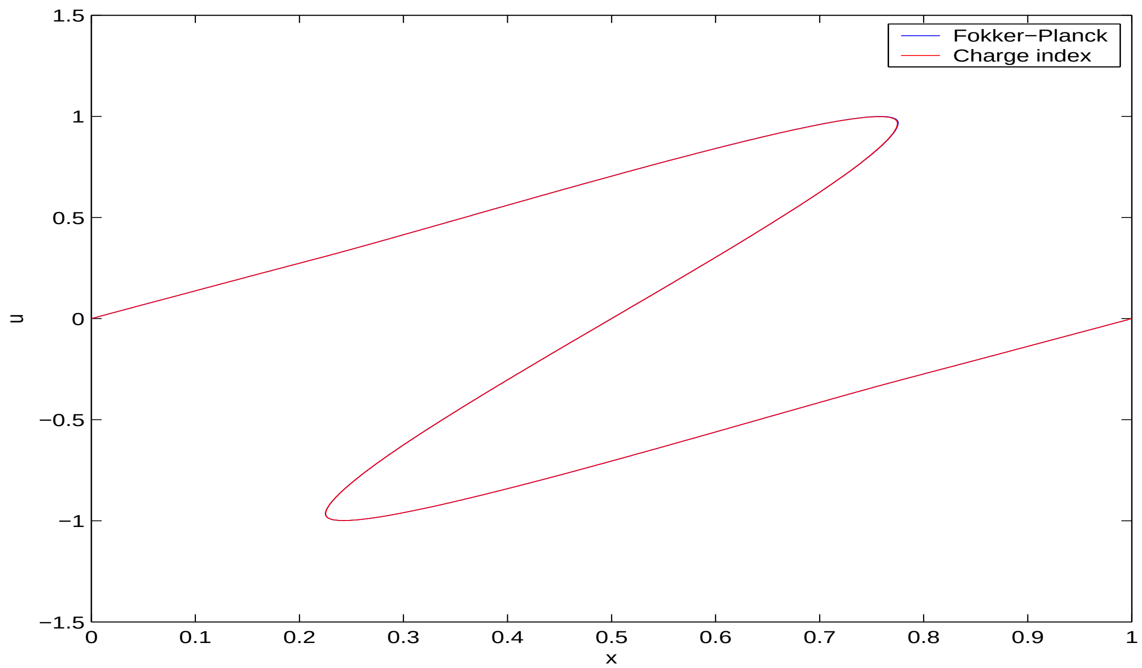
- Solutions to Euler-Poisson, Fokker-Planck, and Moment equations

  Initial data: $\rho_0(x) = 1, \quad u_0(x) = \sin(2\pi x), \quad f_0 = \delta(v - u_0(x))$

  Solution becomes multivalued at $t^* = 0.1598$, we calculate $t = 0.5$

  Charge Index: $\Delta x = \frac{1}{100}, \quad \Delta t = \frac{\Delta x}{5}$

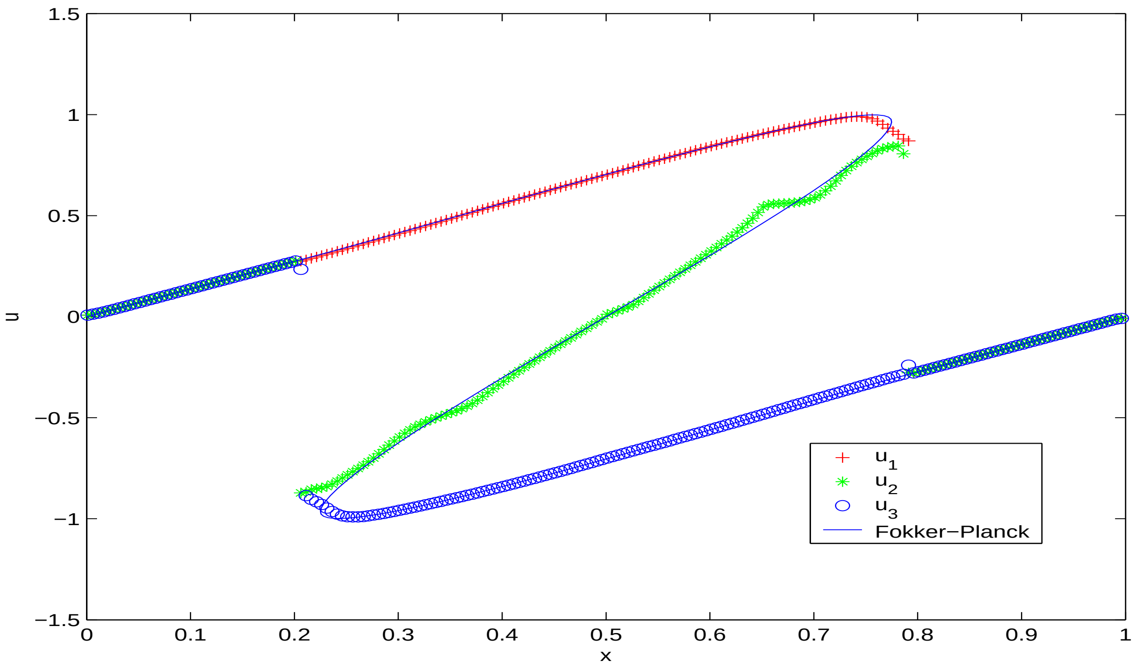
  Fokker-Planck: 8192 particles, $\Delta t = 0.002, \quad \epsilon = 0.01$



Numerical Examples

- Solution to the moment equations

Moment equations: $\Delta x = \frac{1}{400}, \quad \Delta t = \frac{\Delta x}{5}$



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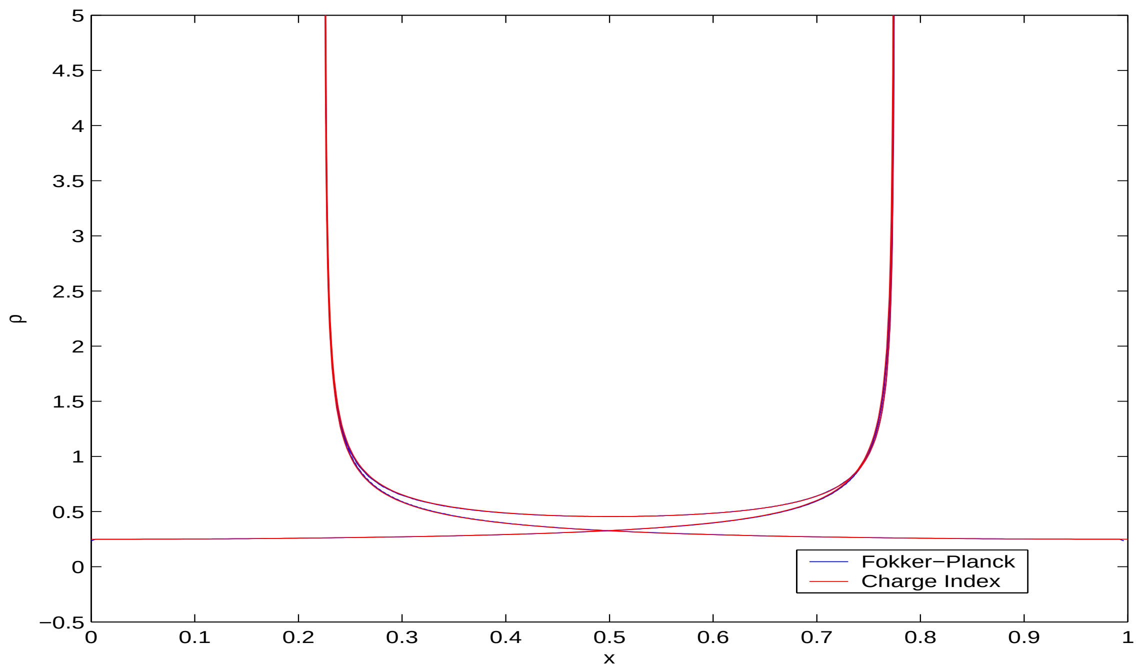

Numerical Examples

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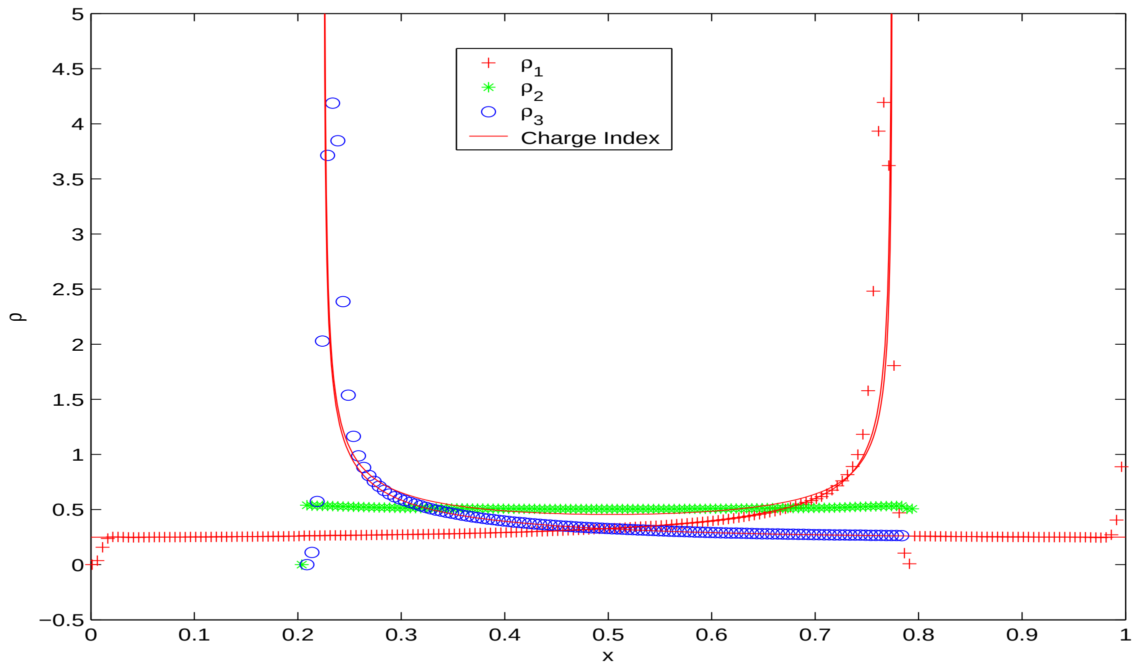

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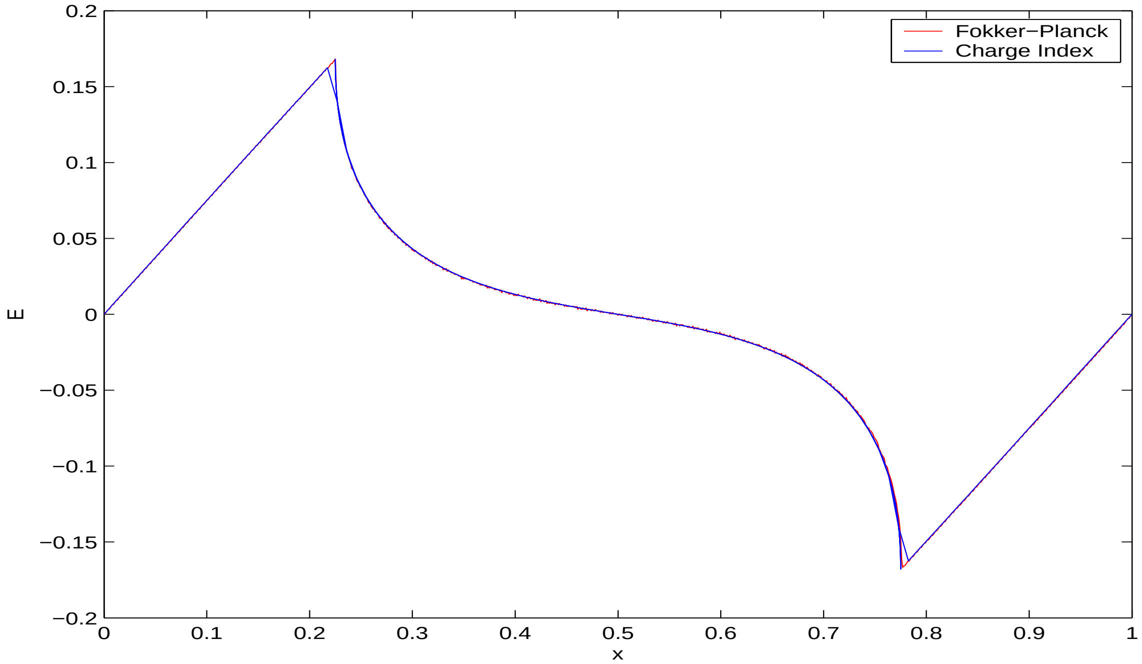
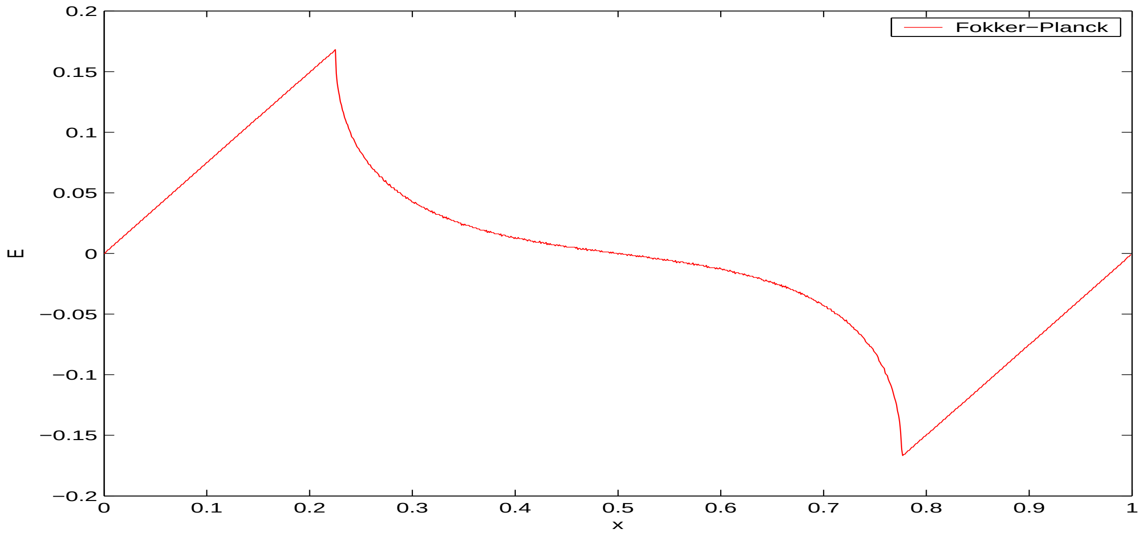
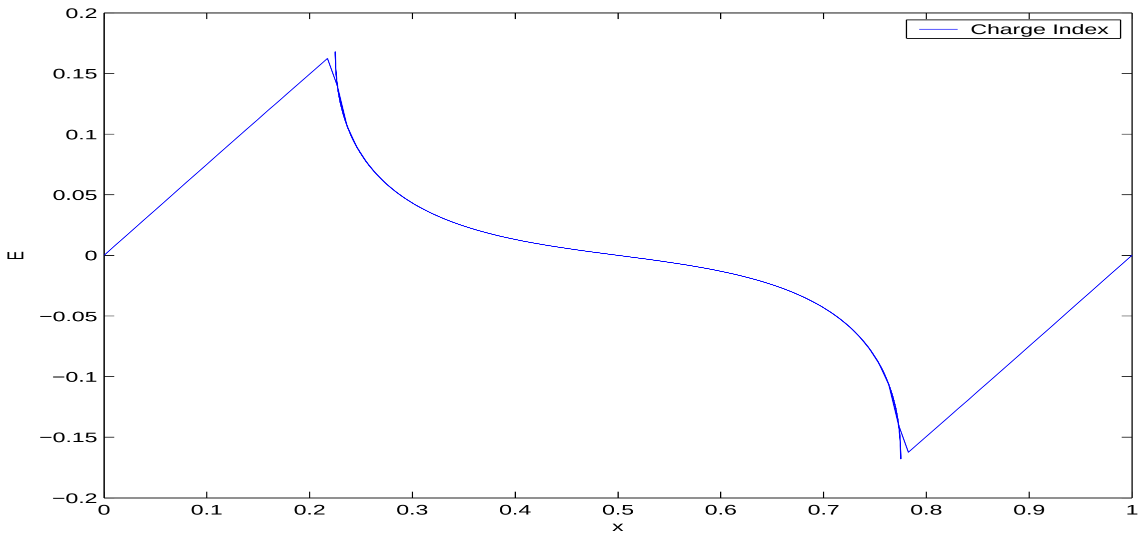


Numerical Examples

Fokker-Planck



Charge Index



Numerical Examples

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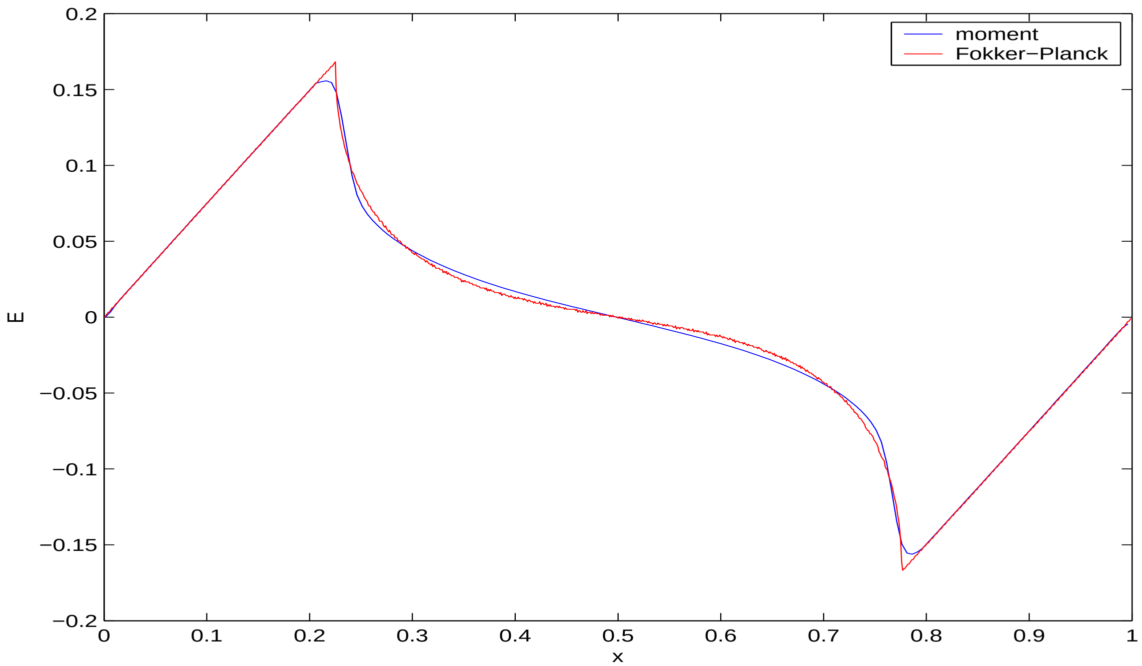

End

Thank You !