Derivation of coarse-grained models from microscopic CA models of traffic flows

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Outline

Main Goal: Systematic Derivation of PDE Models for Traffic Flow

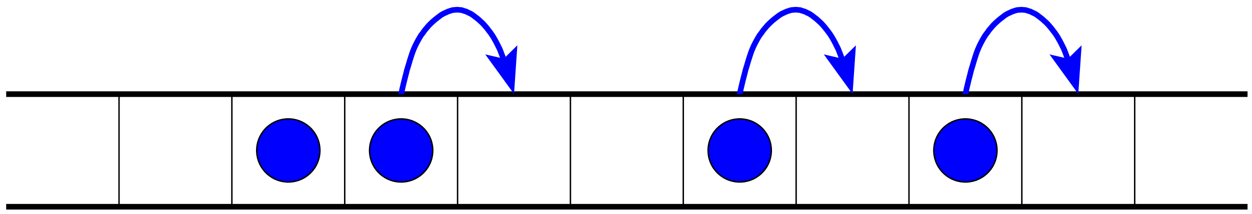
- Microscopic Rules for the Motion of Cars or Pedestrians
- Microscopic CA model for Traffic Flow
- Assumptions for deriving the Coarse-Grained PDE
- Lighthill-Whitham-Richards PDE model
- Equation for the Variance
- Look-Ahead Potential
- One-Dimensional Stochastic Model CA for Pedestrian Traffic
- Explicit Rules for the Interaction of Pedestrians Moving in Opposite Directions

Microscopic Stochastic Lattice Model of Car Traffic

ASEP Dynamics: Spitzer, Derrida, Derrida et al., etc.

L-H Potential: Sopasakis and Katsoulakis, 2006

Approach: One Dimensional $\{0, 1\}$ Lattice Configuration $\sigma_k(t)$, k: cell index. (ASEP: cars moving to the right)



One Step: $k \to k + 1$ during time-interval Δt

Spin Exchange:

$$[\sigma_k(t) = 1,\ \sigma_{k+1}(t) = 0] \to [\sigma_k(t + \Delta t) = 0,\ \sigma_{k+1}(t + \Delta t) = 1]$$

Two cars cannot occupy the same cell

Microscopic Stochastic Lattice Model

Transition Probability: For cells $[\sigma_k, \sigma_{k+1}]$

$$Pr\{[\sigma_k = 0,\, \sigma_{k+1} = 1](t + \Delta t) \mid [\sigma_k = 1,\, \sigma_{k+1} = 0](t)\} \approx c_0 \Delta t$$

where c_0 is the characteristic velocity

Rewrite Transition Probability:

$$Pr\{\text{Transition } \sigma_k(t) \rightarrow \sigma_{k+1}(t + \Delta t)\} \approx \sigma_k(1 - \sigma_{k+1}) c_0 \Delta t$$

Goal: Understand Ensemble Simulations

In particular Traffic Density: $\mathbb{E}\,\sigma_k(t)$

Equation for the Time-Evolution of the Density $\rho_k(t) = \mathbb{E}\,\sigma_k(t)$:
Only two events are important $k \to k+1$ and $k-1 \to k$

$$\frac{d}{dt}\mathbb{E}\,\sigma_k = \mathbb{E}\,[c_0\sigma_{k-1}(1-\sigma_k) - c_0\sigma_k(1-\sigma_{k+1})]$$

The equation is exact, but not closed

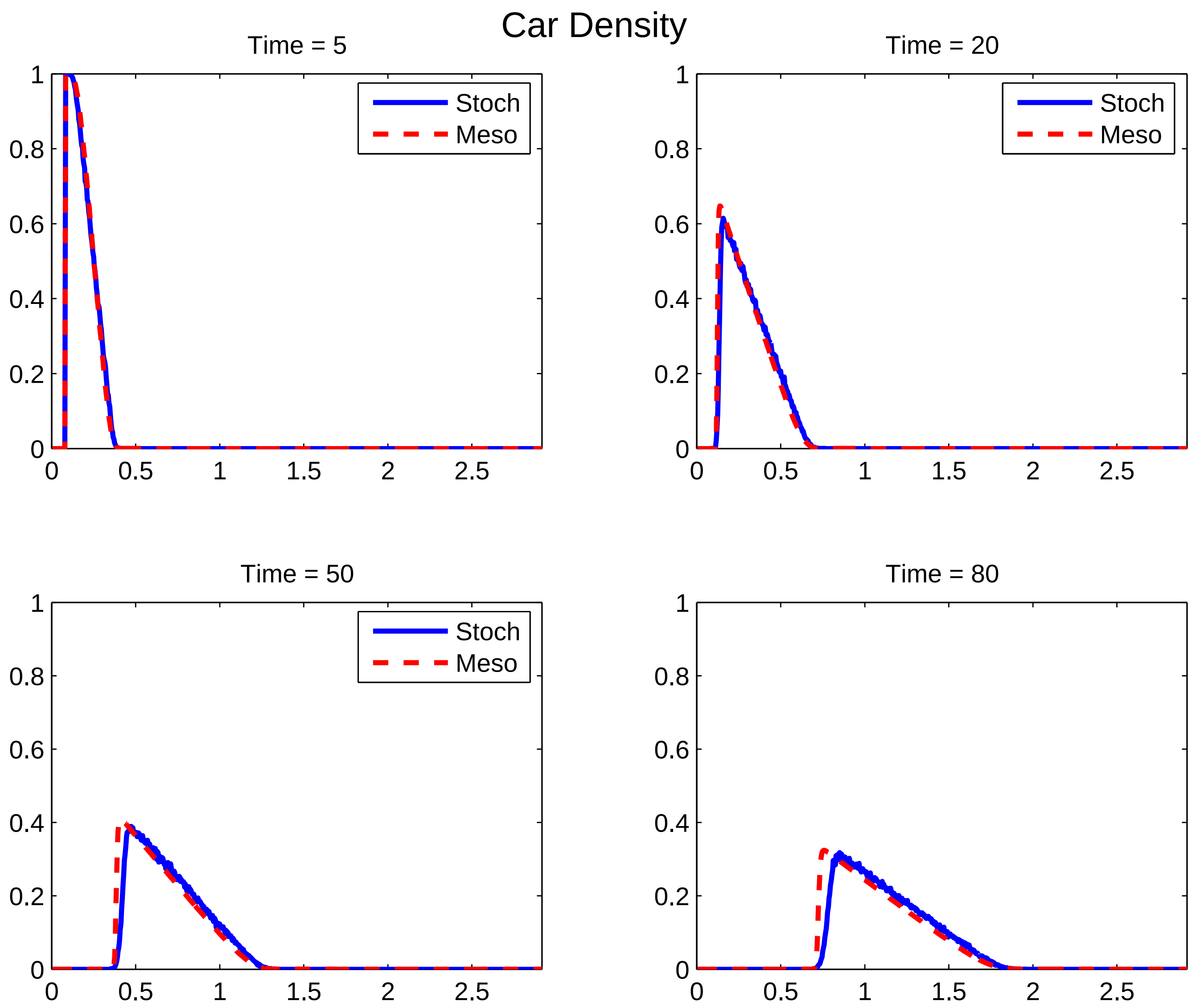
Closure Assumption: Approximate Independence

$$\mathbb{E}[\sigma_k\sigma_{k+1}] \approx \mathbb{E}[\sigma_k]\mathbb{E}[\sigma_{k+1}]$$

Mesoscopic Model: $\rho_k(t) = \mathbb{E}\,\sigma_k(t)$

$$\dot{\rho}_k = c_0\rho_{k-1}(1-\rho_k) - c_0\rho_k(1-\rho_{k+1})$$

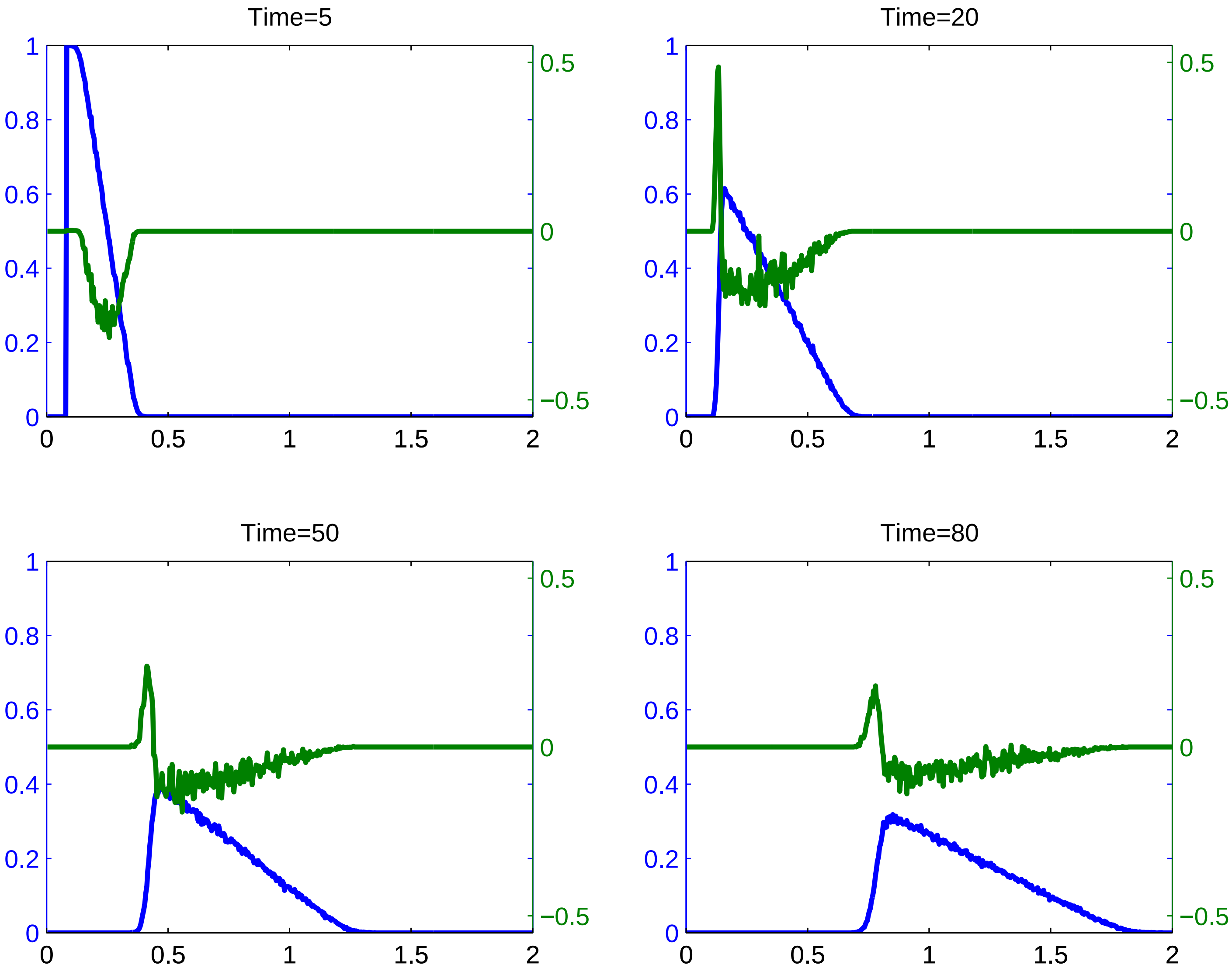
Comparing Microscopic and Mesoscopic Simulations: Car Density



Blue: Stochastic MC Simulations

Red: Mesoscopic Model for $\rho_k(t)$

Testing the Independence Assumption

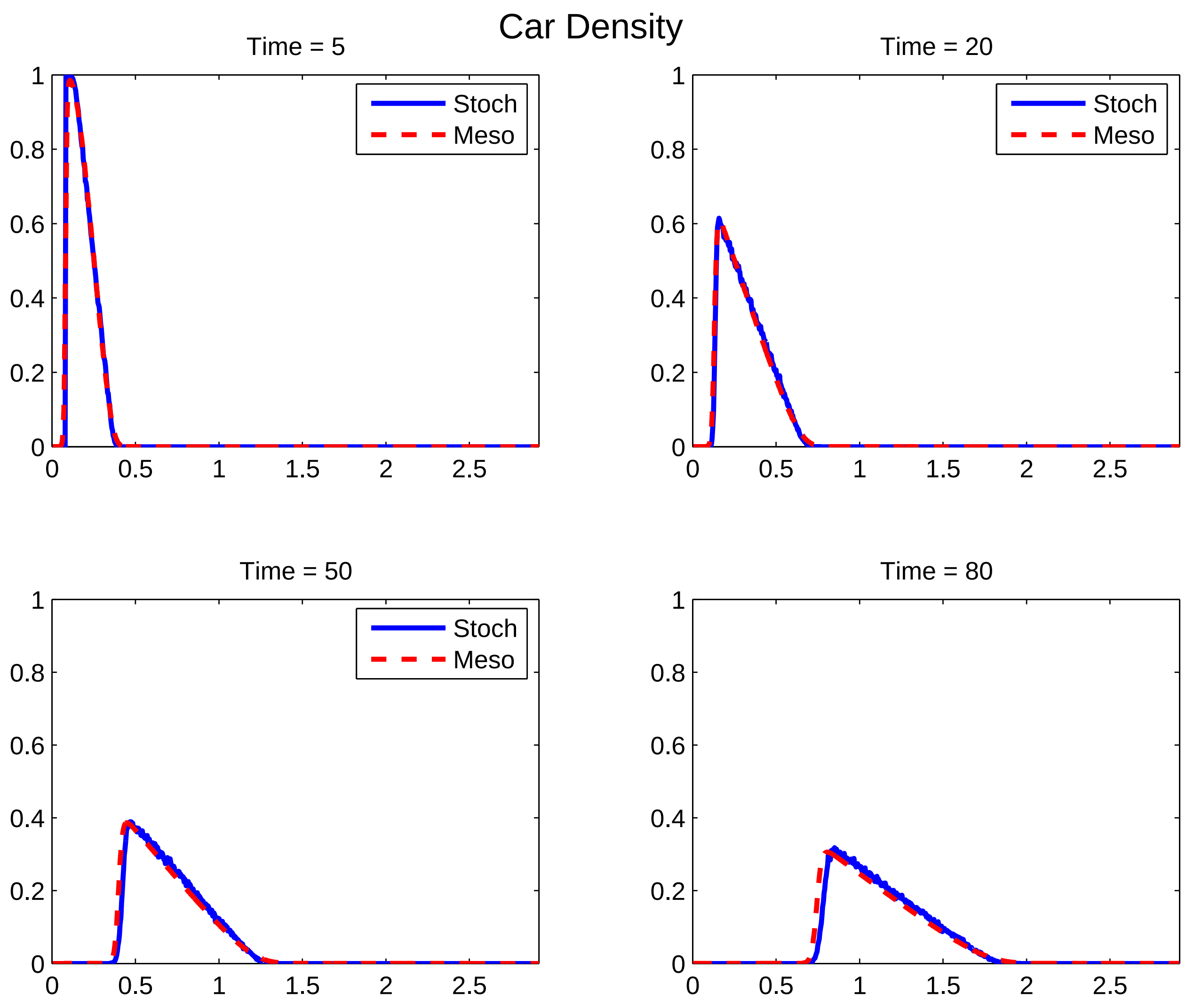


Blue: Car Density

Green: $\mathbb{E}[\sigma_k \sigma_{k+1}] - \mathbb{E}[\sigma_k]\mathbb{E}[\sigma_{k+1}]$

Comparing Microscopic and Mesoscopic Simulations: Car Density

Mesoscopic Model with Diffusion



Blue: Stochastic MC Simulations

Red: Mesoscopic Model for $\rho_k(t)$

Variance in Stochastic Simulations (Hauck, Sun, I.T.)

Consider: $v_k(t) = \mathbb{E}\,\sigma_k^2(t)$

Equations for the Mean and Variance:

$$\dot{\rho}_k = c_0\rho_{k-1}(1-\rho_k) - c_0\rho_k(1-\rho_{k+1})$$

$$\dot{v}_k = c_0\rho_{k-1}(1-\rho_k) - c_0\rho_k(1-\rho_{k+1})$$

Then:

$$v_k = \rho_k + C_k$$

$$Var_k = \rho_k - \rho_k^2 + C_k$$

Derivation Involves:

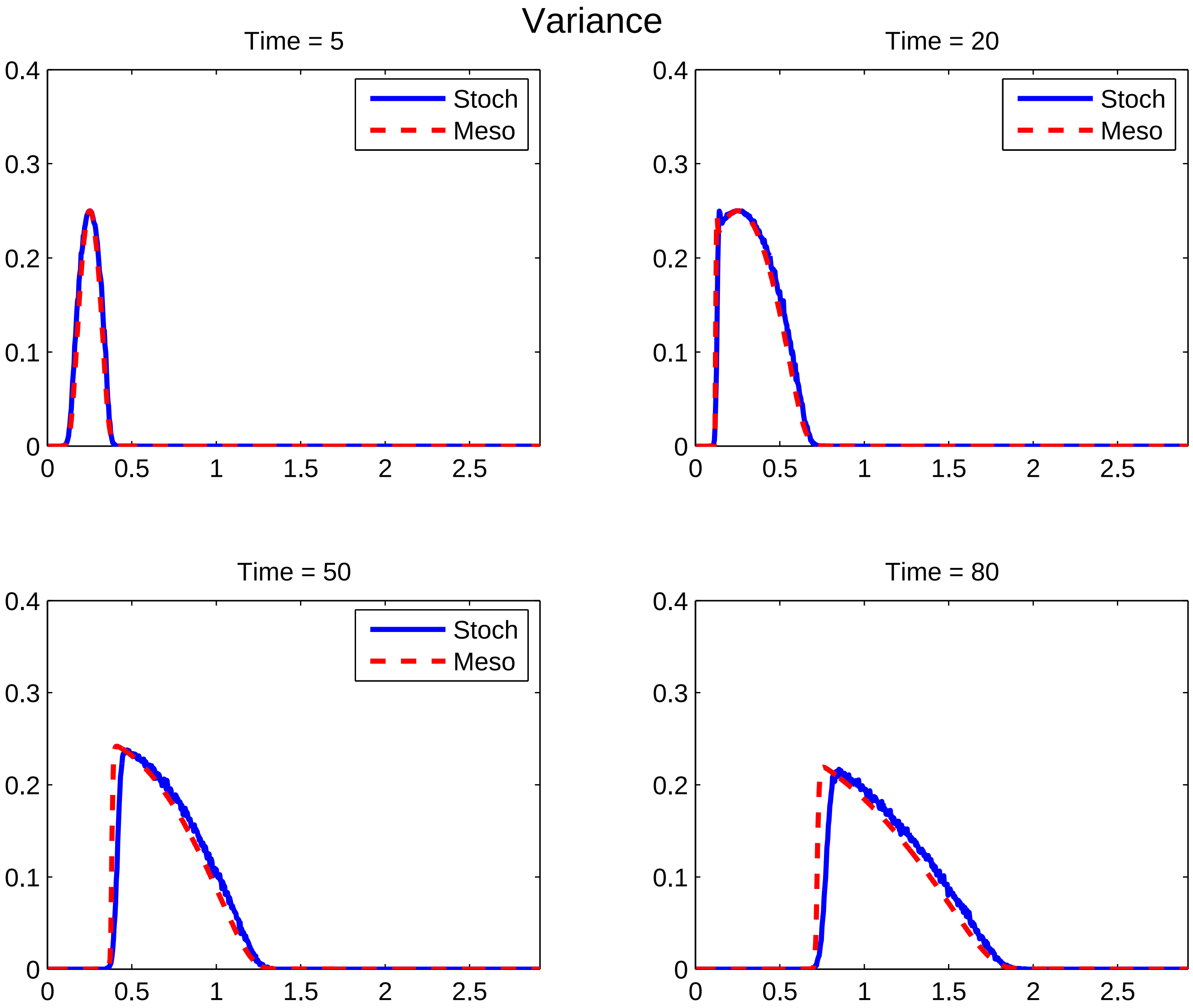
$$Af = \frac{\mathbb{E}[f(\sigma(\Delta t))] - f(\sigma)}{\Delta t}$$

where $f = f(\sigma)$ is any test function on the whole lattice.

To derive equation for ρ_k: $f = \sigma_k$

To derive equation for v_k: $f = \sigma_k^2$

Variance in Stochastic Simulations



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Look-Ahead Potential (Sopasakis & Katsoulakis 06)

Transition Probability: $k \to k+1$

$$Pr\{\text{Transition } \sigma_k(t) \to \sigma_{k+1}(t+\Delta t)\} \approx \sigma_k(1-\sigma_{k+1})c_0 e^{-\beta J_k}\Delta t$$

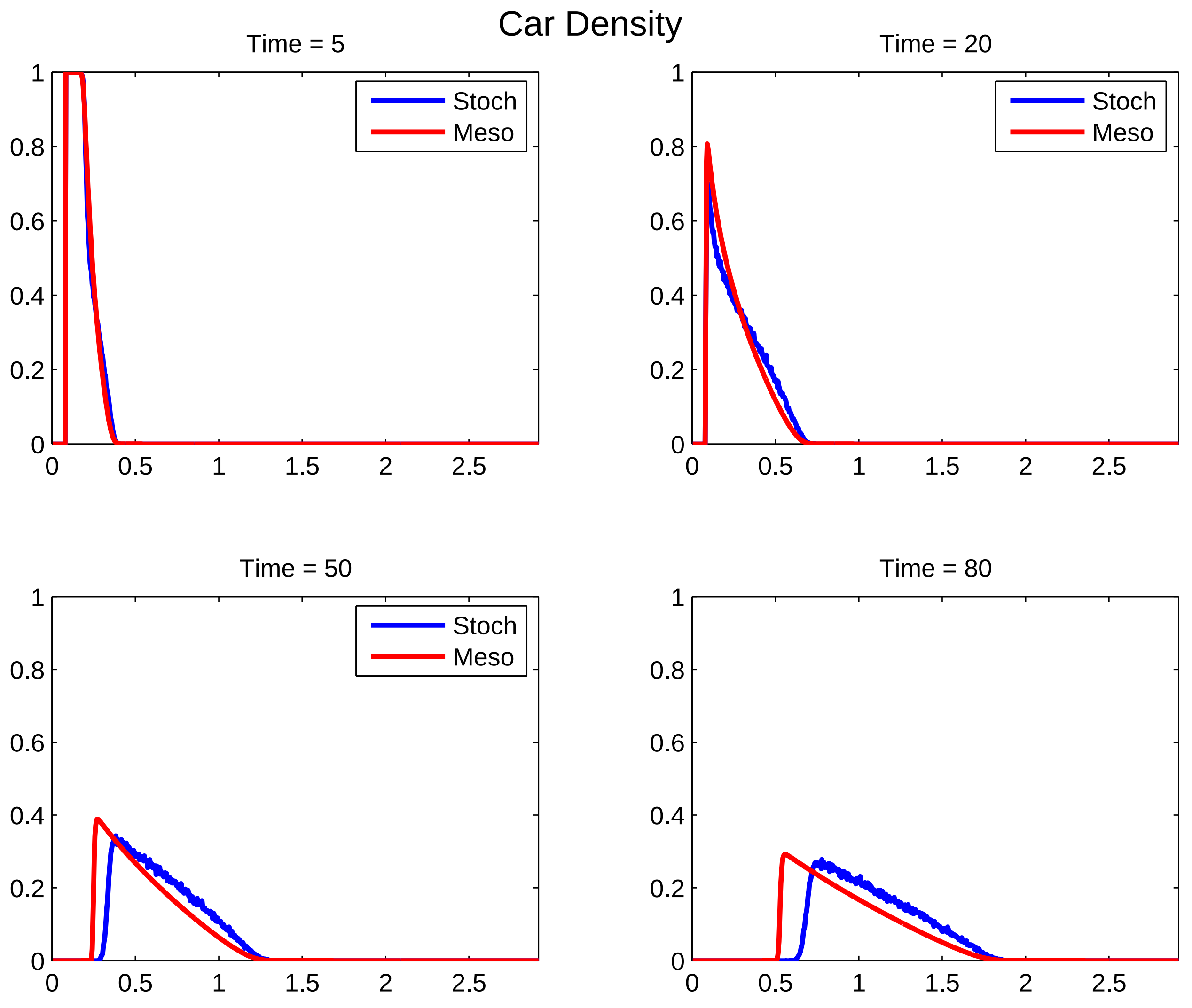
$$J_k = \frac{1}{M}\sum_{l=k+2}^{k+1+M} \sigma_l(t)$$

Takes into account presence/absence of cars in cells $k+2, \ldots, k+1+M$

Mesoscopic Model

$$\begin{aligned}
\dot{\rho}_k \quad = \quad & c_0\rho_{k-1}(1-\rho_k)\prod_{i=1}^{M}\left[1+\rho_{k+i}(e^{-\beta/M}-1)\right] - \\
& c_0\rho_k(1-\rho_{k+1})\prod_{i=1}^{M}\left[1+\rho_{k+i+1}(e^{-\beta/M}-1)\right]
\end{aligned}$$

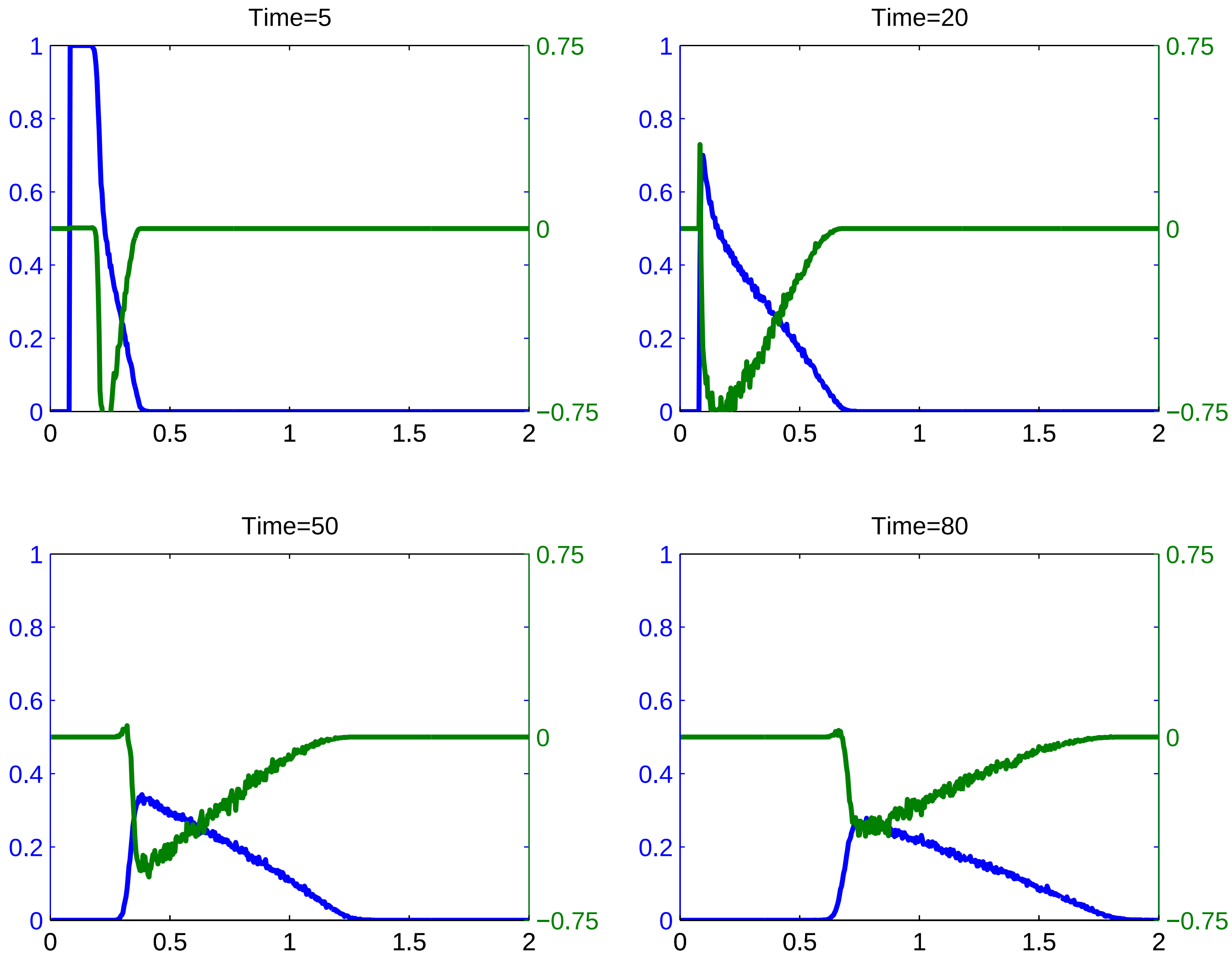
Comparing Microscopic and Mesoscopic Simulations with LH: Car Density



Blue: Stochastic MC Simulations

Red: Mesoscopic Model for $\rho_k(t)$

Testing the Independence Assumption with Look-Ahead Potential

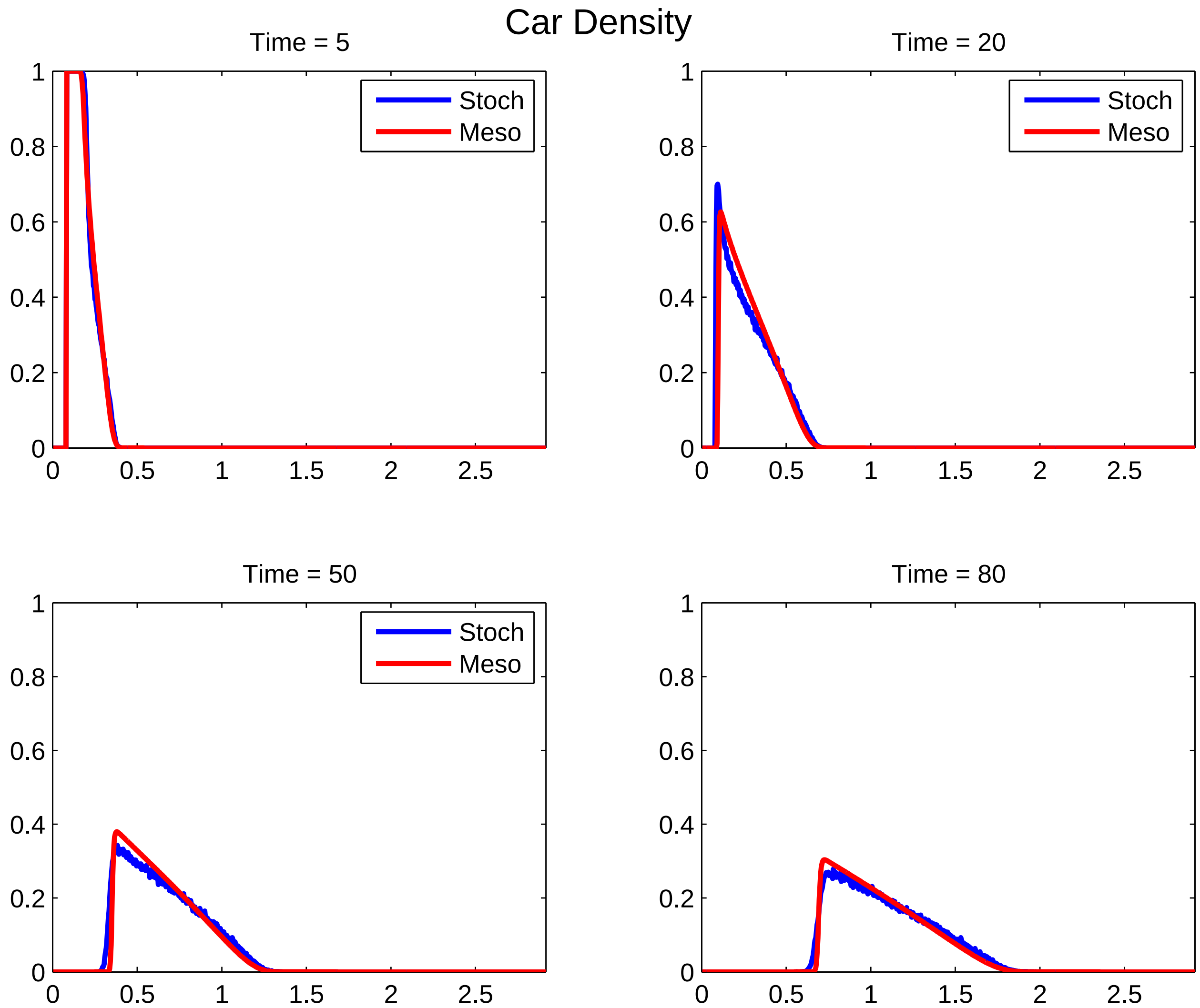


Blue: Car Density

Green: $\mathbb{E}[\sigma_k \sigma_{k+1}] - \mathbb{E}[\sigma_k]\mathbb{E}[\sigma_{k+1}]$

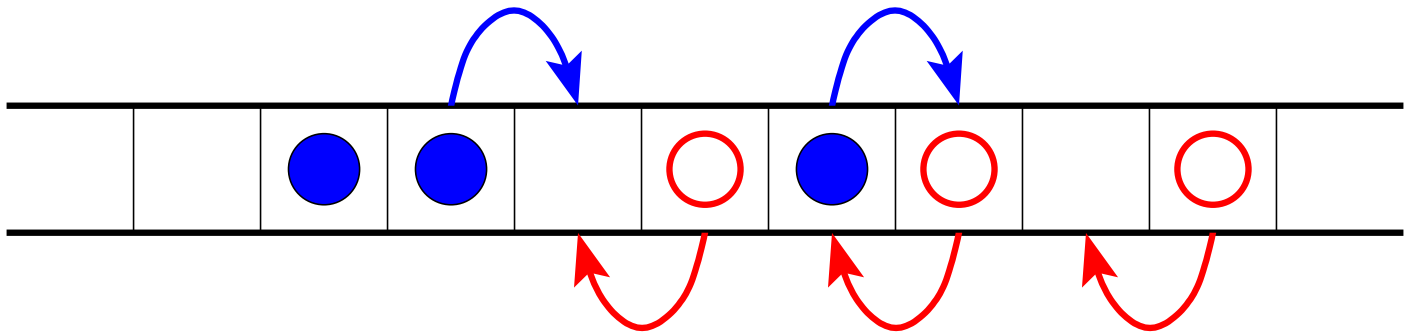
Modified Mesoscopic Model:

$$\dot{\rho}_k = c_0 \rho_{k-1}(1-\rho_k)\left[1+\rho_{k+1}(e^{-\beta\rho_{k+1}^d}-1)\right] - c_0 \rho_k(1-\rho_{k+1})\left[1+\rho_{k+2}(e^{-\beta\rho_{k+2}^d}-1)\right]$$



Microscopic Stochastic Lattice Model of Pedestrian Traffic

Major Difference: Bi-directional model



Approach: One Dimensional $\{0, 1\}$ Lattice Configuration for

$$\sigma_k^+(t) \ - \ \text{pedestrians moving to the right}$$
$$\sigma_k^-(t) \ - \ \text{pedestrians moving to the left}$$

If no pedestrians in the opposite direction: Equivalent to car traffic models

One Step: $k \to k+1$ during time-interval Δt

$$\left[\sigma_k^+(t), \sigma_{k+1}^+(t)\right] = [1, 0] \to \left[\sigma_k^+(t+\Delta t), \sigma_{k+1}^+(t+\Delta t)\right] = [0, 1]$$

Two pedestrians moving in the same direction cannot occupy the same cell

Microscopic Stochastic Lattice Model

Transition Probability: For cells $[\sigma_k^+, \sigma_{k+1}^+]$

$$[\sigma_k^-, \sigma_{k+1}^-] = [0,0] : \quad Pr\{[0,1](t+\Delta t) \mid [1,0](t)\} \approx c_0 \Delta t$$

where c_0 is the characteristic velocity

Pedestrians Moving in the Opposite Direction are Present:

$$[\sigma_k^-, \sigma_{k+1}^-] = [1,0] : \quad Pr\{[0,1](t+\Delta t) \mid [1,0](t)\} \approx c_1 \Delta t$$

$$[\sigma_k^-, \sigma_{k+1}^-] = [0,1] : \quad Pr\{[0,1](t+\Delta t) \mid [1,0](t)\} \approx c_2 \Delta t$$

$$[\sigma_k^-, \sigma_{k+1}^-] = [1,1] : \quad Pr\{[0,1](t+\Delta t) \mid [1,0](t)\} \approx c_3 \Delta t$$

where

$$c_0 > c_1 \approx c_2 > c_3$$

Goal

Understand the Density of the Pedestrian Traffic: $\mathbb{E}\,\sigma_k^+(t)$ and $\mathbb{E}\,\sigma_k^-(t)$

Equations for the Time-Evolution of the Density $\mathbb{E}\,\sigma_k^+(t)$: Only two events are important $k \to k+1$ and $k-1 \to k$

$$
\begin{aligned}
\frac{d}{dt}\mathbb{E}\,\sigma_k^+ \quad = \quad & \mathbb{E}\,\big[c_0\sigma_{k-1}^+(1-\sigma_k^+)(1-\sigma_{k-1}^-)(1-\sigma_k^-))- \\
& \qquad c_0\sigma_k^+(1-\sigma_{k+1}^+)(1-\sigma_k^-)(1-\sigma_{k+1}^-)+ \\
& c_1\sigma_{k-1}^+(1-\sigma_k^+)\sigma_{k-1}^-(1-\sigma_k^-)-c_1\sigma_k^+(1-\sigma_{k+1}^+)\sigma_k^-(1-\sigma_{k+1}^-)+ \\
& c_2\sigma_{k-1}^+(1-\sigma_k^+)(1-\sigma_{k-1}^-)\sigma_k^- - c_2\sigma_k^+(1-\sigma_{k+1}^+)(1-\sigma_k^-)\sigma_{k+1}^- + \\
& c_3\sigma_{k-1}^+(1-\sigma_k^+)\sigma_{k-1}^-\sigma_k^- - c_3\sigma_k^+(1-\sigma_{k+1}^+)\sigma_k^-\sigma_{k+1}^-\big]
\end{aligned}
$$

The equations are exact, but not closed

Closure Assumptions

Approximate Independence:

$$\mathbb{E}[\sigma^+_{k-1}\sigma^+_k\sigma^-_{k-1}\sigma^-_k] \approx \mathbb{E}[\sigma^+_{k-1}]\mathbb{E}[\sigma^+_k]\mathbb{E}[\sigma^-_{k-1}]\mathbb{E}[\sigma^-_k]$$

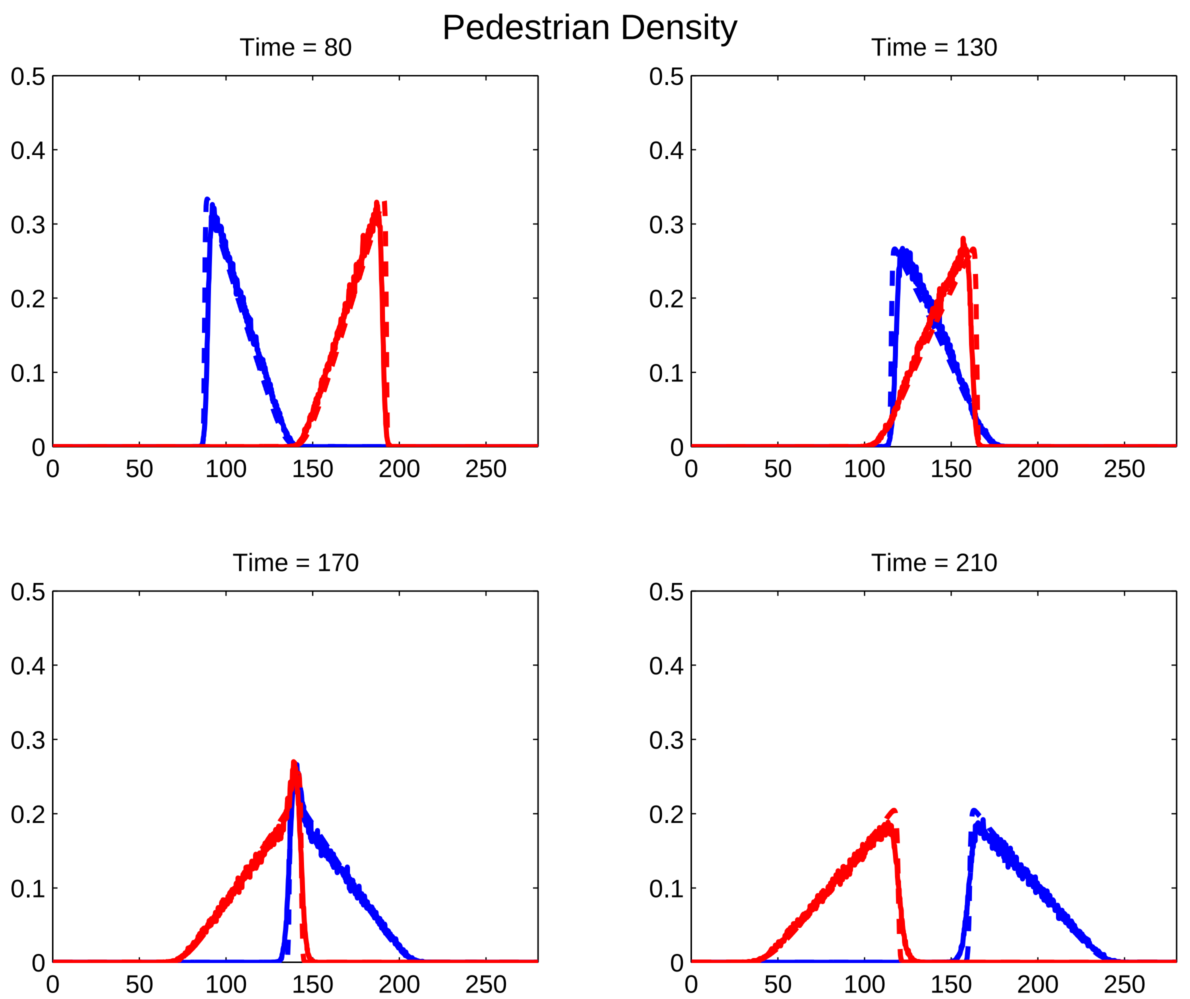
Mesoscopic Model: $\rho^{\pm}_k(t) = \mathbb{E}\,\sigma^{\pm}_k(t)$

$$\begin{aligned}
\frac{d}{dt}\rho^+_k \quad = \quad & c_0\rho^+_{k-1}(1-\rho^+_k)(1-\rho^-_{k-1})(1-\rho^-_k) - \\
& \qquad c_0\rho^+_k(1-\rho^+_{k+1})(1-\rho^-_k)(1-\rho^-_{k+1}) + \\
& c_1\rho^+_{k-1}(1-\rho^+_k)\rho^-_{k-1}(1-\rho^-_k) - c_1\rho^+_k(1-\rho^+_{k+1})\rho^-_k(1-\rho^-_{k+1}) + \\
& c_2\rho^+_{k-1}(1-\rho^+_k)(1-\rho^-_{k-1})\rho^-_k - c_2\rho^+_k(1-\rho^+_{k+1})(1-\rho^-_k)\rho^-_{k+1} + \\
& c_3\rho^+_{k-1}(1-\rho^+_k)\rho^-_{k-1}\rho^-_k - c_3\rho^+_k(1-\rho^+_{k+1})\rho^-_k\rho^-_{k+1}
\end{aligned}$$

Numerical Comparison of Microscopic and Mesoscopic Models

Weak Slowdown: $\alpha = 1.5$

$$c_0 = 0.8m/s, \quad c_1 = c_2 = c_0/\alpha, \quad c_3 = c_0/(2\alpha)$$



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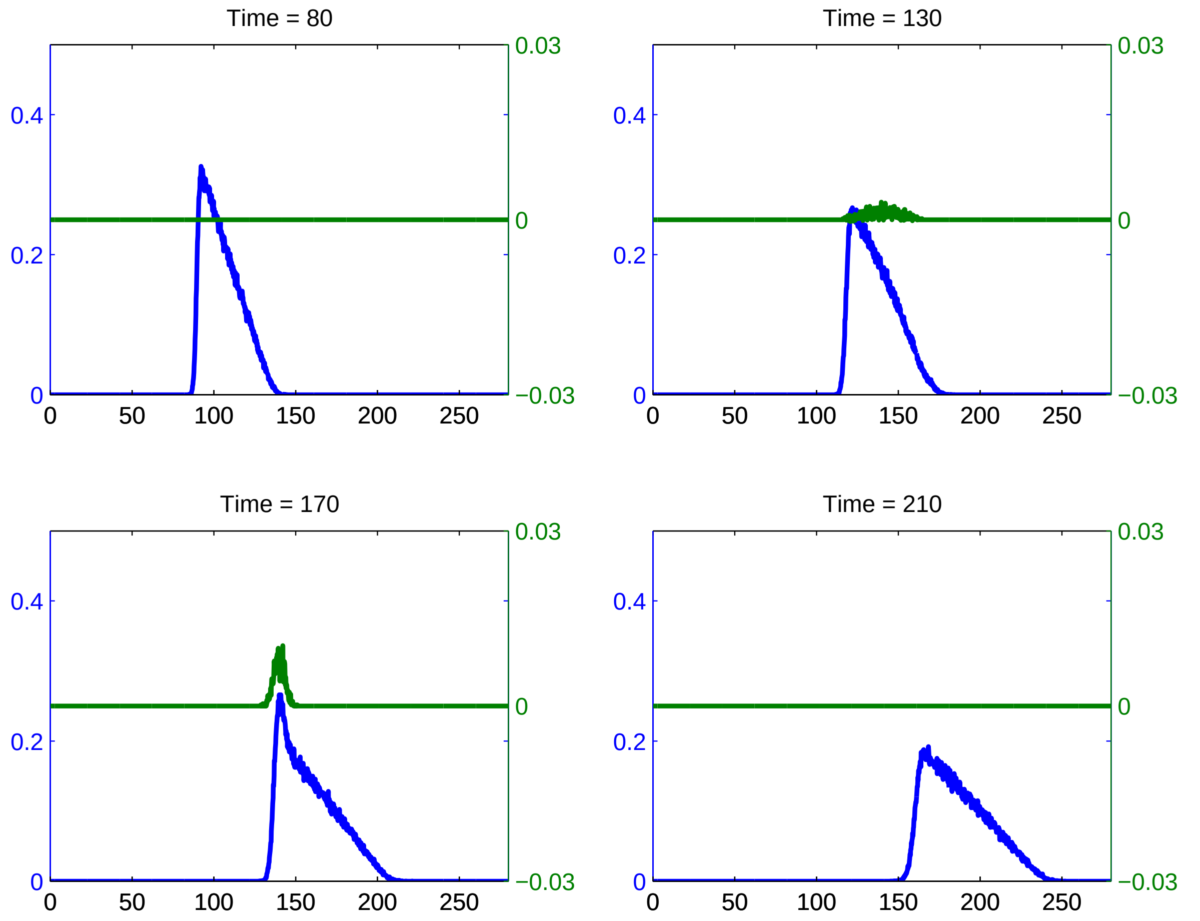
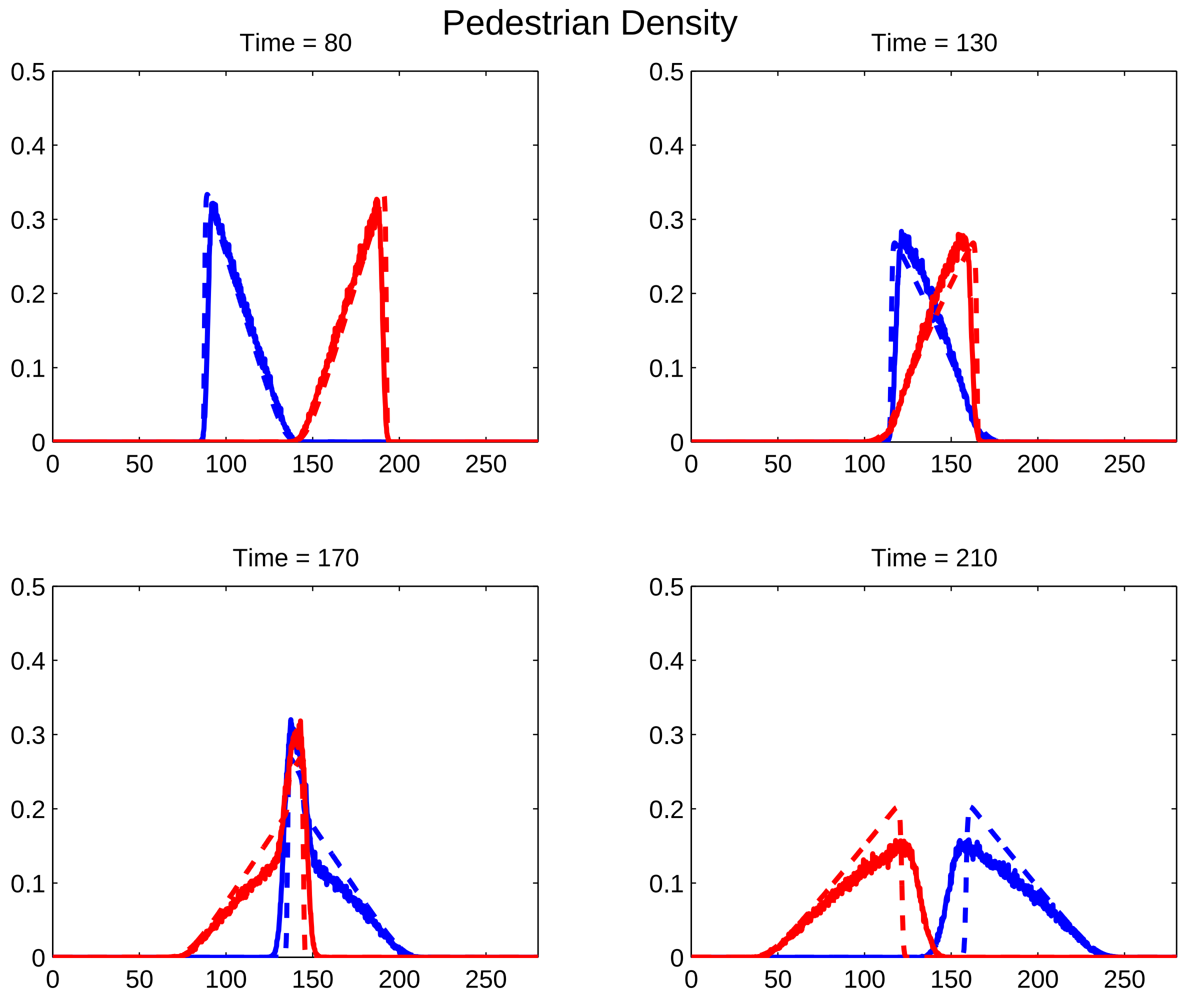


Numerical Comparison of Microscopic and Mesoscopic Models

Intermediate Slowdown: $\alpha = 2$

$$c_0 = 0.8m/s, \quad c_1 = c_2 = c_0/\alpha, \quad c_3 = c_0/(2\alpha)$$



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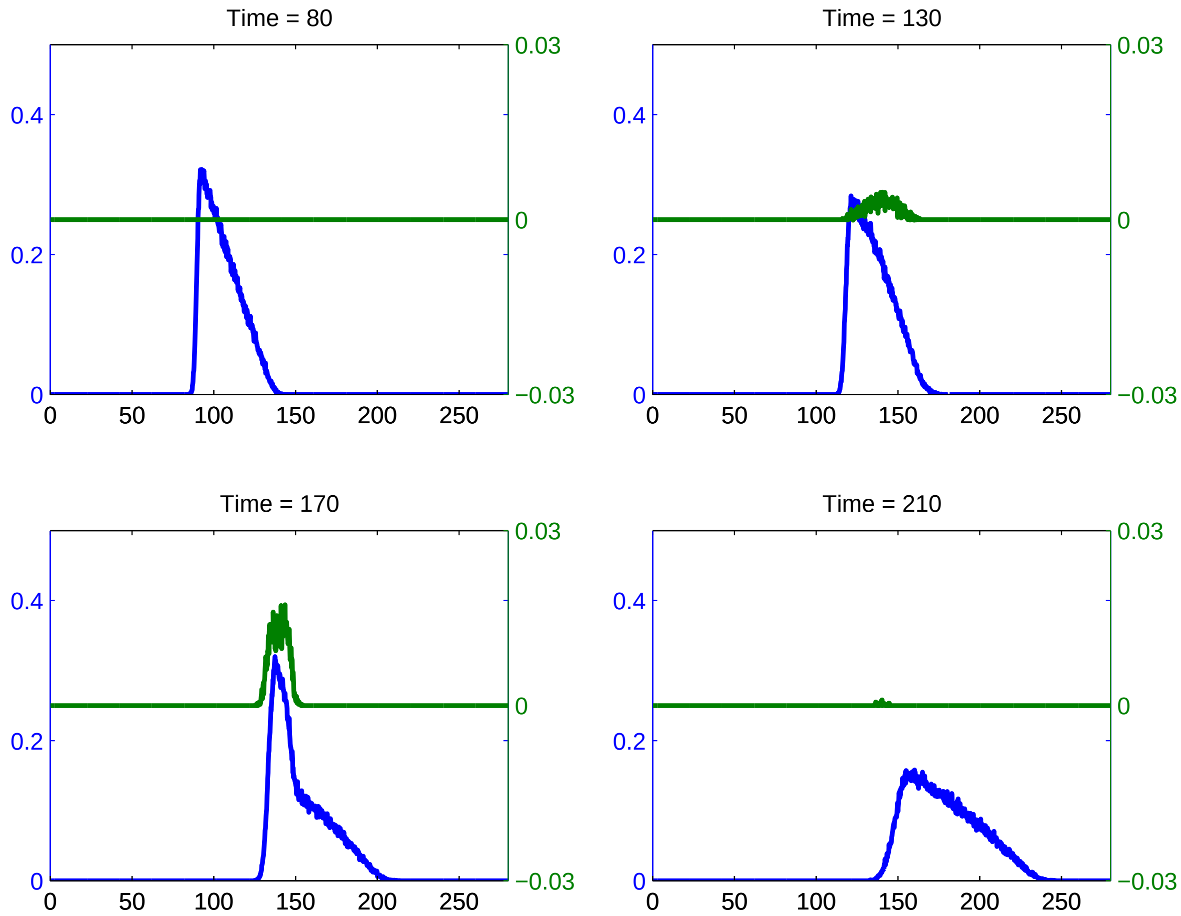
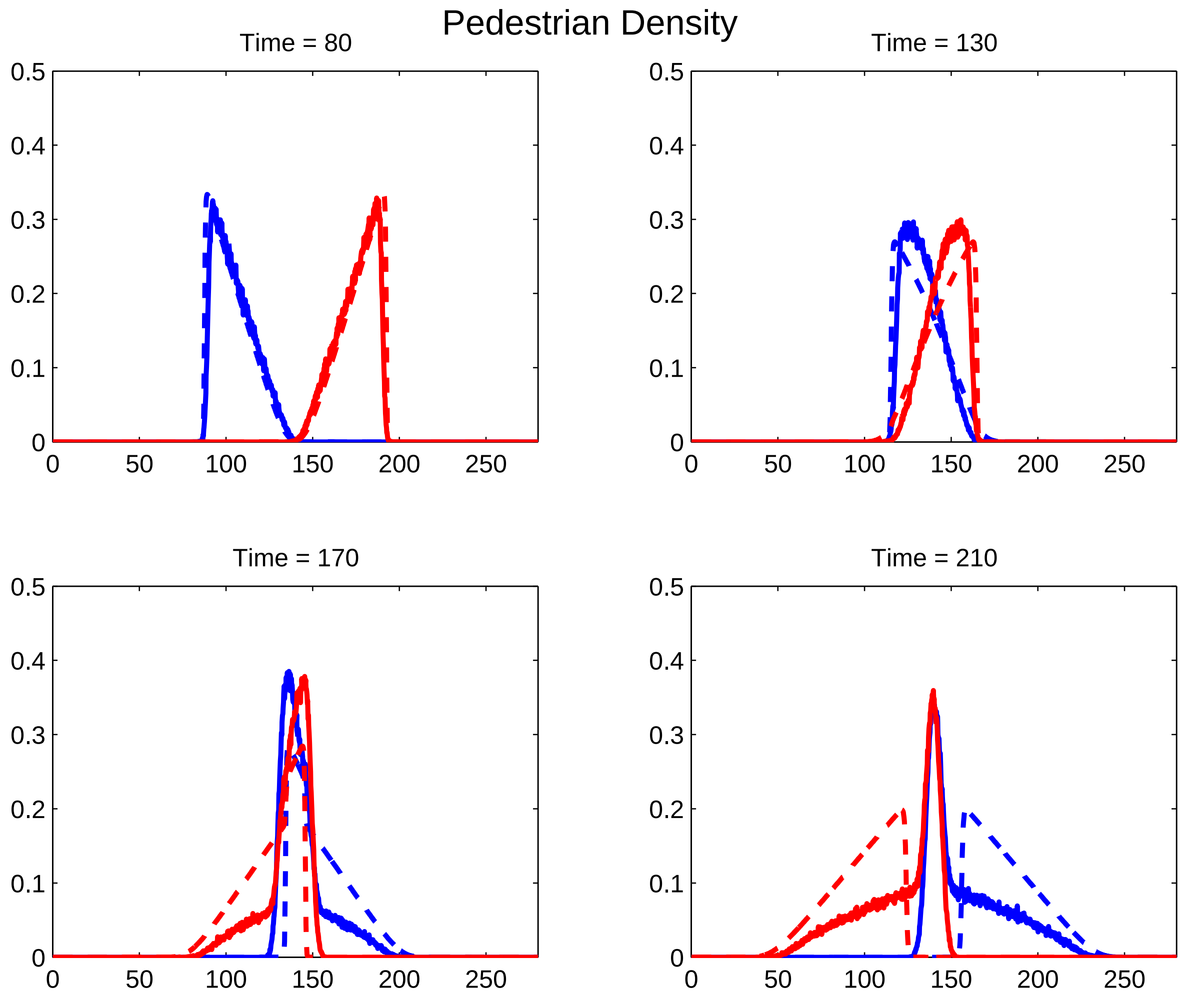


Numerical Comparison of Microscopic and Mesoscopic Models

Strong Slowdown: $\alpha = 3$

$$c_0 = 0.8m/s, \quad c_1 = c_2 = c_0/\alpha, \quad c_3 = c_0/(2\alpha)$$



Numerical Comparison of Microscopic and Mesoscopic Models

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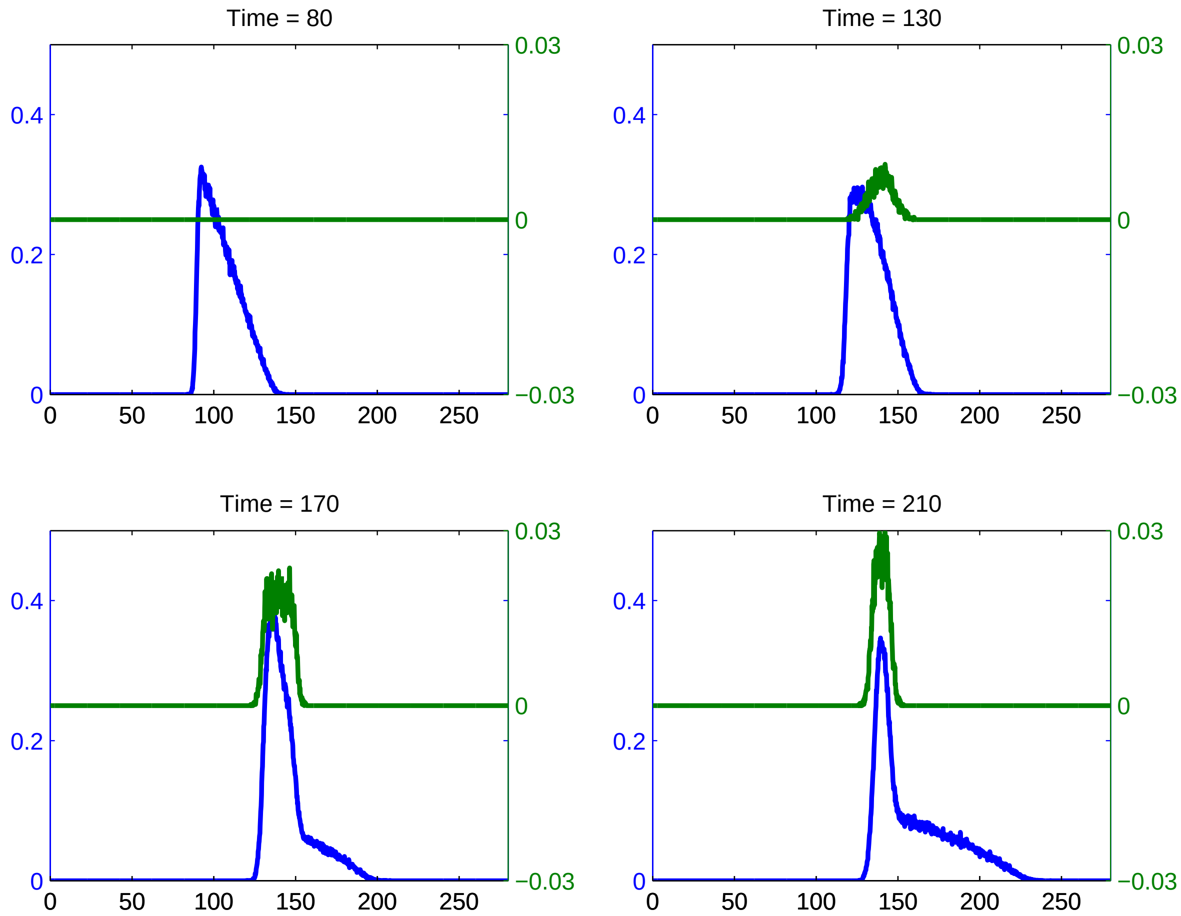
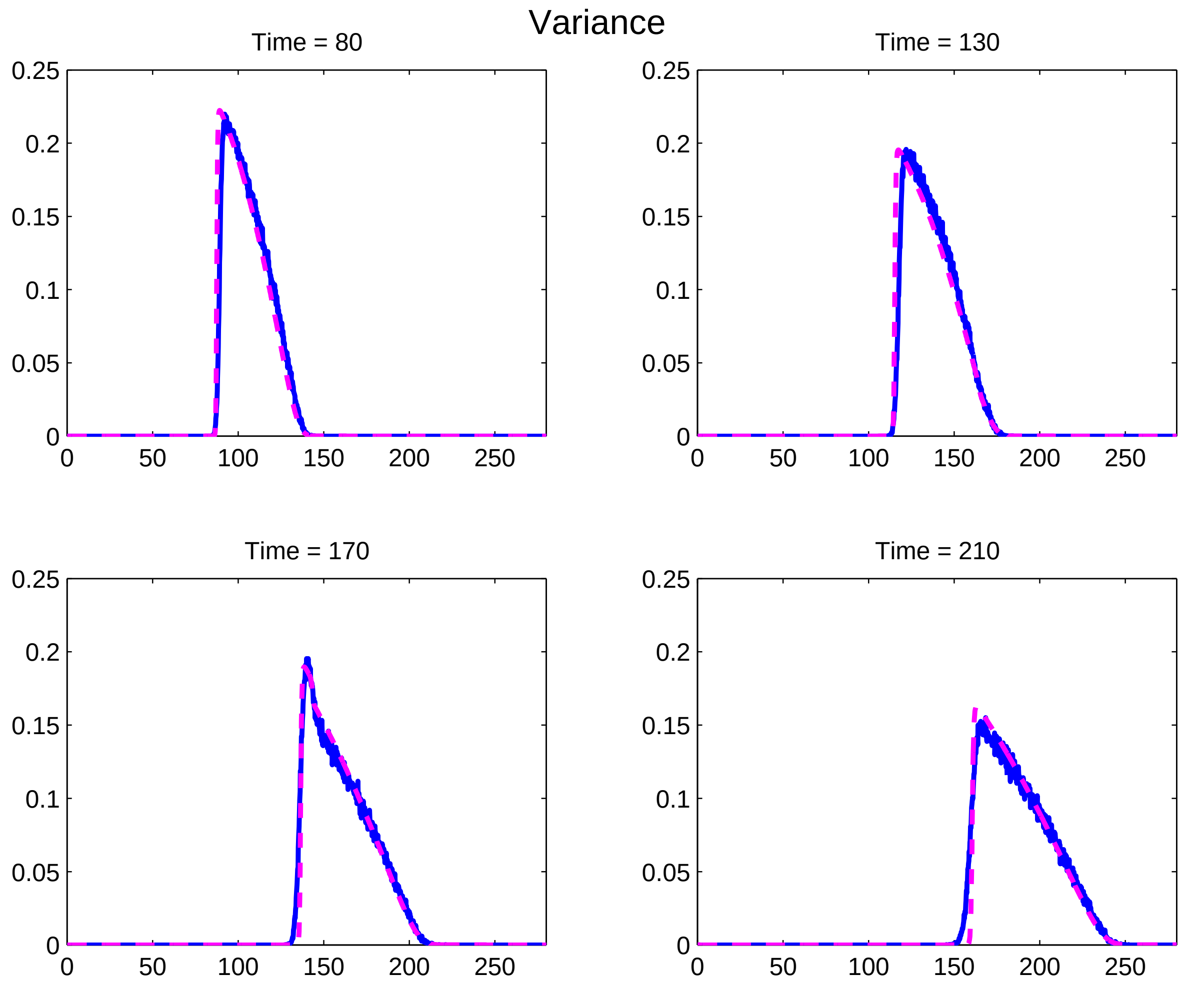


Variance in Microscopic and Mesoscopic Models

Weak Slowdown: $\alpha = 1.5$

$$c_0 = 0.8m/s, \quad c_1 = c_2 = c_0/\alpha, \quad c_3 = c_0/(2\alpha)$$



Conclusions

- Explicit Microscopic Interaction Rules Determine the Functional Form of the corresponding PDE model
- Framework for Systematic Derivation of PDE Models
- Look-Ahead information can also be included in the model
- Important to Understand Regime of Validity
- Variance of Stochastic Simulations can be Easily Inferred from the Density