On selection dynamics with nonlocal competition

Hailiang Liu

Department of Mathematics
Iowa State University

With: Wenli Cai (Tsinghua University)
Pierre Jabin (University of Maryland)

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Plan

- A population model without mutation (linear competition)
  - Relative entropy
  - Discrete selection dynamics
- A population model with mutation (nonlinear competition)
  - Gradient flow structure

- H. Liu, Wenli Cai and Ning Su, SIAM Journal on Numerical Analysis, 53 (2015), 1393-1417.
- Wenli Cai, Pierre-Emmanuel Jabin, and H. Liu, Math. Models Methods Appl. Sci. 25(2015), 1589–.
- H. Liu, Wenli Cai, Mathematical Modeling and Numerical Analysis (M2AN), 2016.
- Pierre-Emmanuel Jabin, and H. Liu, Nonlinearity, 2017.

Background: population adaptive evolution

Darwin (1809-1882) 'On the origin of species' (1859)

Motivation. Analyze self-contained mathematical models for Darwins mechanism at the population scale using only the

Ingredients.

- Population multiplication with heredity
- Natural selection:
  - individuals own a phenotypical trait: ability to use the environment.
  - Because of competition, the individuals that are the most preforment are selected.
- Mutations can modify the trait from parents to off-springs.

A direct selection model

We consider a structured population model

$$\partial_t f(t,x) = f(t,x)R, \quad t > 0, x \in X.$$

- Population structured by a continuous trait variable $x \in X$
- Reproduction (or fitness) R includes both growth a and competition ($b > 0$):

$$R = a(x) - \int_X b(x,y) f(t,y) dy.$$

- The competition $b > 0$ means that the individual with trait y only has a negative effect on the one with trait x, therefore leading to selection!

$$f \to \sum_{j=1}^{n} \rho_j \delta(x - x_j)?$$

- see Desvillettes, Gyllenberg, Jabin, Mischler, Perthame, Raoul, ...

Selection or no selection

As an example, we consider

$$a(x) = G(x, \sigma_1), \quad b(x, y) = G(x - y, \sigma_2),$$

where

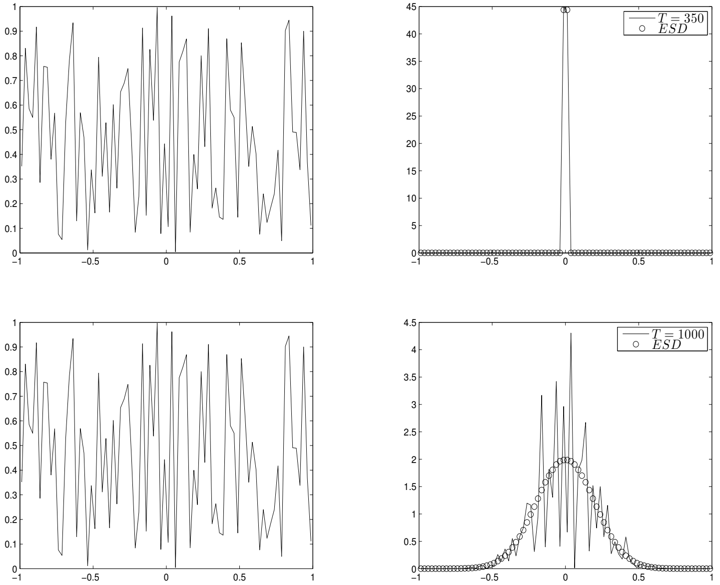
$$G(x, \sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}.$$

- For $\sigma_1 < \sigma_2$, the Dirac mass is a stable steady state.
- One can verify that for $\sigma_1 > \sigma_2$ there is a smooth steady state which is given by

$$f_{eq} = G(x, \sigma), \quad \sigma = \sigma_1 - \sigma_2.$$

Selection or no selection

The first row $\sigma_1 = 0.01 < \sigma_2 = 0.05$; the second row: $\sigma_1 = 0.05 > \sigma_2 = 0.01$.

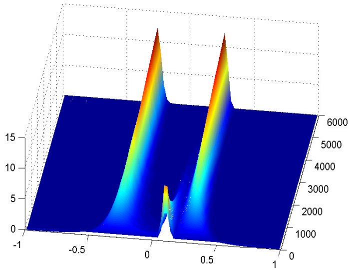


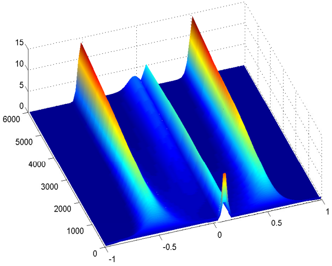
Branching

We test initial data of delta-like function with

$$a(x) = A - x^2, \quad b(x, y) = \frac{1}{1 + (x - y)^2}.$$

(1) branching into two subspecies for $A = 1.5$.
(2) $A = 2.5$, branching into two subspecies and then a new trait appears in the middle.





Model description

$$\partial_t f(t,x) = f(t,x)R, \quad t > 0,\ x \in X.$$

- Wellposedness in $C([0,\infty); L^1(X))$ is known for $f_0 \in L^1(X)$, provided

$$a \in L^\infty(X), \quad |\{x; \quad a(x) > 0\}| \neq 0;$$
$$b \in L^\infty(X \times X), \quad \inf_{x,x' \in X} b(x,x') > 0.$$

  Desvillettes L, Jabin PE, Mischler S, Raoul G (2008)

- The model is interesting from the point of view of large-time behavior.
  Natural questions appear, such as

– does the population really converge to an equilibrium?

– Is this equilibrium an evolutionarily stable strategy or distribution (ESS or ESD)?

– Does this limit depend on the initial population distribution?

Evolutionary Stable Distribution (ESD)

- Solutions are expected to converge toward the stationary states ...

$$\left\{\tilde{f}(x) \middle| \quad \tilde{f}(x)\left(a(x) - \int_X b(x,y)\tilde{f}(y)dy\right) = 0\right\}$$

- However, there are many stationary states!

A special class of stationary states features a particular sign property characterized by the ESD:

$$\forall x \in \text{supp}\tilde{f}, \ \ R = 0,$$
$$\forall x \in X, \ \ R \leq 0.$$

Jabin and Raoul (JMB 2011)

- Existence of ESD is known only for some *a* and *b* (Raoul 2009)
- In general case, the ESD is not necessarily unique!

Model parameters

The basic assumptions for some existing results:

$$(i) a \in L^\infty(X), \quad |\{x; \quad a(x) > 0\}| \neq 0,$$
$$(ii) b \in L^\infty(X \times X), \quad \inf_{x, x' \in X} b(x, x') > 0.$$

The uniqueness of the ESD is ensured if

$$\forall g \in L^1(X) \backslash \{0\}, \quad \int \int b(x, y) g(x) g(y) dxdy > 0.$$

Convergence to ESD (when time becomes large) toward a singular ESD is rather complex.

Partial results under additional symmetry assumption on b, say

$$b(x, y) = b(y, x).$$

Jabin and Raoul (JMB2011)

Relative entropy

The proof of global convergence to the ESD relies on a Lyapunov functional of the form

$$F(t) = \int_X \left[\tilde{f}(x) \log \frac{\tilde{f}(x)}{f(t,x)} + f(t,x) - \tilde{f}(x) \right] dx,$$

which is dissipating in time and serves as a relative entropy.

The obtained convergence rate (with no selection) is

$$\| f(t,\cdot) - \tilde{f}(\cdot) \|_b = O\left(\frac{\ln t}{t} \right),$$

where

$$\|g\|_b = \left(\int \int b(x,y) g(x) g(y) dx dy \right)^{1/2}.$$

Semi-discrete scheme

Let $f_j(t)$ denote the approximation of cell averages

$$f_j(t) \sim \frac{1}{h} \int_{I_j} f(t,x)\, dx,$$

then we have the following semi-discrete scheme

$$\frac{d}{dt} f_j = f_j \left(\bar{a}_j - h \sum_{i=1}^{N} \bar{b}_{ji} f_i \right), \quad j = 1, \cdots, N, \tag{1}$$

where

$$\bar{a}_j = \frac{1}{h} \int_{I_j} a(x) dx, \quad \bar{b}_{ji} = \frac{1}{h^2} \int_{I_i} \int_{I_j} b(x,y) dxdy.$$

The basic assumptions can be carried over to the discrete level:

$$|\bar{a}_j| \le \|a\|_{L^\infty}, \quad \{1 \le j \le N, \bar{a}_j > 0\} \ne \emptyset;$$
$$0 \le \bar{b}_{ji} \le \|b\|_{L^\infty} \text{ and } \bar{b}_{ji} = \bar{b}_{ij}, \text{ for } 1 \le i,j \le N;$$
$$\sum_{j=1}^{N} \sum_{i=1}^{N} \bar{b}_{ji} g_i g_j > 0 \text{ for any } g_j \text{ such that } \sum_{j=1}^{N} |g_j|^2 \ne 0.$$

Discrete ESD

- (Discrete ESD) For initial data $f_j(0) > 0$ for all $j = 1, 2, \cdots, N$, the corresponding discrete ESD $\tilde{f} = \{\tilde{f}_j\}$ (still called ESD) may be defined as

$$\forall j \in \{1 \le i \le N, \tilde{f}_i \neq 0\}, \quad R_j[\tilde{f}] := \bar{a}_j - h \sum_{i=1}^{N} \bar{b}_{ji} \tilde{f}_i = 0,$$

$$\forall j \in \{1 \le i \le N, \tilde{f}_i = 0\}, \quad \bar{a}_j - h \sum_{i=1}^{N} \bar{b}_{ji} \tilde{f}_i \le 0.$$

  This ESD is shown to be unique!
- Questions:

- Can we come up with an independent solver to produce the discrete ESD?
- Does the numerical scheme preserve: positivity and the relative entropy dissipation law?
- Does the numerical solution converge toward the discrete ESD?
- What are the time-asymptotic convergence rates?

How to generate ESD?

We prove that finding the ESD is equivalent to solving the following problem

$$\min_{f \in \mathbb{R}^N} H, \tag{2a}$$

$$\text{subject to} \quad f \in S = \{f \geq 0\}, \tag{2b}$$

where

$$H(f) = \frac{f^{\mathrm{T}} B f}{2} - a^{\mathrm{T}} f,$$

with $f = (f_1, f_2, \cdots, f_N)^{\mathrm{T}}$, $B = (\bar{b}_{ij})$, and $a = (\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_N)^{\mathrm{T}}/h$.

- B is positive definite, symmetric, hence problem (2) has a unique solution.
- A good quadratic programing algorithm can be used to produce the ESD!

Proven properties of the semi-discrete scheme

We define the discrete entropy functional as follows

$$F = \sum_{j=1}^{N} \left(\tilde{f}_j \log \left(\frac{\tilde{f}_j}{f_j} \right) + f_j - \tilde{f}_j \right) h.$$

Theorem

Let $f_j(t)$ be the numerical solution to the semi-discrete scheme. Then
(i) If $f_j(0) > 0$ for every $1 \leq j \leq N$, then $f_j(t) > 0$ for any $t > 0$;
(ii) F is non-increasing in time. Moreover,

$$\frac{dF}{dt} \leq -h^2 \sum_{j=1}^{N} \sum_{i=1}^{N} \bar{b}_{ji} \left(f_i - \tilde{f}_i \right) \left(f_j - \tilde{f}_j \right) \leq 0.$$

Positivity and entropy satisfying property

$$\frac{f_j^{n+1} - f_j^n}{\Delta t} = f_j^{n+1}(\bar{a}_j - h\sum_{i=1}^{N} \bar{b}_{ji} f_i^n) \qquad (3)$$

Theorem

Assume $F^0 < \infty$, and let f_j^n be the numerical solution to the fully-discrete scheme (3) with time step satisfying

$$\Delta t \le \frac{\lambda_{\min}}{4\lambda_{\max}\left[\|a\|_{L^\infty} + \|b\|_{L^\infty}\|\tilde{f}\|_1 + \lambda_{\max} S(F^0)\right]},$$

where S is a monotone function. Then,
(i) $f_j^{n+1} = 0$ for $f_j^n = 0$, and $f_j^{n+1} > 0$ for $f_j^n > 0$ for any $n \in \mathbb{N}$;
(ii) F^n is a decreasing sequence in n. Moreover,

$$F^{n+1} - F^n \le -\frac{1}{2}\Delta t\|f^n - \tilde{f}\|_b^2.$$

Note: $F^n = \sum_{j=1}^{N}\left(\tilde{f}_j \log\left(\frac{\tilde{f}_j}{f_j^n}\right) + f_j^n - \tilde{f}_j\right) h$. $\lambda_{\min}(\lambda_{\max})$ denotes the smallest (largest) eigenvalue of $B = (\bar{b}_{ji})_{N\times N}$.

Convergence rates

- A strict ESD: if it also satisfies the following strict sign condition,

$$R_j[\tilde{f}] < 0 \quad \text{for } j \in \{i : \ \tilde{f}_i = 0\}.$$

- The strict ESD is both linearly and non linearly stable, with perturbations decaying to zero exponentially in time.
- In order to quantify the exponential decay of the perturbations, we use the following notation,

$$I = \{j \mid \tilde{f}_j = 0 \text{ and } R_j < 0\}, \quad I^c = \{j, 1 \leq j \leq N\} - I,$$

and

$$s = \min_{j \in I}(-R_j[\tilde{f}]) > 0, \quad f_m = \min_{j \in I^c} \tilde{f}_j > 0.$$

$$\mu = hf_m\lambda_{\min}, \quad r = \min\{s, \mu\}$$

Convergence rates

Theorem

Let $f_j(t)$ be the solution to the semi-discrete scheme, associated with the strict ESD, then there exists $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$ if

$$\|f(0) - \tilde{f}\|_2 \leq \delta,$$

then

$$\|f(t) - \tilde{f}\|_p \leq C(1+t)^{\xi} e^{-rt}, \quad \xi = 1_{\{s=\mu\}},$$

where $1 \leq p \leq 2$,

$$\delta^* = \frac{\alpha^2 \min\{1, \sqrt{f_m}\}}{\sqrt{2}\max\{1, \alpha\}}, \quad \alpha = \sqrt{\frac{r}{\|b\|_{L^\infty}} + \frac{\|\tilde{f}\|_1}{2}} - \sqrt{\frac{\|\tilde{f}\|_1}{2}},$$

and C may depend on the parameters and the norms of the initial data but not explicitly on N or h.

Convergence rates

Another objective is to establish an algebraic convergence rate but with parameters uniform in the mesh size, thus extending the rates known at the continuous limit.

Theorem

Let f_j^n be the numerical solution generated from fully discrete scheme with positive initial data $f_j^0 > 0$ for all $j = 1, \cdots, N$, with $\tilde{f} = \{\tilde{f}_j\}$ as its associated ESD. If

$$F^0 := \sum_{j=1}^{N} \left(\tilde{f}_j \log \left(\frac{\tilde{f}_j}{f_j^0} \right) + f_j^0 - \tilde{f}_j \right) h < +\infty,$$

then

$$\|f^n - \tilde{f}\|_b^2 \leq \frac{2F^0}{n\Delta t},$$

provided that Δt is suitably small.

Conclusion I

- Rich dynamic behavior in discrete models.
- Convergence rates:
  - For the strict discrete ESD, we establish the exponential convergence rate of numerical solutions towards such a strict ESD. However, the convergence rate is typically mesh dependent, as a similar result is not expected for the continuous model.
  - For general discrete ESD, we prove that numerical solutions of the fully discrete scheme converge towards the discrete ESD at a rate $1/n$, which is faster than the rate $O(logt/t)$ obtained for the continuous model
- Open questions:
  - Characterize (a, b) that generate Dirac concentrations
  - How to connect operator positivity $\int b(x, y)n(x)n(y)dxdy \geq 0$ to scaling limits.

Models with mutation

Off-springs undergo mutations that change slightly the trait. Two models are

$$\partial_t f(t,x) = f(t,x)R + \Delta f.$$

$$\partial_t f(t,x) = f(t,x)R + \mu \left(\int_X f(t,y)M(x,y)dy - f(t,x) \right).$$

Depending on the scales of mutations, both models can de derived from

- Stochastic models, Individual Based Models
  - N individuals,
  - rescale mutation, birth, death rates
  - U. Dieckmann- R. Law, R. Ferriere
  - N. Champagnat, S. Meleard

A special case

When $b \equiv 1$, the competition is uniform with same strength. The model becomes

$$\partial_t f(t,x) = f(t,x) R(x, \rho(t)) + \Delta f(t,x),$$
$$R = a(x) - \rho(t), \quad \rho = \int f(t,x) dx.$$

This special model was well studied.

Theorem (B. Perthame, et al) Let f be the solution of

$$\partial_t f(t,x) = f(t,x) R(x, \rho(t))$$

Suppose $X = \mathbb{R}$, $R_\rho < 0$ and $R(x, \rho_{max}) < 0, \forall x$. Then,

$$\rho(t) \to \rho_\infty, \quad \text{as } t \to \infty,$$
$$\lim_{t \to \infty} f(t,x) \to \rho_\infty \delta(x = x_\infty), \text{ (Competitive Exclusion Principle)}$$

and $\min_\rho \max_x R(x, \rho) = 0 = R(x_\infty, \rho_\infty)$ (pessimism principle)

However, when $b \neq const$, the problem is much more challenging!

Asymptotic approach

We assume that mutations are RARE and introduce a scale ϵ for small mutations, so that

$$\epsilon \partial_t f(t,x) = f(t,x) R(x, \rho(t)) + \epsilon^2 \Delta f(t,x).$$

Theorem (B. Perthame, et al) Suppose $X = \mathbb{R}$, $R_\rho < 0$. Then, as $\epsilon \to 0$, we have

$$f(t,x) \to \bar{\rho}(t)\delta(x = \bar{x}(t)), \quad \rho \to \bar{\rho} = \int_X f(t,x)dx,$$

and the 'fittest' trait $\bar{x}(t)$ is characterised by the Eikonal equation with constraints

$$\partial_t \phi(t,x) = R(x, \bar{\rho}) + |\nabla_x \phi(t,x)|^2$$
$$\max_x \phi(t,x) = 0 = \phi(t, \bar{x}(t)).$$

- This is not far from Fisher/KPP equation for invasion fronts/chemical reaction:

$$\epsilon \partial_t f(t,x) = f(t,x)(1 - f(t,x)) + \epsilon^2 \Delta f(t,x).$$

- Tools: WKB approach, level set, geometric motion.

A new model

There are also other models featuring balance between evolutionary forces.

- We are concerned with the problem governed by

$$\partial_t f(t,x) = \Delta f(t,x) + \frac{1}{2} f(t,x) \left(a(x) - \int_X b(x,y) f^2(t,y) dy \right), \text{ for } t > 0,\ x \in X, \tag{4a}$$

$$f(0,x) = f_0(x) \geq 0, \quad x \in X, \tag{4b}$$

$$\frac{\partial f}{\partial \nu} = 0, \quad x \in \partial X, \tag{4c}$$

  where $f(t,x)$ denotes the density of individuals with trait x, X is a subdomain of $\mathbb{R}^d$, ν is the unit outward normal at a point x on the boundary ∂X.

- The nonlinear competition effect does appear in the model for fish species:

$$\partial_t f(t,x) = \frac{1}{2} f(t,x) \left(a(x) - \int_X b(x,y)(f(t,y) - d(x,y))^2 dy \right).$$

  K. Shirakihara, S. Tanaka (1978)

Gradient flow structure

- The model can be expressed as

$$\partial_t f = -\frac{1}{2}\frac{\delta F}{\delta f}$$

where the corresponding energy functional is

$$F[f] = \frac{1}{4}\int\int b(x,y)f^2(t,x)f^2(t,y)dxdy - \frac{1}{2}\int a(x)f^2(t,x)dx + \int |\nabla_x f(t,x)|^2 dx$$

so that the energy dissipation law $\frac{d}{dt}F[f] = -2\int |\partial_t f|^2 dx \leq 0$ holds for all $t > 0$, at least for classical solutions.

- Under the transformation $u = f^2$, the resulting equation becomes

$$\partial_t u(t,x) = \Delta u - \frac{|\nabla u|^2}{2u} + u(t,x)\left(a(x) - \int_X b(x,y)u(t,y)dy\right).$$

Issues and questions

- Numerical approximation to capture the time-dynamics (w/ Wenli Cai, 2016)
- Theory for the continuous model (w/ P.E. Jabin)
- – Well-posedness in $C([0, \infty); L^2(X))$ can be established for $f_0 \in L^2(X)$.
- – Other questions

a does the population converge to a nontrivial equilibrium?

b Is this equilibrium globally stable?

c Does this limit depend on the initial population distribution?

Basic assumptions

In order to analyze the solution behavior at large times, we make the following assumptions:

$$a \in L^\infty(X), \quad |\{x; \quad a(x) > 0\}| \neq 0; \tag{5a}$$

$$b \in L^\infty(X \times X), \quad b_m = \inf_{x,x' \in X} b(x, x') > 0. \tag{5b}$$

$$b(x, y) = b(y, x), \forall g \in L^1(X)\backslash\{0\}, \quad \int\int b(x, y)g(x)g(y)dxdy > 0. \tag{5c}$$

One can check that b defines then a scalar product over $L^1(X)$,

$$\langle g, h \rangle_b = \int\int b(x, y)g(x)h(y)dxdy$$

with corresponding norm

$$\|g\|_b = \left(\int\int b(x, y)g(x)g(y)dxdy\right)^{1/2}.$$

In what follows we also use the notation

$$H[h] = \frac{1}{2}h\left(a - \int b(x, y)h^2(y)dy\right).$$

Well-posedness

Existence and uniqueness of the solution can be obtained without much effort.

Theorem

Let $f_0 \in L^2(X)$, and both a and b satisfy the first two assumptions of (5). Then (4) admits a global weak solution

$$f \in L^\infty(\mathbb{R}^+; L^2(X)).$$

Moreover, we have
(a) $\|f\| := \sup_{t>0} \|f(t,\cdot)\|_{L^2(X)} \leq M, \quad (t,x) \in \mathbb{R}^+ \times X.$
(b) f is stable and depends continuously on f_0 in the following sense: if $\tilde{f}$ is another solution with initial data $\tilde{f}_0$, then for every $t > 0$,

$$\int |f - \tilde{f}|^2 dx \leq e^{\lambda t} \int |f_0 - \tilde{f}_0|^2 dx,$$

where λ depends only on a, b and $\|f_0\|$.

The proof of this result is classical: (i) the a priori estimate of $\|f\|$; (ii) fixed point argument in a ball within $C([0,T], L^2(X))$; (iii) extension to all time.

Steady solutions

The steady problem:

$$\Delta g + H[g] = 0, \quad x \in X \quad \partial_\nu g = 0, \quad \text{on } \partial X. \tag{6}$$

Theorem

There exists $g \geq 0$ solution in the sense of distribution to (6). Moreover,
(i) If $\int adx \geq 0$ or $\int adx < 0$ with $\lambda_1 < 1/2$, then there exists a unique positive solution such that $0 < g_{\min} \leq g \leq g_{\max} < \infty$ in X.
(ii) If $\int adx < 0$ with $\lambda_1 \geq 1/2$, there is no positive steady solution.

Remarks: If $\int adx \geq 0$, the steady state is strictly positive. The case $\int adx < 0$ is less obvious. Brown and Lin (1980) proved that there exists a unique positive λ_1 and the positive function $\psi \in D(L_1)$ such that $\int a\psi^2 dx > 0$ and

$$\lambda_1 = \frac{\int |\nabla_x \psi|^2 dx}{\int a\psi^2 dx} = \inf \left\{ \frac{\int |\nabla_x v|^2 dx}{\int av^2 dx} : v \in D(L_1) \text{ and } \int av^2 dx > 0 \right\}, \tag{7}$$

where $D(L_1) = \{u \in H^2(X) : \partial_n u|_{\partial X} = 0\}$ is the domain of the Laplace operator $L_1 u = -\Delta u$.

Steps in the proof

- Existence of the weak solution by a variational construction: The weak solution in distributional sense is shown to be equivalent to the nonzero critical point of the functional

$$F[w] = \int \left[\frac{1}{4}(b * w^2)w^2 - \frac{1}{2}aw_+^2 + |\nabla_x w|^2 \right] dx, \quad w_+ = \max(w, 0).$$

  There exists $g \in A := \{g \in H^1(X),\ g \geq 0\}$, such that

$$F(g) = \inf_{w \in H^1(X)} F[w].$$

  (i) If $\int adx \geq 0$ or $\int adx < 0$ with $\lambda_1 < 1/2$, then g is not identically 0;
  (ii) If $\int adx < 0$ with $\lambda_1 \geq 1/2$, $g \equiv 0$.

- Regularity and positivity: elliptic theory and the standard Harnack inequality.
- Uniqueness is more interesting

Proof of uniqueness

Let g_1 and g_2 be two positive solutions of (6), then using the positivity of b, third assumption in (5)

$$
\begin{aligned}
0 &\leq \int\int (g_1^2 - g_2^2)(x) b(x,y)(g_1^2 - g_2^2)(y) dy dx \\
&= \int (g_1 - g_2^2/g_1) g_1(x) \int b(x,y) g_1^2(y) dy dx - \int (g_1^2/g_2 - g_2) g_2(x) \int b(x,y) g_2^2(y) dy dx \\
&= \int (g_1 - g_2^2/g_1)(2\Delta g_1(x) + a(x) g_1(x)) dx + \int (g_2 - g_1^2/g_2)(2\Delta g_2(x) + a(x) g_2(x)) dx \\
&= 2\int (g_1 - g_2^2/g_1)\, \Delta g_1(x) + 2 \int (g_2 - g_1^2/g_2)\, \Delta g_2(x),
\end{aligned}
$$

by using the equation (6). Hence by integrating by part

$$
\begin{aligned}
0 \leq -2 &\int \left(\nabla_x g_1 - \frac{2 g_1 g_2 \nabla_x g_2 - g_2^2 \nabla_x g_1}{g_1^2} \right) \cdot \nabla_x g_1 dx \\
- 2 &\int \left(\nabla_x g_2 - \frac{2 g_1 g_2 \nabla_x g_1 - g_1^2 \nabla_x g_2}{g_2^2} \right) \cdot \nabla_x g_2 dx \\
= -2 &\int \left(\left| \nabla_x g_1 - \frac{g_1}{g_2} \nabla_x g_2 \right|^2 + \left| \nabla_x g_2 - \frac{g_2}{g_1} \nabla_x g_1 \right|^2 \right) dx \leq 0.
\end{aligned}
$$

As a conclusion $g_1^2 = g_2^2$, leading to $g_1 = g_2$.

Main result

We can show the convergence of $f(t,\cdot)$ towards g:

Theorem

Assume both a and b satisfy (5). Consider any non-negative $f^0 \in L^1(X) \cap L^\infty(X)$*. Then the corresponding solution* $f(t,\cdot)$ *of (4) is such that*

$$\frac{d}{dt} F[f(t,\cdot)] < 0 \ \text{ as long as } f \text{ is not a steady solution.} \tag{8}$$

As a consequence

$$\lim_{t\to\infty} \|f(t,\cdot) - g(\cdot)\|_{L^2(X)} = 0. \tag{9}$$

And moreover, there exists C depending on initial data f_0 *and* $g \geq 0$ *such that*

$$\int |f(t,x) - g(x)|^2 dx \leq Ce^{-rt} \quad \forall t > 0,$$

for $\int a dx \geq 0$ *or* $\int a dx < 0$ *with* $\lambda_1 \neq \frac{1}{2}$*, where of course* $g = 0$ *if* $\lambda_1 > 1/2$*.*
For $\int a dx < 0$ *and* $\lambda_1 = \frac{1}{2}$*,*

$$\int |f(t,x)|^2 dx \leq \frac{C}{1+t} \quad \forall t > 0.$$

Proof of convergence

- Since F is non-increasing, it only remains to show that for some $t_0 \geq 0$, if $\partial_t f(t_0, x) \equiv 0$ for all $x \in X$, then $\partial_t f(t, x) \equiv 0$ for all $x \in X$ and $t \geq 0$.
- By uniqueness we have $f(t, x) = f(t_0, x)$ for all $t > t_0$.
- For $0 \leq t \leq t_0$, we prove by a contradiction argument based on a key quantity

$$\Lambda(t) = \frac{\int_X |\nabla_x w|^2 dx}{\int |w|^2 dx}$$

with $w = f(t, x) - f(t_0, x)$. Key estimates are

– On one hand

$$\frac{d}{dt}\Lambda(t) \leq \frac{1}{2}\lambda^2, \quad \lambda := \frac{1}{2}(\|a\|_\infty + 3\|b\|_\infty M^2).$$

– On the other hand,

$$\begin{aligned} \frac{d}{dt}\left(\log \frac{1}{\int w^2 dx}\right) &= -\frac{2}{\int w^2 dx}\int w\partial_t w dx \\ &\leq 2\Lambda(t) + 2\lambda. \end{aligned}$$

Exponential convergence

In the case $g > 0$, we introduce the auxiliary functional

$$G = \int \left[\frac{f^2 - g^2}{2} - g^2 \log\left(\frac{f}{g}\right) \right] dx,$$

which is bounded from below

$$G \geq \int \left[\frac{f^2 - g^2}{2} - g^2 \left(\frac{f}{g} - 1\right) \right] dx = \frac{1}{2} \int (f-g)^2 dx.$$

A direct calculation gives

$$\frac{d}{dt} G \leq -D(f,g),$$

where

$$D(f,g) = \int g^2 \left| \nabla_x \left(\frac{f}{g}\right) \right|^2 dx + \frac{1}{2} \int\int (f^2 - g^2)(x) b(x,y) (f^2 - g^2)(y) dy dx.$$

The key is to show that that there exists $\mu > 0$ such that

$$D(f,g) \geq \mu \|f/g - 1\|_{L^2}^2. \tag{10}$$

which gives

$$\frac{d}{dt} G \leq -\mu \int \left(\frac{f}{g} - 1\right)^2 dx \leq -\frac{2\mu}{g_{max}^2} G.$$

By Gronwall lemma

$$\|f(t,\cdot) - g(\cdot)\|_{L^2} \leq \sqrt{2G(t)} \leq \sqrt{2G(0)} \left(-\frac{\mu}{g_{max}^2} t\right).$$

A new functional inequality

Due to the Poincare inequality it suffices to find μ independent of $c \geq 0$ such that

$$I := C_X g_{\min}^2 \int \left| \frac{f}{g} - c \right|^2 dx + \frac{1}{2} \|f^2 - g^2\|_b^2 \geq \mu \| \frac{f}{g} - 1 \|_{L^2}^2 .$$

- find ϵ so that

$$I \geq \frac{1}{2} C_X\, g_{\min}^2 \, \|f/g - c\|^2 + \epsilon \, (c^2 - 1)^2 \, \|g\|_b^2 .$$

- For any $\eta > 0$

$$\int \left| \frac{f}{g} - c \right|^2 dx \geq \eta \int \left| \frac{f}{g} - 1 \right|^2 dx - \frac{\eta |X|}{1 - \eta} \, |c - 1|^2 .$$

- Together

$$I \geq \frac{\eta}{2} C_X \; g_{\min}^2 \int \left| \frac{f}{g} - 1 \right|^2 dx + |c - 1|^2 \left(\epsilon \, (c + 1)^2 \, \|g\|_b^2 - \frac{\eta |X|}{1 - \eta} \right) .$$

Conclusion II

- The self-contained population models with three simple ingredients:
  - growth and death: trait dependent
  - limited resources: selection through competition
  - mutations

  is able to express selection and branching.
- Open questions
  - Does the entropy method hold in the case with mutation?
  - For the new model, how to characterize *a* more explicitly that generate positive concentrations
  - Whether similar results hold true for corresponding discrete models.

THANK YOU ALL