Swarming models with local alignment effects: phase transitions & hydrodynamics

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Outline

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Individual Based Models (Particle models)

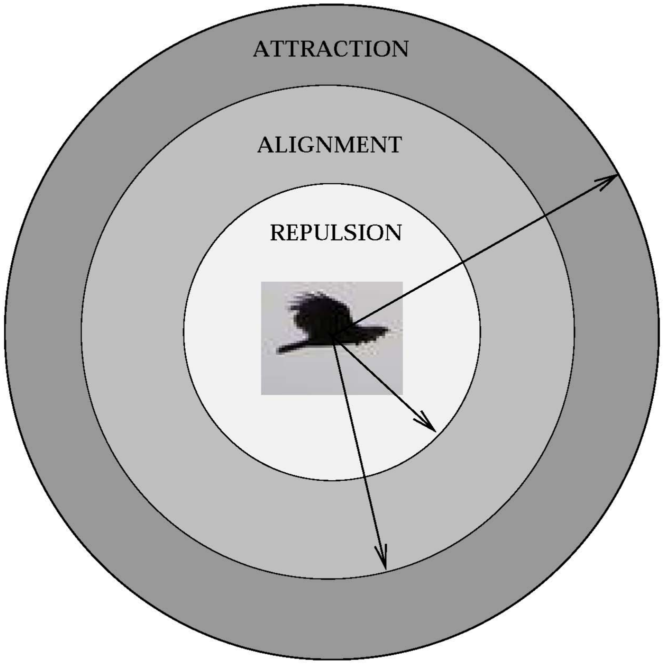
Swarming = Aggregation of agents of similar size and body type generally moving in a coordinated way.

Highly developed s ocial organization: insects (locusts, ants, bees ...), fish, birds, micro-organisms (myxo-bacteria, ...) and artificial robots for unmanned vehicle operation.

Interaction regions between individuals[a]

[a] Aoki, Helmerijk et al., Barbaro, Birnir et al.

- Repulsion Region: R_k.
- Attraction Region: A_k.
- Orientation Region: O_k.



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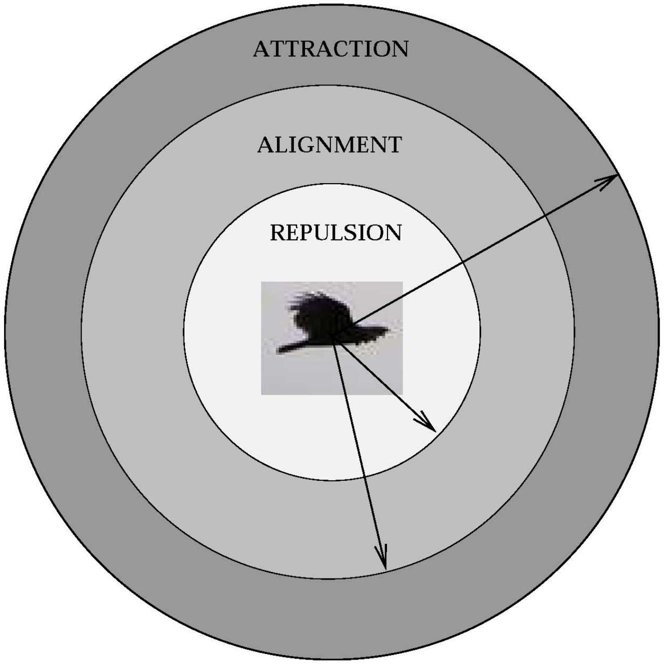

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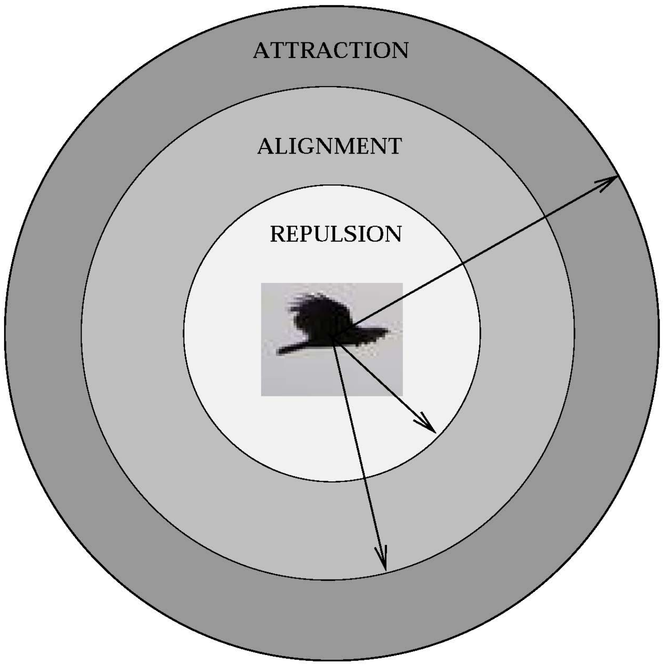

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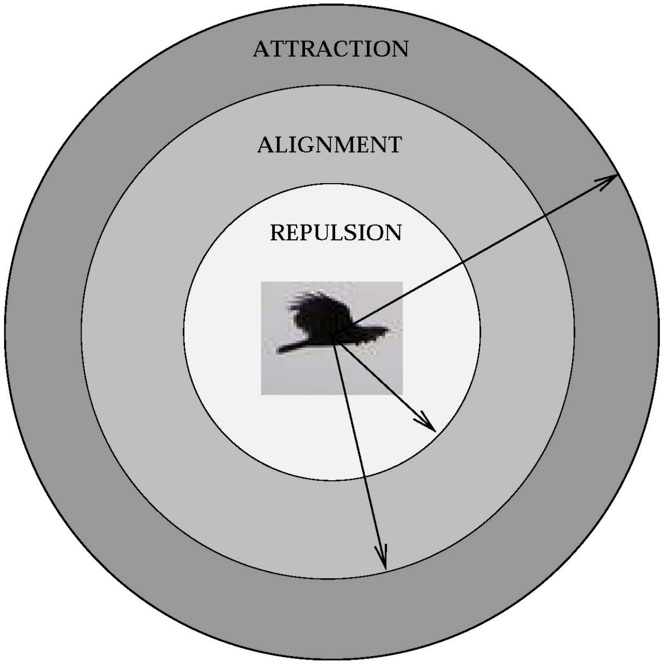
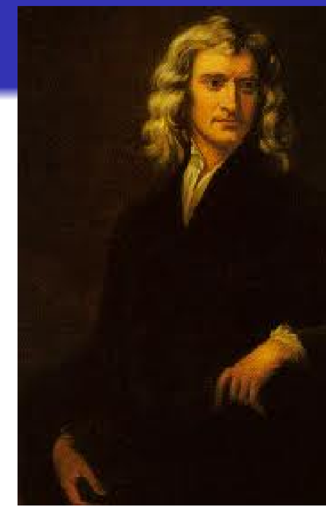


2nd Order Model: Newton's like equations



D'Orsogna, Bertozzi et al. model (PRL 2006):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ m\dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \sum_{j\neq i} \nabla U(|x_i - x_j|). \end{cases}$$

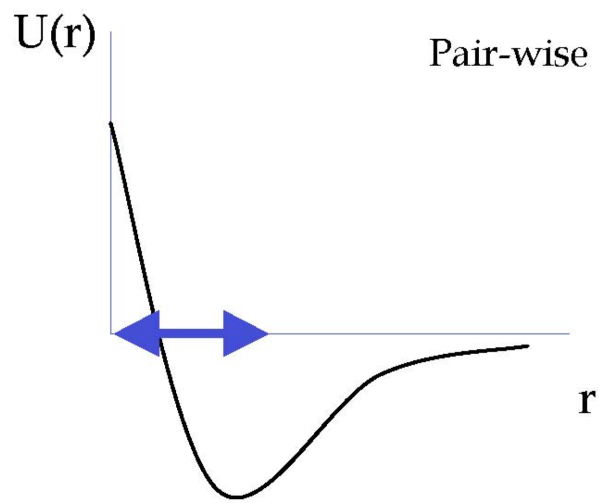
Model assumptions:

- Self-propulsion and friction terms determines an asymptotic speed of $\sqrt{\alpha/\beta}$.
- Attraction/Repulsion modeled by an effective pairwise potential $U(x)$.

$$U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}.$$

One can also use Bessel functions in 2D and 3D to produce such a potential.

$C = C_R/C_A > 1$, $\ell = \ell_R/\ell_A < 1$ and $C\ell^2 < 1$:



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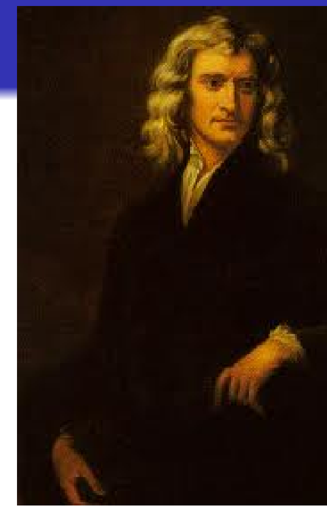

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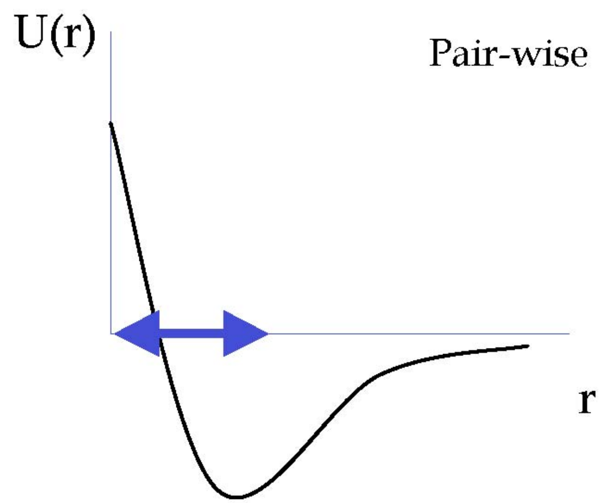

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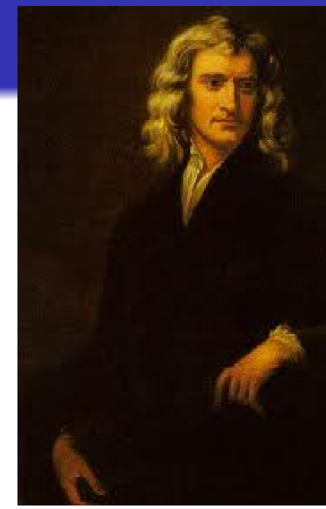

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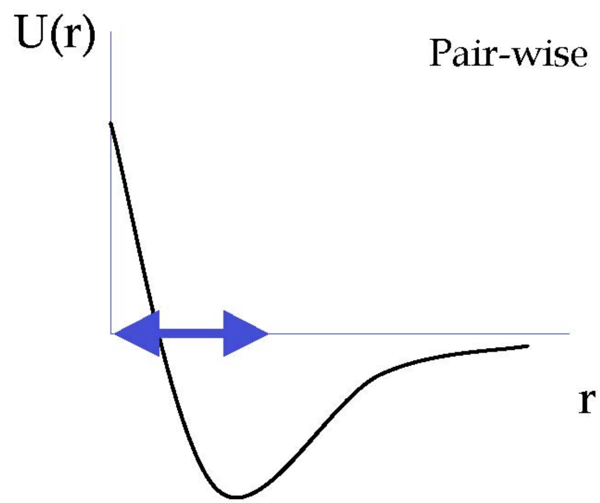

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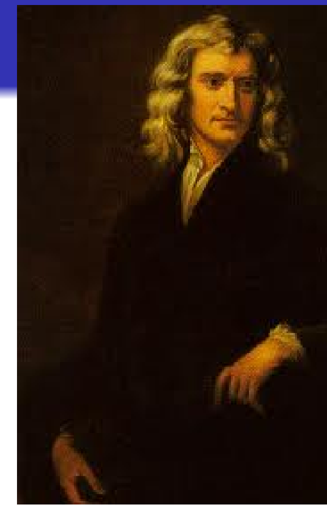

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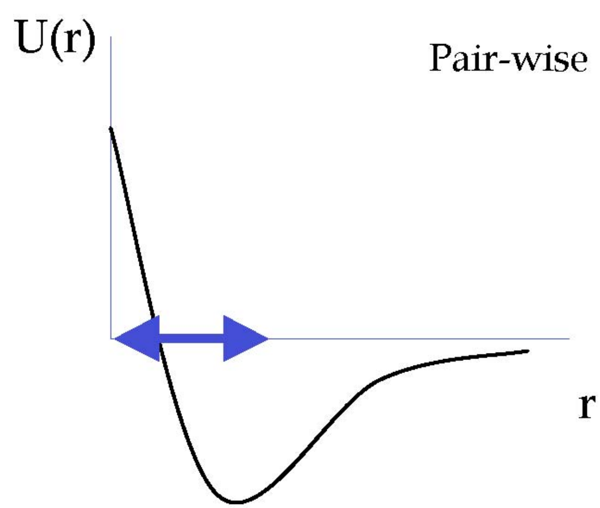
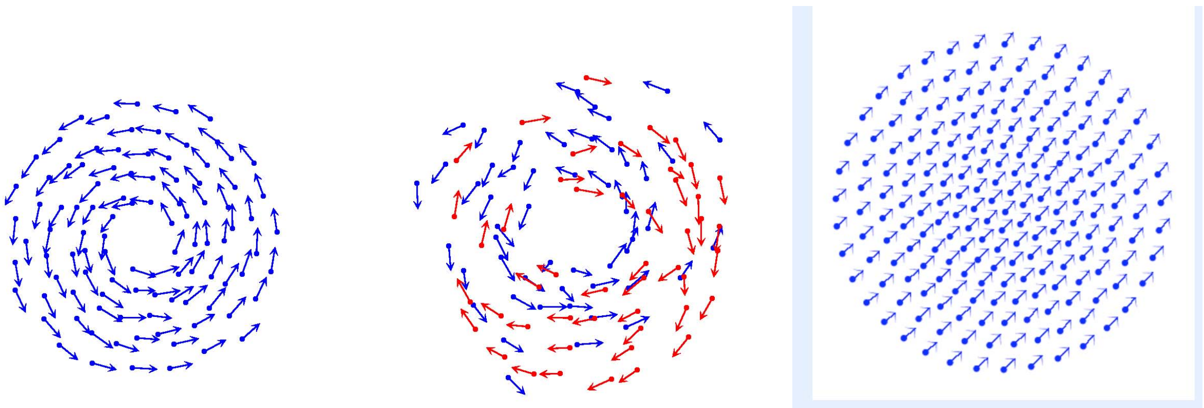


Model with an asymptotic speed

Typical patterns: milling, double milling or flocking:



Velocity consensus model

Cucker-Smale Model (IEEE Automatic Control 2007):

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j=1}^{N} a_{ij}\left(v_j - v_i\right), \end{cases}$$

with the communication rate, $\gamma \geq 0$:

$$a_{ij} = a(|x_i - x_j|) = \frac{1}{(1 + |x_i - x_j|^2)^{\gamma}}.$$

Asymptotic flocking: $\gamma < 1/2$; Cucker-Smale.
General Proof for $\gamma \leq 1/2$; C.-Fornasier-Rosado-Toscani.

Global Stability for the full model: Albi-Balague-C.-VonBrecht (SIAM J. Appl. Math. 2014), C.-Huang-Martin (Nonlinear Analysis: Real World Applications 2014).

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Leadership, Geometrical Constraints, and Cone of Influence

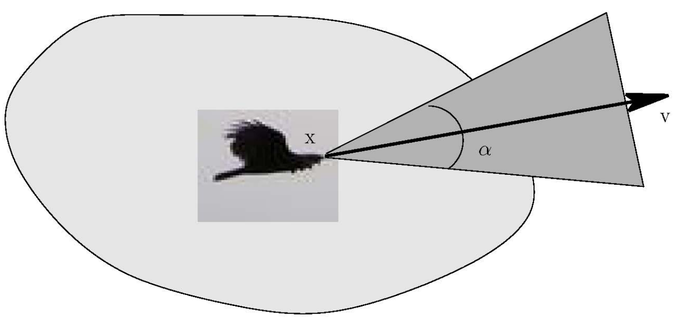
Cucker-Smale with local influence regions:

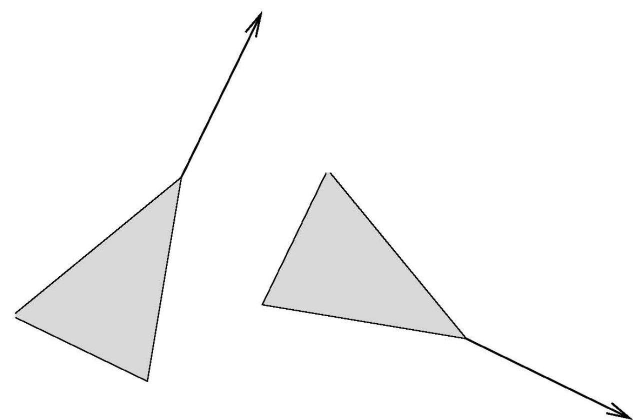
$$\begin{cases} \dfrac{dx_i}{dt} = v_i \ , \\ \dfrac{dv_i}{dt} = \displaystyle\sum_{j \in \Sigma_i(t)} a(|x_i - x_j|)(v_j - v_i) \ , \end{cases}$$

where $\Sigma_i(t) \subset \{1, \dots, N\}$ is the set of dependence, given by

$$\Sigma_i(t) := \left\{ 1 \le \ell \le N : \frac{(x_\ell - x_i) \cdot v_i}{|x_\ell - x_i||v_i|} \ge \alpha \right\}.$$

Cone of Vision:





Rigorous Mean-Field Limit: C.-Choi-Hauray-Salem, to appear in JEMS.

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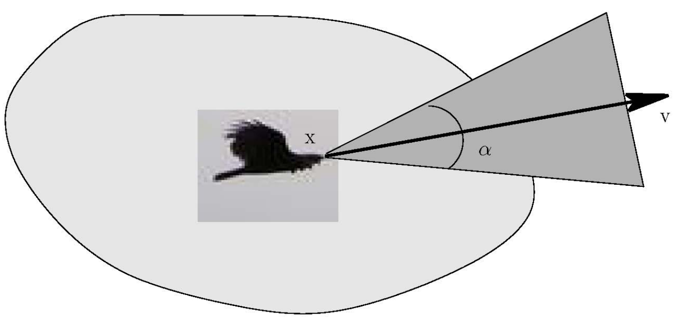

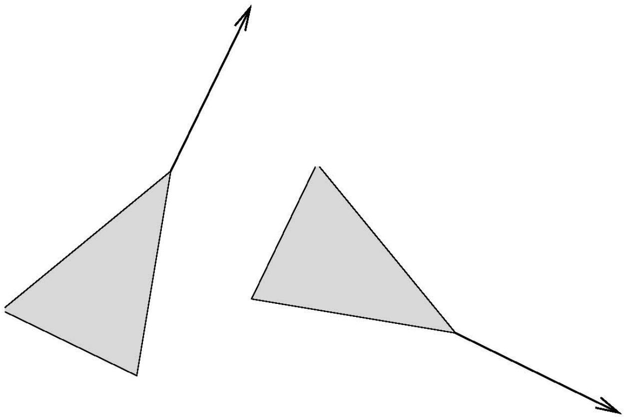

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Adding Noise

Self-Propelling/Friction/Interaction with Noise Particle Model:

$$\begin{cases} \dot{x}_i = v_i, \\ dv_i = \left[(\alpha - \beta\, |v_i|^2) v_i - \nabla_{x_i} \sum_{j \neq i} U(|x_i - x_j|) \right] dt + \sqrt{2D}\, d\Gamma_i(t) \ , \end{cases}$$

where $\Gamma_i(t)$ are N independent copies of standard Wiener processes with values in $\mathbb{R}^d$ and $\sigma > 0$ is the noise strength. The Cucker–Smale Particle Model with Noise:

$$\begin{cases} dx_i = v_i dt \ , \\ dv_i = \sum_{j=1}^{N} a(|x_j - x_i|)(v_j - v_i)\, dt + \sqrt{2D \sum_{j=1}^{m} a(|x_j - x_i|)}\, d\Gamma_i(t) \ . \end{cases}$$

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Outline

Vicsek's model

Assume N particles moving at unit speed: reorientation & diffusion:

$$\begin{cases} dX_t^i = V_t^i\,dt, \\ dV_t^i = \sqrt{2D}\,P(V_t^i)\circ dB_t^i - P(V_t^i)\left(\frac{1}{N}\sum_{j=1}^{N} K(X_t^i - X_t^j)(V_t^i - V_t^j)\right)dt. \end{cases}$$

Here $P(v)$ is the projection operator on the tangent space at $v/|v|$ to the unit sphere in $\mathbb{R}^d$, i.e.,

$$P(v) = I - \frac{v\otimes v}{|v|^2}.$$

Noise in the Stratatonovich sense: imposed by the rigorous construction of the Brownian motion on a manifold. Rigorous derivation: Bolley-Cañizo-C.

Main issue: phase transition driven by noise D: Degond-Liu-Frouvelle.

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Convergence of the particle method

Empirical measures: if $x_i, v_i : [0,T) \to \mathbb{R}^d$, for $i = 1, \dots, N$, is a solution to the ODE system,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \overbrace{(\alpha - \beta |v_i|^2) v_i}^{\text{propulsion-friction}} - \overbrace{\sum_{j \neq i} m_j \nabla U(|x_i - x_j|)}^{\text{attraction-repulsion}} + \overbrace{\sum_{j=1}^{N} m_j a_{ij} (v_j - v_i)}^{\text{orientation}}. \end{cases}$$

then the $f_N : [0,T) \to \mathcal{P}_1(\mathbb{R}^d)$ given by

$$f_N(t) := \sum_{i=1}^{N} m_i \delta_{(x_i(t), v_i(t))} \qquad \text{with} \sum_{i=1}^{N} m_i = 1 \,,$$

is expected to be the solution corresponding to initial atomic measures.

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Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2)vf] - \mathrm{div}_v\left[(\nabla_x U \star \rho)f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Big[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x - y|^2)^\gamma} f(y, w, t)\, dy\, dw\right)}_{:=\xi(f)(x,v,t)} f(x, v, t)\Big]$$

Orientation, Attraction and Repulsion:

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Rigorous proofs of the mean field limit: Cañizo-C.-Rosado (M3AS 2010), Bolley-Cañizo-Rosado (M3AS 2011), C.-Choi-Hauray (Springer Verlag 2012).

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Rigorous proofs of the mean field limit: Cañizo-C.-Rosado (M3AS 2010), Bolley-Cañizo-Rosado (M3AS 2011), C.-Choi-Hauray (Springer Verlag 2012).

Mesoscopic models

Model with asymptotic velocity + Attraction/Repulsion:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \mathrm{div}_v[(\alpha - \beta|v|^2) v f] - \mathrm{div}_v \left[(\nabla_x U \star \rho) f\right] = 0.$$

Velocity consensus Model:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot \Bigg[\underbrace{\left(\int_{\mathbb{R}^{2d}} \frac{v - w}{(1 + |x-y|^2)^\gamma} f(y,w,t)\, dy\, dw \right)}_{:=\xi(f)(x,v,t)} f(x,v,t) \Bigg]$$

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Outline

Short Relaxation towards Cruising Speed

Scaled Vlasov equation in $d = 2, 3$ dimensions:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + a^\varepsilon(t,x) \cdot \nabla_v f^\varepsilon + \frac{1}{\varepsilon} \mathrm{div}_v \{ f^\varepsilon (\alpha - \beta |v|^2) v \} = 0, \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^{2d}$$

with $a^\varepsilon(t,\cdot) = -\nabla_x U \star \rho^\varepsilon(t,\cdot) - H \star f^\varepsilon(t,\cdot)$.

This asymptotic limit enforces that particles move at cruising speed $\sqrt{\alpha/\beta}$. If one formally does the expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \ldots$$

we get

$$\mathrm{div}_v \{ f(\alpha - \beta |v|^2) v \} = 0$$

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v(fa(t,x)) + \mathrm{div}_v \{ f^{(1)} (\alpha - \beta |v|^2) v \} = 0 \, ,$$

up to first order.

To eliminate the higher order term we use the invariants of the flow associated to the field $(\alpha - \beta |v|^2) v \cdot \nabla_v$, functions of x and $v/|v|$.

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Vicsek Model as Asymptotic Limit

Bostan-C. (M3AS 2013)

Assume that $U \in C_b^2(\mathbb{R}^d)$, $H(x,v) = h(x)v$ with $h \in C_b^1(\mathbb{R}^d)$ nonnegative, $f^{\text{in}} \in \mathcal{P}_1(\mathbb{R}^d \times \mathbb{R}^d)$, supp$f^{\text{in}} \subset \{(x,v) : |x| \leq L_0, r_0 \leq |v| \leq R_0\}$.

Then for all $\delta > 0$, the sequence $(f^\varepsilon)_\varepsilon$ converges towards the measure solution $f(t,x,\omega)$ on $(x,\omega) \in \mathbb{R}^d \times \sqrt{\alpha/\beta}\mathbb{S}$ of the problem

$$\partial_t f + \text{div}_x(f\omega) - \text{div}_\omega \left\{ f \left(I - \frac{1}{r^2}(\omega \otimes \omega) \right) (\nabla_x U \star \rho + H \star f) \right\} = 0$$

with initial data $f(0) = \langle f^{\text{in}} \rangle$.

Remarks:

- Adding noise we get from $\Delta_v f$ to the Laplace-Beltrami operator on the sphere $\Delta_\omega f$. We only know how to perform the formal expansion but not the rigorous limit.
- This formally shows that the fixed speed limit of the Cucker-Smale's model is the Vicsek's model.

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Outline

The Local Cucker-Smale model with noise

- We consider the following kinetic flocking model:

$$\partial_t f + v\nabla_x f = \nabla_v \cdot \Big((v - u_f) f - \alpha v(1 - |v|^2) f + D\nabla_v f \Big) ,$$

where

$$u_f(t,x) = \frac{\int v f(t,x,v)\,dv}{\int f(t,x,v)\,dv}$$

- The first term is a Cucker-Smale-like term, encourages the velocity to align with the mean velocity
- The second term provides self-propulsion and friction, encouraging unit velocities
- The last term captures the influence of noise in the velocity

Outline

The homogeneous problem

- Looking at the spatially homogeneous problem:

$$\partial_t f = \nabla_v \cdot \Big((v - u_f) f - \alpha v (1 - |v|^2) f + D \nabla_v f \Big)$$

- We have a gradient flow structure: write the equation as $\partial_t f = \nabla_v \cdot (f \nabla_v \xi)$ with $\xi = \Phi(v) + W * f + D \log f$
  - Confinement in v: $\Phi(v) = \alpha \left(\frac{|v|^4}{4} - \frac{|v|^2}{2} \right)$
  - Interaction potential of the form $W(v) = \frac{|v|^2}{2}$
  - Linear diffusion.
- Our model is continuity equation with velocity field of the form $-\nabla_v \xi$
- Natural entropy for this equation given by the free energy of the system:

$$\begin{aligned}\mathcal{F}[f] &:= \int_{\mathbb{R}^d} \Phi(v) f(v)\, dv + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(v-w) f(v) f(w)\, dw\, dv + D \int_{\mathbb{R}^d} f(v) \log f(v)\, dv \\ &= \int_{\mathbb{R}^d} \left(\alpha \frac{|v|^4}{4} + (1-\alpha) \frac{|v|^2}{2} \right) f(v)\, dv - \frac{1}{2} |u_f|^2 + D \int_{\mathbb{R}^d} f \log f(v)\, dv \,,\end{aligned}$$

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The stationary solutions

- We consider stationary solutions of the form:

$$f(v) = \frac{1}{Z}\exp\left(\frac{-1}{D}\left[\alpha\frac{|v|^4}{4} + (1-\alpha)\frac{|v|^2}{2} - u_f\cdot v\right]\right)$$

- We see that in order for the stationary solution to exist, u_f must be a root of the equation:

$$\mathcal{H}(u,D) = \int (v-u)f(v)dv$$

- We prove that, in any dimension[1]
  - There is a region of parameter space with only one such root, namely $u=0$
  - There is another region of parameter space with more than one root, $u=0$ and $|u| = C_{\alpha,D} \neq 0$

[1] 1D case was proven independently in J. Tugaut's *Phase transitions of McKean-Vlasov processes in symmetric and asymmetric multi-wells landscape*, and S. Herrmann and J. Tugaut. *Non-uniqueness of stationary measures for self-stabilizing processes*

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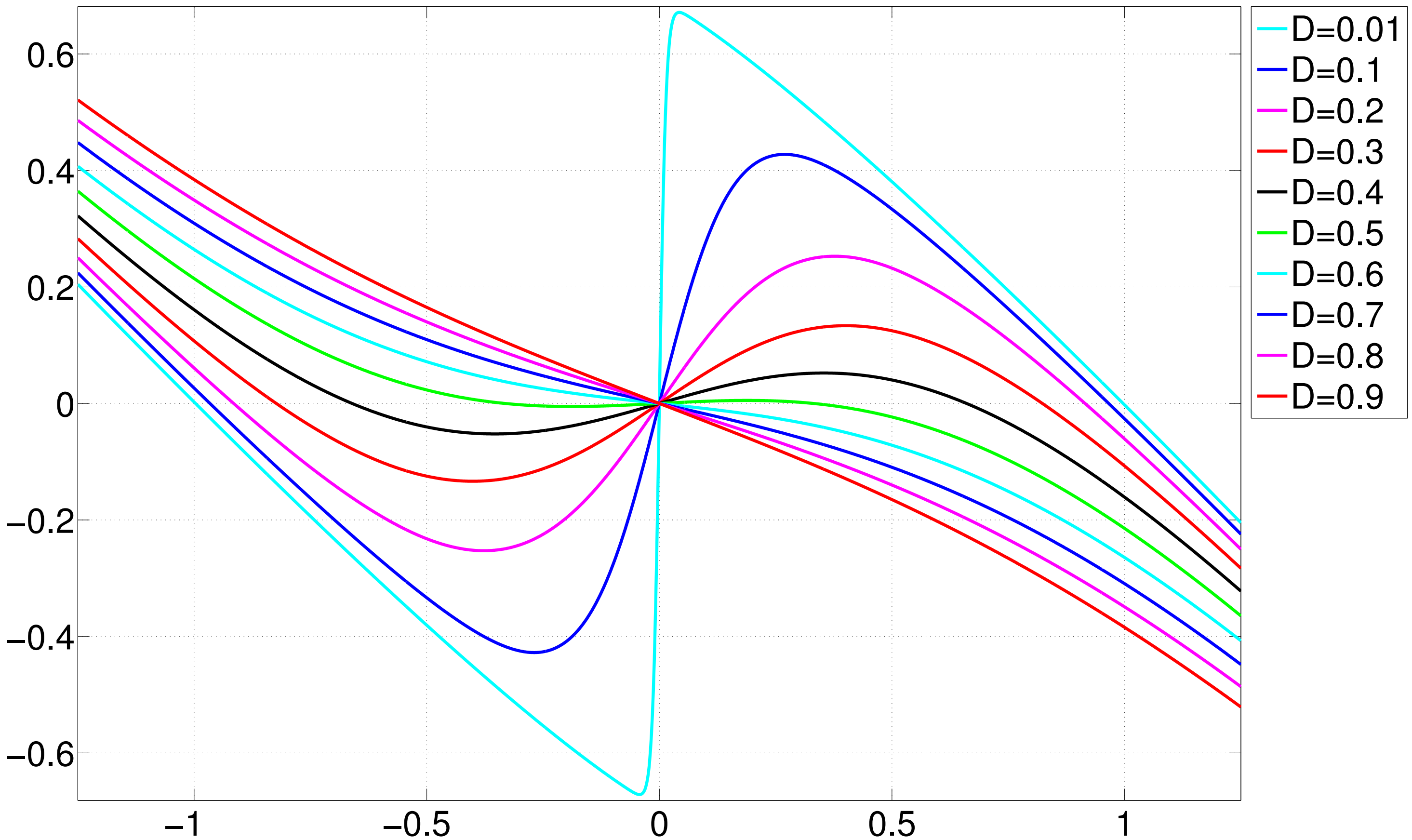
$\mathcal{H}(u, D)$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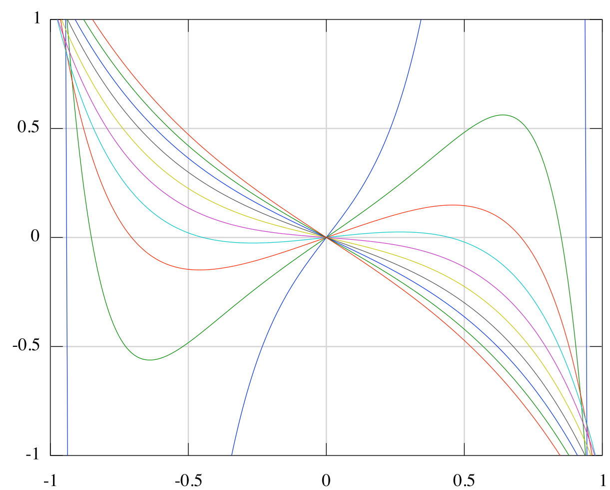
Main idea of our proof



- Our proof hinges Laplace's method and the behavior of $\mathcal{H}(u, D)$ as D varies:
  - For small D, we are able to use Laplace's Method to show that there is a nonzero stationary solution
  - For large D, $\frac{\partial \mathcal{H}}{\partial u}$ is negative for all u.
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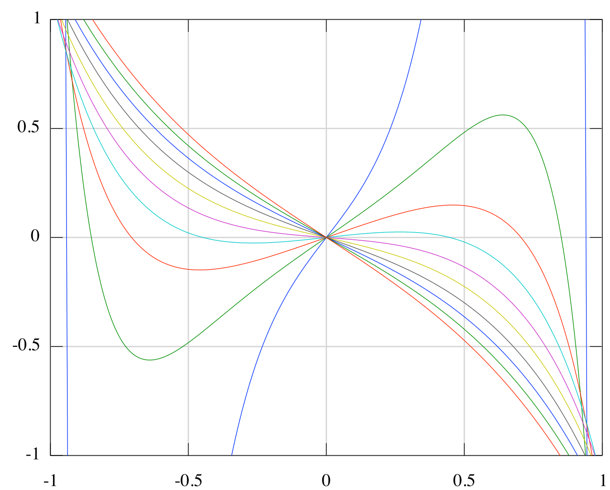

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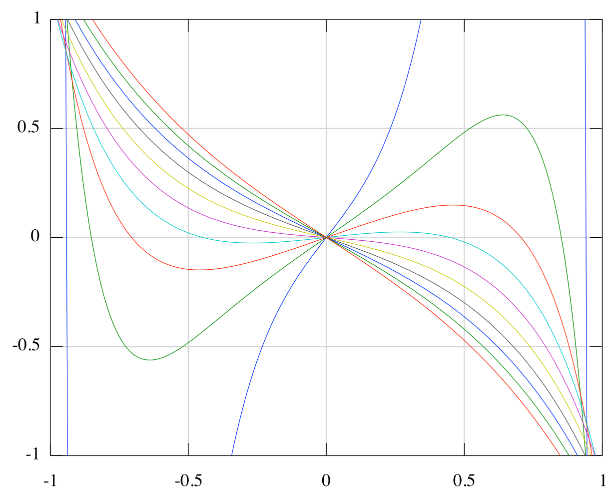

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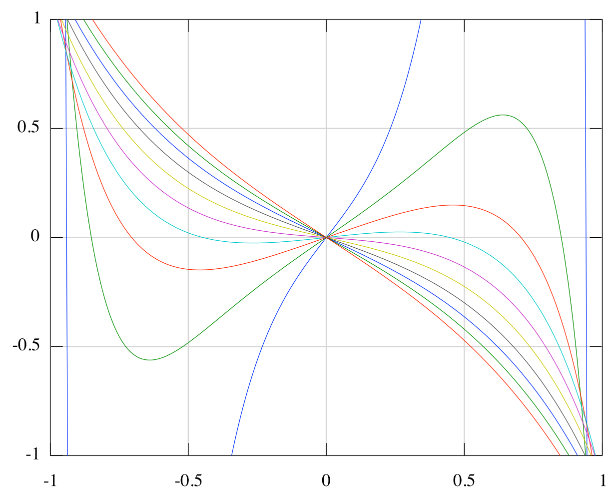

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The case of small D

Find u such that it is a root of $\mathcal{H}(u, D)$, i.e. as $D \to 0$,

$$u = \frac{\int \exp\left(-\frac{1}{D} P_u(v)\right) v_1 dv}{\int \exp\left(-\frac{1}{D} P_u(v)\right) dv} \tag{1}$$

Laplace's Method tells us that this u must be such that

$$u \approx \frac{(2\pi D)^{\frac{d}{2}} \left|H(P_u(\tilde{v}))\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{D} P_u(\tilde{v})\right) \tilde{v}_1}{(2\pi D)^{\frac{d}{2}} \left|H(P_u(\tilde{v}))\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{D} P_u(\tilde{v})\right)} \tag{2}$$

where $\tilde{v}$ is the global minimum of $P_u(v)$.

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- This global minimum is strictly positive
- Hence, there is a nonzero stationary solution in addition to u=0

In order to prove this rigorously, we need to apply an implicit function theorem from the positive root for $D = 0$, this needs to compute next orders in the expansion of Laplace's theorem and their limits as the noise $D \to 0$. These expansions are not standard since we need to track carefully the powers of D involved in each term.

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The case of $D \to \infty$

- We show that $\mathcal{H}$ is strictly decreasing in u for $D \to \infty$
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Outline

Varying α and D

- We have proven analytically that for small D, there is more than one stationary solutions, while for large D, there is only one
- Now, numerically consider where in parameter space each of these situations occur
  - Vary α and D and count the number of roots of $\mathcal{H}$
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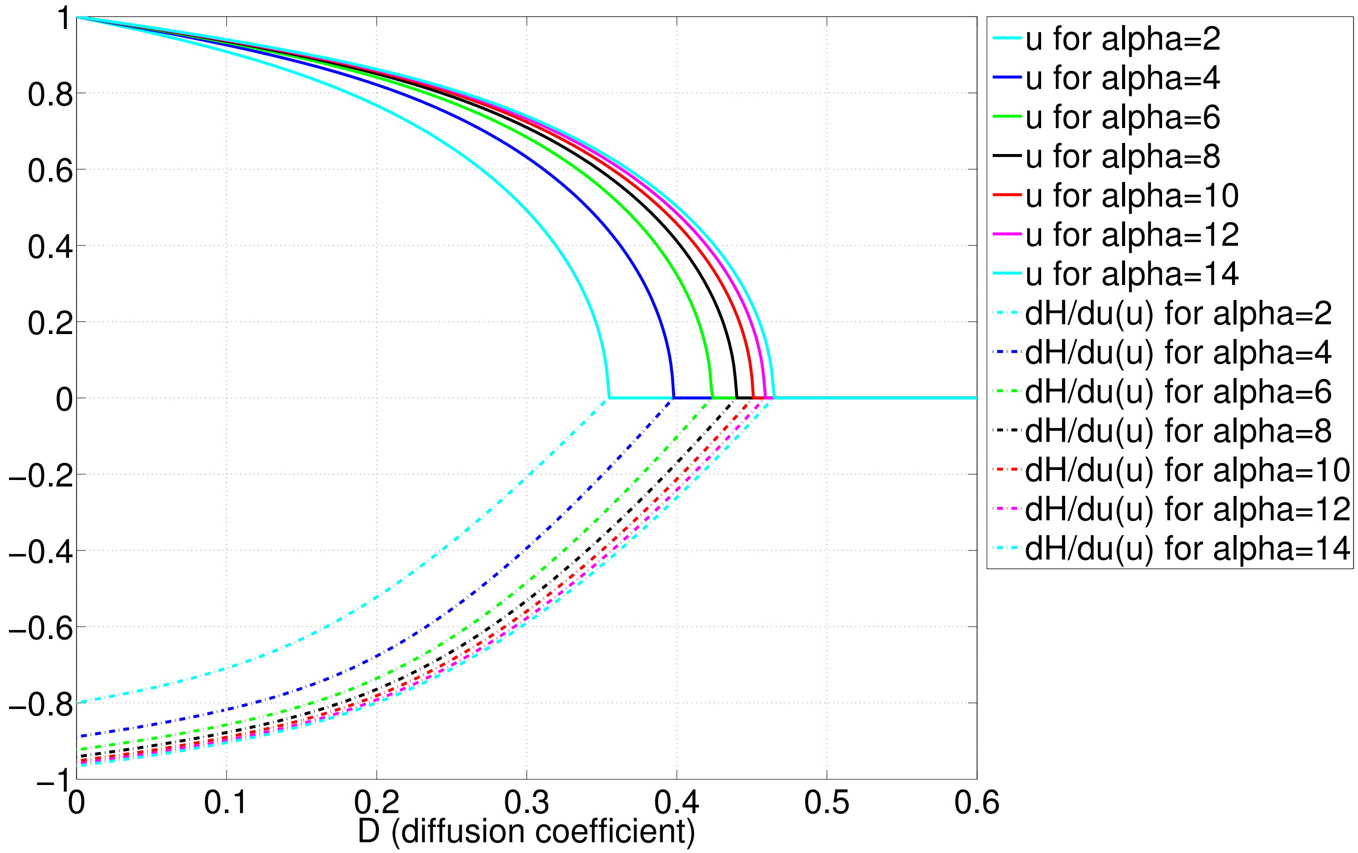
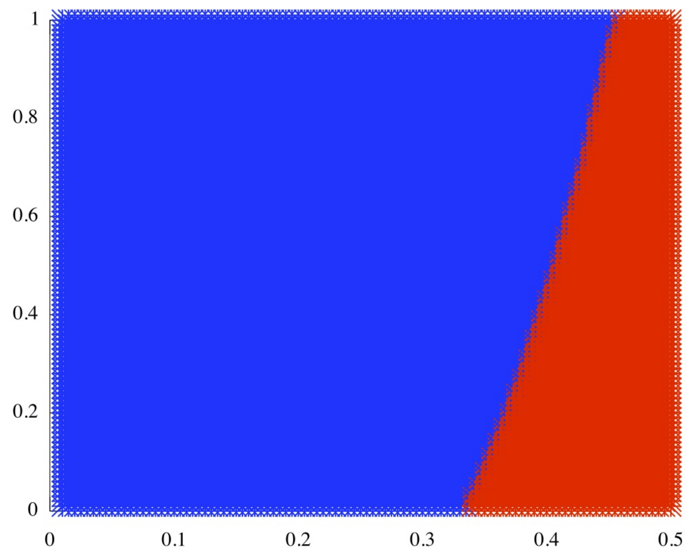
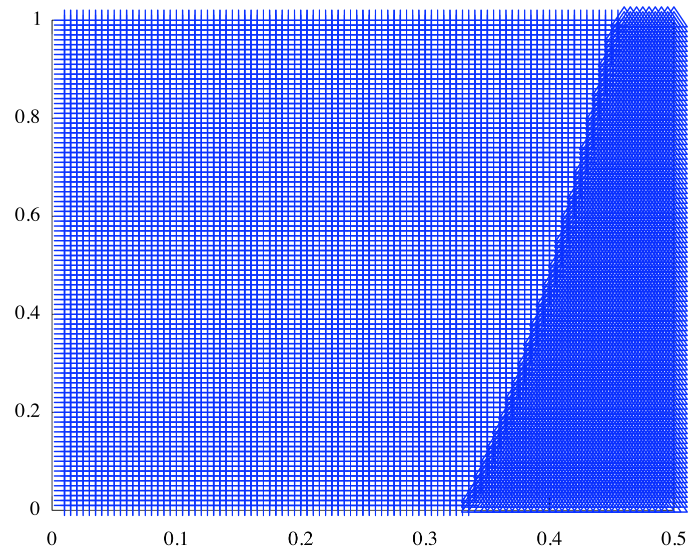
The roots of $\mathcal{H}$ plotted against D in 2D



Numerical exploration, varying α and D in 2D

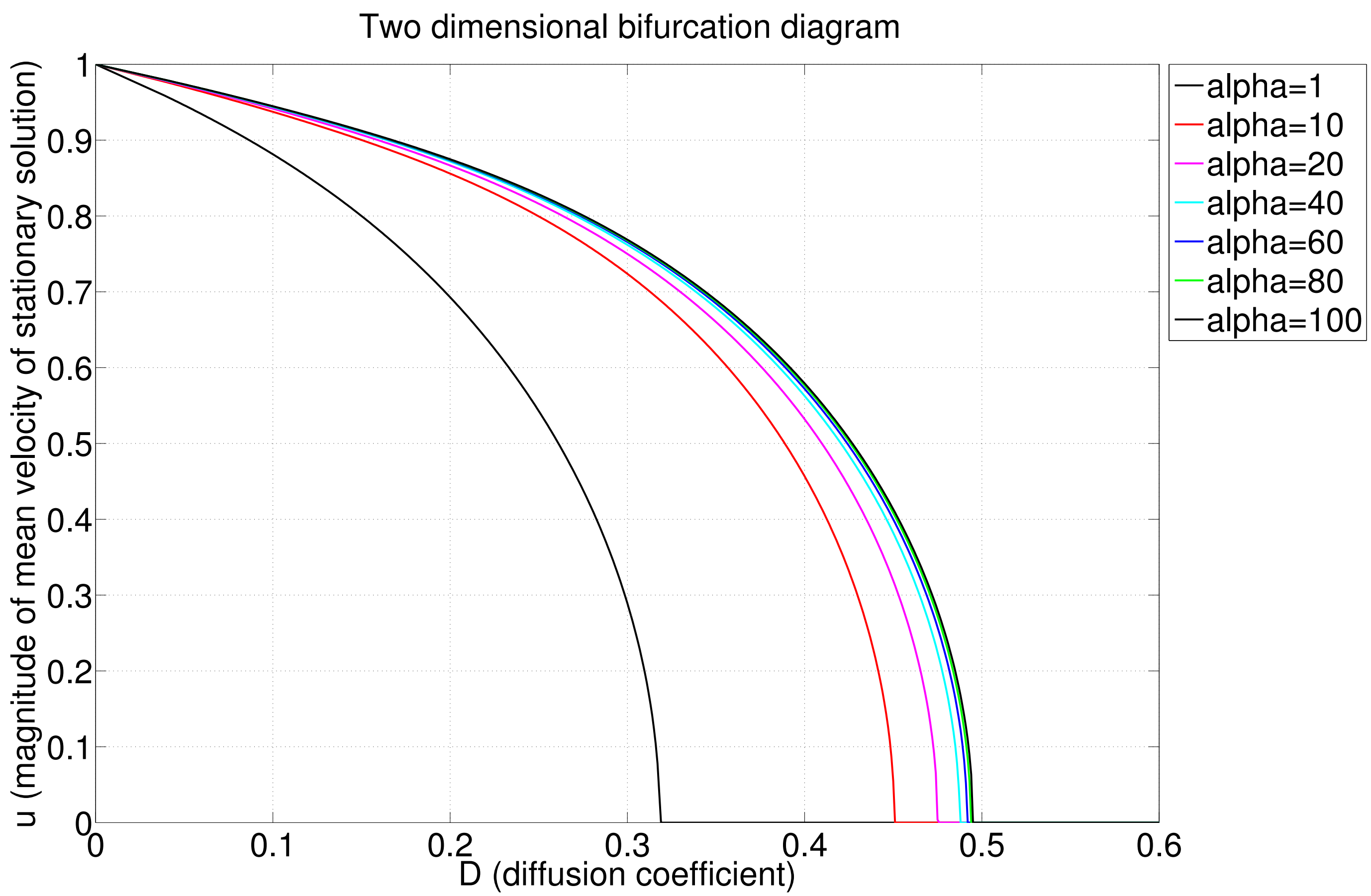


The number of roots at $u = 0$

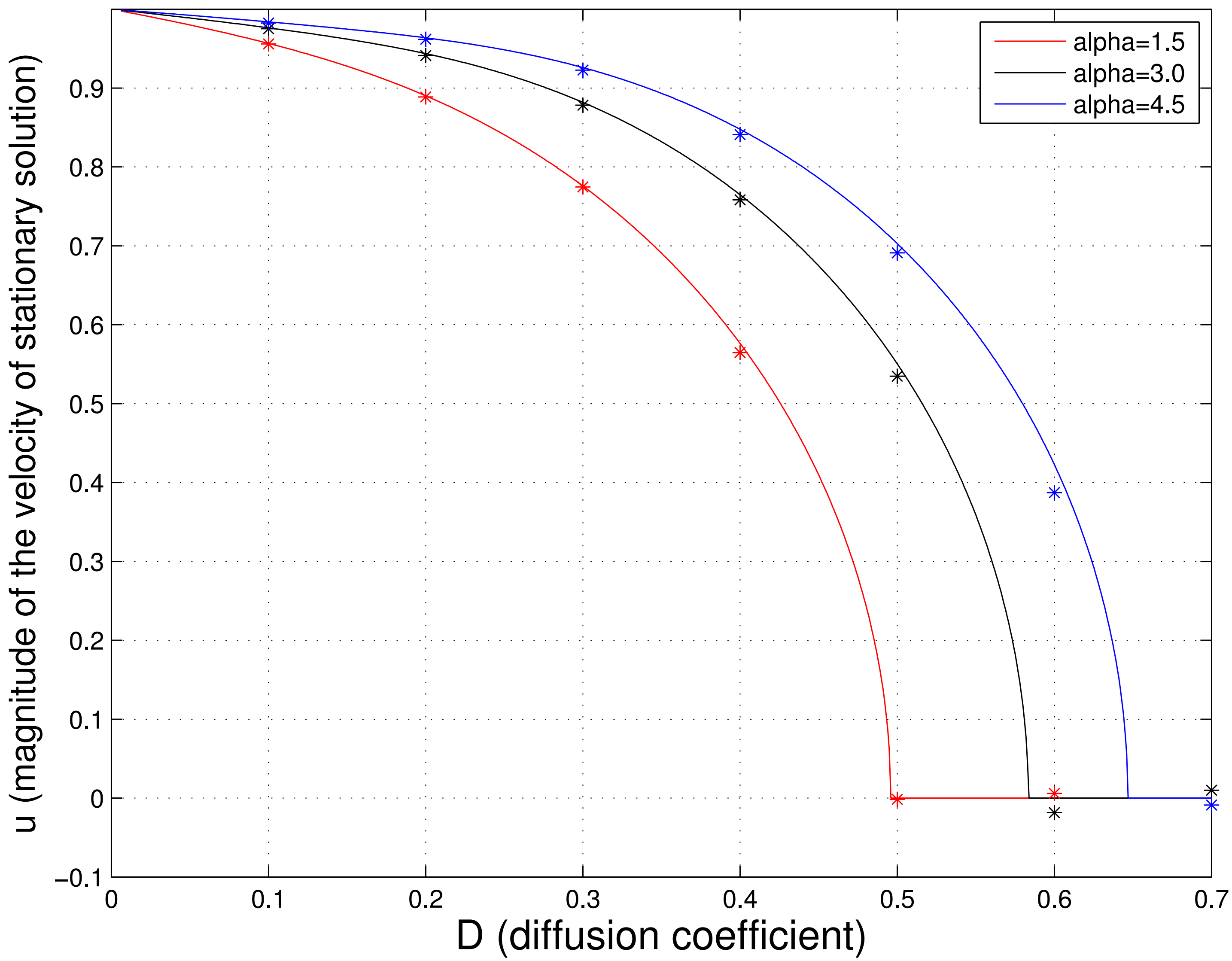


The sign of $\frac{\partial \mathcal{H}}{\partial u}$ at $u = 0$

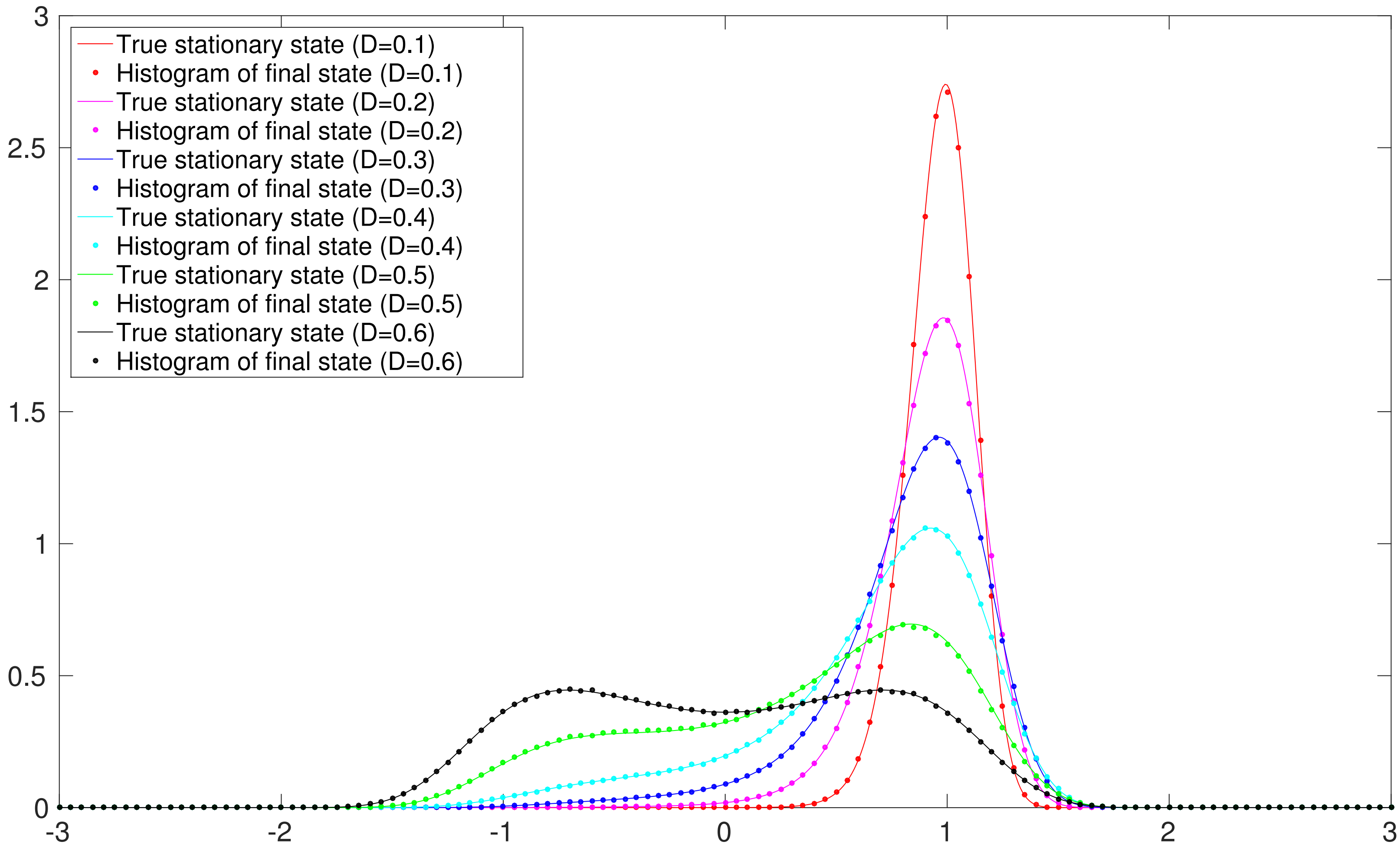
Exploring the limit $\alpha \to \infty$ in 2D



Stability of the stationary solutions in 1D



Comparing particles to f in 1D



Outline

Hydrodynamics via Asymptotic Limit

Bostan-C. (M3AS 2017)

Given a solution to

$$\partial_t f^\varepsilon + \mathrm{div}_x(f^\varepsilon v) + \frac{1}{\varepsilon^2}\mathrm{div}_v(f^\varepsilon(\alpha - \beta|v|^2)v) = \frac{1}{\varepsilon}\mathrm{div}_v\{f^\varepsilon(v - u[f^\varepsilon]) + \sigma\nabla_v f^\varepsilon\}$$

for any σ, r such that $\frac{\sigma}{r^2} \in]0, \frac{1}{d}[$, we denote by $l = l\left(\frac{\sigma}{r^2}\right)$ the unique positive solution of $\lambda(l) = \frac{\sigma}{r^2} l$ with

$$\lambda(l) = \frac{\int_0^\pi \cos\theta e^{l\cos\theta} \sin^{d-2}\theta \, \mathrm{d}\theta}{\int_0^\pi e^{l\cos\theta}\sin^{d-2}\theta \, \mathrm{d}\theta}, \quad l \in \mathbb{R}_+, \quad d \geq 2.$$

Then the limit distribution $f = \lim_{\varepsilon \searrow 0} f^\varepsilon$, is a von Mises-Fisher equilibrium $f = \rho M_{l\Omega}(\omega)\,\mathrm{d}\omega$ on $r\mathbb{S}^{d-1}$, where the density $\rho(t,x)$ and the orientation $\Omega(t,x)$ satisfy the macroscopic equations (SOH)

$$\partial_t \rho + \mathrm{div}_x\left(\rho \frac{l\sigma}{r}\Omega\right) = 0, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$$

$$\partial_t \Omega + k_d \, r(\Omega \cdot \nabla_x)\Omega + \frac{r}{l}(I_d - \Omega \otimes \Omega)\frac{\nabla_x \rho}{\rho} = 0$$

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Expansion

The behavior of the family $(f^\varepsilon)_{\varepsilon>0}$, as the parameter ε becomes small, follows by analyzing the formal expansion

$$f^\varepsilon = f + \varepsilon f^{(1)} + \varepsilon^2 f^{(2)} + \dots$$

Plugging the above Ansatz into the kinetic equation, leads to the constraints

$$\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} = 0$$

$$\mathrm{div}_v\{f^{(1)}(\alpha - \beta|v|^2)v\} = \mathrm{div}_v\{f(v - u[f]) + \sigma\nabla_v f\} := Q(f)$$

and to the time evolution equations

$$\partial_t f + \mathrm{div}_x(fv) + \mathrm{div}_v\{f^{(2)}(\alpha - \beta|v|^2)v\} = \mathcal{L}_f(f^{(1)})$$

with

$$\mathcal{L}_f(f^{(1)}) := \mathrm{div}_v\{f^{(1)}(v - u[f]) + \sigma\nabla_v f^{(1)}\} - \mathrm{div}_v\left\{f\,\frac{\int_{\mathbb{R}^d} f^{(1)}(v' - u[f])\,\mathrm{d}v'}{\int_{\mathbb{R}^d} f\,\mathrm{d}v'}\right\}$$

cutting the development at second order.

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0th-order term in expansion

Assume that $(1+|v|^2)F \in \mathcal{M}_b^+(\mathbb{R}^d)$. Then F solves $\mathrm{div}_v\{F(\alpha - \beta|v|^2)v\} = 0$ in $\mathcal{D}'(\mathbb{R}^d)$ *i.e.*,

$$\int_{\mathbb{R}^d} (\alpha - \beta|v|^2)v \cdot \nabla_v \varphi \,\mathrm{d}F(v) = 0, \quad \text{for any } \varphi \in C_c^1(\mathbb{R}^d)$$

if and only if $\mathrm{supp} F \subset \{0\} \cup r\mathbb{S}$.

Let $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ be a non negative bounded measure on $\mathbb{R}^d$. We denote by $\langle F \rangle$ the measure corresponding to the linear application

$$\psi \to \int_{\mathbb{R}^d} \psi(v)\, \mathbf{1}_{v=0} F(v) + \int_{\mathbb{R}^d} \psi\left(r\frac{v}{|v|}\right) \mathbf{1}_{v\neq 0} F(v)\,,$$

for all $\psi \in C_c^0(\mathbb{R}^d)$.

Elimination

For any $f \in \mathcal{M}_b^+(\mathbb{R}^d \times \mathbb{R}^d)$ such that $\mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} \in \mathcal{M}_b(\mathbb{R}^d \times \mathbb{R}^d)$, we have $\langle \mathrm{div}_v\{f(\alpha - \beta|v|^2)v\} \rangle = 0$.

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For any $l \in \mathbb{R}_+, \Omega \in \mathbb{S}$, we introduce the von Mises-Fisher distribution

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Kernel of the averaged collision operator

Let $F \in \mathcal{M}_b^+(\mathbb{R}^d)$ be a non negative bounded measure on $\mathbb{R}^d$, supported in $r\mathbb{S}^{d-1}$. The following statements are equivalent:

1. $\langle Q(F)\rangle = 0$, that is

$$\int_{v\neq 0} \left\{ -(v-u[F]) \cdot \nabla_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] + \sigma \Delta_v \left[\widetilde{\psi}\left(r\frac{v}{|v|}\right)\right] \right\} F \ \mathrm{d}v = 0,$$

for all $\widetilde{\psi} \in C^2(r\mathbb{S}^{d-1})$.

2. There are $\rho \in \mathbb{R}_+, \Omega \in \mathbb{S}$ such that $F = \rho M_{l\Omega} \mathrm{d}\omega$ where $l \in \mathbb{R}_+$ satisfies

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- Uniqueness of the non symmetric equilibria except symmetries is open.
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