Multi-scale hyperbolic relaxation system, kinetic-fluid domain coupling, and singular limit analysis

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Set up of the problem

Jin-Xin hyperbolic relaxation system:

$$\begin{aligned} \partial_t u^\epsilon + \partial_x v^\epsilon &= 0, \\ \partial_t v^\epsilon + a^2 \partial_x u^\epsilon &= \lambda_\epsilon(x)\big(f(u^\epsilon) - v^\epsilon\big), \end{aligned}$$

with a two-scale relaxation rate ($0 < \epsilon \ll 1$):

$$\lambda_\epsilon(x) = \begin{cases} 1, & x < 0, \\ \frac{1}{\epsilon}, & x > 0. \end{cases}$$

Domain decomposition method:

- overlap vis non-overlap
- data transmission and derivatives matching
- hamiltonian system vs dissipation system
- steady calculations vs dynamical calculations

Motivation:

- Analysis of Asymptotic Preserving schemes,
- Kinetic-fluid domain coupling, localized kinetic upscaling
- Related to Milne problem in kinetic theory.

Set up of the problem

Assume a sub-characteristic condition:

$$|f'(u)| < a,$$

with compactly supported initial data $u_0(x) \in L^\infty(\mathbb{R}) \cap \mathrm{BV}(\mathbb{R})$.
For simplest, assume $f(0) = 0$ and $v_0(x) = f(u_0(x))$.
We assume either

$$0 < C_0 \le f'(u), \quad \text{for all } u \in \mathbb{R},$$

or

$$f'(u) \le -C_0 < 0, \quad \text{for all } u \in \mathbb{R}.$$

Under this assumption, no shock wave can stick to the interface.

A decoupled zero ϵ limit system

When $f'(u(t,0^+)) < 0$,

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x > 0,\ t > 0, \\ u(0,x) = u_0(x), \end{cases}$$

and set $v(t,x) = f(u(t,x))$.

$$\begin{cases} \partial_t u + \partial_x v = 0, & x < 0,\ t > 0, \\ \partial_t v + a^2 \partial_x u = f(u) - v, \\ v(t,0^-) = v(t,0^+) \\ u(0,x) = u_0(x), v(0,x) = v_0(x). \end{cases}$$

When $f'(u(t,0^+)) > 0$,

$$\begin{cases} \partial_t u + \partial_x v = 0, & x < 0,\ t > 0, \\ \partial_t v + a^2 \partial_x u = f(u) - v, \\ v(t,0^-) = f(u(t,0^-)), \\ u(0,x) = u_0(x), v(0,x) = v_0(x), \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & x > 0,\ t > 0, \\ u(t,0^+) = u(t,0^-), \\ u(0,x) = u_0(x), \end{cases}$$

and set $v(t,x) = f(u(t,x))$.

We will only analyze these two cases.

Coupled system for general flux f (characteristic)

$$\begin{cases} \partial_t u + \partial_x v = 0, \;\; x < 0, \;\; t > 0 \\ \partial_t v + a^2 \partial_x u = f(u) - v, \\ v(t, 0^-) = f(u(t, 0^+)), \\ u(0, x) = u_0(x), \;\; v(0, x) = v_0(x) \end{cases}$$

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \;\; x > 0, \;\; t > 0 \\ v(t, x) = f(u(t, x)), \\ u(t, 0^+) \overset{BLN}{:=} u(t, 0^-), \\ u(0, x) = u_0(x) \end{cases}$$

Bardos-Lerous-Nédélec condition: for all k between $u(t, 0^-)$ and $u(t, 0^+)$.

$$\mathrm{sgn}\big(u(t, 0^+) - u(t, 0^-)\big) \, \big(f(u(t, 0^+)) - f(k)\big) \leq 0,$$

If $f' > 0$, then BLN reduced to $u(t, 0^+) = u(t, 0^-)$
If $f' < 0$, then BLN is always hold.
Hence, when $f' \neq 0$, this system is reduced to the previous decoupled system.

Connection profile for general flux f (characteristic)

The connection profile is given by

$$a^2 \frac{d}{dy}\mathcal{U}(t,y) = f(\mathcal{U}(t,y)) - v(t,0^+),\ y > 0, \quad \mathcal{U}(t,0) = u(t,0^-)$$

and it is possible to connected to a middle value $u(t,0^m)$ with $f(u(t,0^m)) = f(u(t,0^+))$ and there is a stationary shock at the end of profile connecting $u(t,0^m)$ and $u(t,0^+)$.

Example:

Take $f(u) = \frac{1}{2}u^2$, $a = 1$, $u(t,x) \equiv \left\{ \begin{array}{ll} 1 & \text{for } x \leq \delta \\ -1 & \text{for } x > \delta \end{array} \right.$, $\delta \geq 0$
$v(t,x) = f(u(t,x)$, $\mathcal{U}(t,y) \equiv 1$ is an exact solution to the zero ϵ-limit coupled system.

Vasseur's result on coupled kinetic and Burgers' eq

Perthame-Tadmor kinetic model with a two-scale relaxation rate

$$\partial_t f^\epsilon + \xi \partial_x f^\epsilon = \lambda_\epsilon(x)\big(\mathcal{M} f^\epsilon - f^\epsilon\big)$$

$$u = \int_{\mathbb{R}} f \, d\xi, \quad \mathcal{M} f := M(u,\xi) := \mathbf{1}_{\{0 \le \xi \le u\}} - \mathbf{1}_{\{u \le \xi \le 0\}}$$

compatibility condition

$$0 \le \ \mathrm{Sign}(\xi)\, f(\xi) \le 1$$

For $0 < u^+ < L$, $f_0 \in L^\infty(\mathbb{R}, [-L, L])$
Initial data case I:

$$f_0(x,\xi) \ge \begin{cases} M(u^+,\xi), & \text{for } x < 0, \\ M(-u^+,\xi), & \text{for } x > 0, \end{cases} \quad \xi \in [-L, L]$$

Initial data case II:

$$\begin{cases} \mathrm{Supp}\ f_0 \in [-L, u^+ - \eta], & \text{for } x < 0, \\ f_0 = M(-u^+,\xi), & \text{for } x > 0, \ \xi \in [-L, L] \end{cases}$$

The coupled zero ϵ limit system

$$\begin{cases} \partial_t f + \xi \partial_x f = \mathcal{M} f - f, \; x < 0, \; \xi \in \mathbb{R} \\ f(t, 0_-, \xi) = F(t, 0_+, \xi), \; \xi \le 0 \\ f(0, x, \xi) = f_0(x, \xi) \end{cases}$$

$$\begin{cases} \xi \partial_y F = \mathcal{M} F - F, \; y > 0, \; \xi \in \mathbb{R} \\ F(t, 0_+, \xi) = f(t, 0_-, \xi), \; \xi \ge 0 \\ \int_{\mathbb{R}} \xi F(t, y, \xi) \, d\xi = \frac{1}{2} u(t, 0_+)^2, \; y > 0 \end{cases}$$

$$\begin{cases} \partial_t u + \frac{1}{2} \partial_x u^2 = 0, \; x > 0, \; t > 0 \\ v(t, x) = f(u(t, x)), \\ u(t, 0^+) \overset{BLN}{:=} \sqrt{2 \int_{-\infty}^{\infty} \xi f(t, 0^-, \xi) \, d\xi}, \\ u(0, x) = u_0(x) \end{cases}$$

Vasseur's Results: Well-posedness and stability of the limit system; Convergence to the limit system for the two class of initial data.

Existence and stability for the ϵ system

For any $\epsilon > 0$ and $T > 0$, there is a unique weak solution (u^ϵ, v^ϵ) in $L^1\big((0,T), L^1(\mathbb{R})\big)$ and satisfies the following *a priori* estimates:

$$(i)\quad \| u^\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C, \quad \| v^\epsilon(t,\cdot)\|_{L^\infty(\mathbb{R})} \le C;$$

$$(ii)\quad TV(v^\epsilon(t,\cdot)) \le C;$$

$$(iii)\quad TV_{\{x<0\}}(u^\epsilon(t,\cdot)) \le C(1+T),$$

$$TV_{\{x>0\}}(u^\epsilon(t,\cdot)) \le C\left(1+\frac{T}{\epsilon}\right);$$

$$(iv)\quad ||u^\epsilon - f^{-1}(v^\epsilon)||_{L^1((0,T);L^2(\mathbb{R}_x^+))} \le C_T\sqrt{\epsilon};$$

for all time $T > 0$ while

$$(v)\int_{\mathbb{R}} |u^\epsilon(t_2,x) - u^\epsilon(t_1,x)|dx \le C|t_2 - t_1|, \quad 0 \le t_1 \le t_2 \le T;$$

$$\int_{\mathbb{R}} |v^\epsilon(t_2,x) - v^\epsilon(t_1,x)|dx \le C|t_2 - t_1|.$$

Outline of the proof

- Smoothing the data $u_{0,\delta}(x) = \rho_\delta * u_0(x)$, $\lambda_{\epsilon,\delta}(x) = \rho_\delta * \lambda_\epsilon(x)$. We skip subscripts ϵ, δ below for clearlity in expression.
- Use the characteristic variables,

$$r_\pm(t,x) = a\ u(t,x) \pm v(t,x),$$

one has L^1-contraction:

$$\int_{|x|<M} \left(|r_+ - \bar{r}_+| + |r_- - \bar{r}_-| \right) dx \\ \leq \int_{|x|<M+at} \left(|(r_+)_0 - (\bar{r}_+)_0| + |(r_-)_0 - (\bar{r}_-)_0| \right)(x) dx.$$

for any two classical solutions $(u^{\epsilon,\delta}, v^{\epsilon,\delta})$ and $(\bar{u}^{\epsilon,\delta}, \bar{v}^{\epsilon,\delta})$, vanishing outside the cone $\bigcup_{t\geq 0}[-(M+at),(M+at)]$.

Outline of the proof

- Uniform L^1 estimate for the time derivative $s_\pm = a\partial_t u \pm \partial_t v$

$$\begin{cases} \partial_t s_- - a\partial_x s_- = -\lambda_{\epsilon,\delta}(x)\mathcal{R}(s_-, s_+), \\ \partial_t s_+ + a\partial_x s_+ = \lambda_{\epsilon,\delta}(x)\mathcal{R}(s_-, s_+), \end{cases}$$

with

$$\mathcal{R}(s_-, s_+) = f'(u)\frac{s_- + s_+}{2a} - \frac{s_+ - s_-}{2}, \tag{8}$$

The sub-characteristic condition implies

$$\partial_{s_-}\mathcal{R}(s_-, s_+) > 0, \;\; \partial_{s_+}\mathcal{R}(s_-, s_+) < 0$$

$$\int_{|x|<M} \big(|s_+(t,x)| + |s_-(t,x)|\big)\,dx \le \int_{|x|<M+at} \big(|s_+(0,x)| + |s_-(0,x)|\big)\,dx$$

Outline of the proof

- BV estimate of $v^{\epsilon,\delta}$ immediately follows from $\partial_x v = -\partial_t u$.
- Entropy dissipation estimate. For Chen-Levermore-Liu entropy pair Φ, Ψ,

$$\partial_{vv}\Phi(u,v) \geq C, \Phi(0,0) = 0, \partial_v \Phi(u, f(u)) = 0, \forall u \in \mathbb{R},$$

$$\partial_t \Phi(u,v) + \partial_x \Psi(u,v) = \lambda_{\epsilon,\delta}(x)\partial_v \Phi(u,v)\Big(f(u) - v\Big).$$

$$\partial_t \Phi(u,v)+\partial_x \Psi(u,v) = -\lambda_{\epsilon,\delta}(x)\partial_{vv}\Phi(u,\theta_1)f'(\theta_2)^2|f^{-1}(v)-u|^2$$

for some θ_1 in between v and $f(u)$ and θ_2 in between u and $f^{-1}(v)$

Strong convergence in $L^1((0,T)\times\mathbb{R})$

For any $T>0$, (u^ϵ, v^ϵ) converges to (u,v) in $L^1((0,T)\times\mathbb{R})$.

- (u,v) is a weak solution in $(0,T)\times\mathbb{R}_x^-)$.
- u is an entropy solution in $(0,T)\times\mathbb{R}_x^+)$ and $v=f(u)$ a.e.
- $u_t+v_x=0$ in $\mathcal{D}'((0,T)\times\mathbb{R})$.
- $(u,v)\in BV((0,T)\times\mathbb{R})$. For a.e. $t>0$, there exist left and right traces $u(t,0^-)$, $u(t,0^+)$, and $v(t,0^-)=v(t,0^+)$.
- L^1 contraction:

$$\frac{1}{2a}||(r_+-\bar{r}_+)(t,.)||_{L^1(\mathbb{R}_x^-)}+\frac{1}{2a}||(r_--\bar{r}_-)(t,\cdot)||_{L^1(\mathbb{R}_x^-)}$$
$$+||(u-\bar{u})(t,.)||_{L^1(\mathbb{R}_x^+)}\leq \| u_0-\bar{u}_0\|_{L^1(\mathbb{R}_x)},$$

and the estimates reduced from ϵ-system.

Matched asymptotic analysis

- Blow-up: $y = \frac{x}{\epsilon}$, $(\mathcal{U}^{\epsilon,\delta}, \mathcal{V}^{\epsilon,\delta})(t,y) = (u^{\epsilon,\delta}, v^{\epsilon,\delta})(t, \epsilon y)$,

$$a^2 \partial_y \mathcal{U}^{\epsilon,\delta} = \epsilon \lambda_{\epsilon,\delta}(\epsilon y)(f(\mathcal{U}^{\epsilon,\delta}) - \mathcal{V}^{\epsilon,\delta}) - \epsilon \partial_t \mathcal{V}^{\epsilon,\delta},$$

- For any $L > 0$, $TV_{(-L,L)}\Big(\mathcal{U}^{\epsilon,\delta}(t,\cdot)\Big) \le CL$
- Let $\delta \to 0$, then let $\epsilon \to 0$, $(\mathcal{U}^{\epsilon,\delta}(t,\cdot), \mathcal{V}^{\epsilon,\delta}(t,\cdot))$ converges to $(\mathcal{U}(t,\cdot), \mathcal{V}(t,\cdot))$ in $L^1_{\mathrm{loc}}(\mathbb{R}_y)$ for any given $t > 0$ and satisfy

$$\partial_y \mathcal{V} = 0, \quad a^2 \partial_y \mathcal{U} = \mathbf{1}_{\{y>0\}}(f(\mathcal{U}) - \mathcal{V}), \text{ in } \mathcal{D}'(-L, L)$$

- $\mathcal{U}(t,\cdot)$, $\mathcal{V}(t,\cdot)$ Lipschitz; admit $\mathcal{U}(t,\pm\infty)$, $\mathcal{V}(t,\pm\infty)$;
  $\mathcal{V}(t,y) = \mathcal{V}(t,+\infty) = \mathcal{V}(t,-\infty)$, $y \in \mathbb{R}$;
  $\mathcal{U}(t,y) = \mathcal{U}(t,-\infty)$, $y \le 0$; $f(\mathcal{U}(t,+\infty)) = \mathcal{V}(t,-\infty)$;

$$a^2 d_y \mathcal{U}(t,y) = f(\mathcal{U}(t,y)) - \mathcal{V}(t,+\infty), \quad y > 0,$$

  If $\mathcal{U}(t,\cdot)$ is not locally constant for $y > 0$, then it is strictly monotone for $y > 0$ and $f'(\mathcal{U}(t,+\infty)) < 0$.

Matched asymptotic analysis

- For a.e. $t > 0$, $\mathcal{V}$ and v perfectly match

$$\mathcal{V}(t,y) = v(t,0^-) = v(t,0^+), \quad \text{for all } y \in \mathbb{R}.$$

$\mathcal{U}$ and u are linked according to

$$\mathcal{U}(t,y) = u(t,0^-), \quad y < 0.$$

Defining $R_\pm(t,y) = a\mathcal{U}(t,y) \pm \mathcal{V}(t,y)$, then

$$\frac{1}{2}\Big(|R_+(t,y) - \ell| - |R_-(t,y) - h(\ell)|\Big) \\ \geq \operatorname{sgn}(u(t,0^+) - k)(f(u(t,0^+)) - f(k)), \quad y > 0,$$

for any $\ell \in R$ s.t. $k = (\ell + h(\ell))/2a$ verifies $|k| < ||u_0||_{L^\infty(\mathbb{R})}$.

Matched asymptotic analysis

- For a.e. $t > 0$, the following Kružkov inequalities hold

$$\operatorname{sgn}\big(\mathcal{U}(t,+\infty) - k\big)\big(f(\mathcal{U}(t,+\infty)) - f(k)\big)$$
$$\geq \operatorname{sgn}(u(t,0^+) - k)(f(u(t,0^+)) - f(k)),$$

for all $k \in \lfloor \mathcal{U}(t,+\infty), u(t,0^+) \rceil$. In particular, we have:

$$f(\mathcal{U}(t,+\infty)) = f(u(t,0^+)) = v(t,0^-).$$

hence

$$\mathcal{U}(t,+\infty) = u(t,0^+).$$

Theorem:

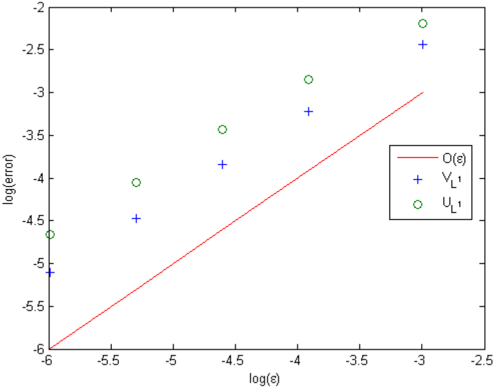
1. There exists a unique solution (u^ϵ, v^ϵ) to the original two-scale problem.
2. There exists a unique solution $(u, v) \in L^\infty \cap BV(\mathbb{R}^+ \times \mathbb{R})$ for the zero-ϵ limit system and satisfying L^1-contraction,

$$\frac{1}{2a} \| r_+^1(t,\cdot) - r_+^2(t,\cdot) \|_{L^1(\mathbb{R}_x^-)} + \frac{1}{2a} \| r_-^1(t,\cdot) - r_-^2(t,\cdot) \|_{L^1(\mathbb{R}_x^-)} + \| u^1(t,\cdot) - u^2(t,\cdot) \|_{L^1(\mathbb{R}_x^+)} \leq \| u_0^1 - u_0^2 \|_{L^1(\mathbb{R}_x)},$$

   for any two solutions (u^1, v^1) and (u^2, v^2). Here $r_\pm^i = (au^i \pm v^i)(t,x),\ i = 1, 2.$
3. For any $T > 0$, (u^ϵ, v^ϵ) converges strongly to (u, v) in $L^1((0,T) \times \mathbb{R})$.

Numerical example 1

Let $f(u^\epsilon) = \frac{1}{4}(e^{-u^\epsilon} - 1)$, with initial condition $u^\epsilon(x,0) = \sin(\pi x)^3$. In this case, $f'(u) < 0$, so there will be an interface layer at the interface $x = 0$.



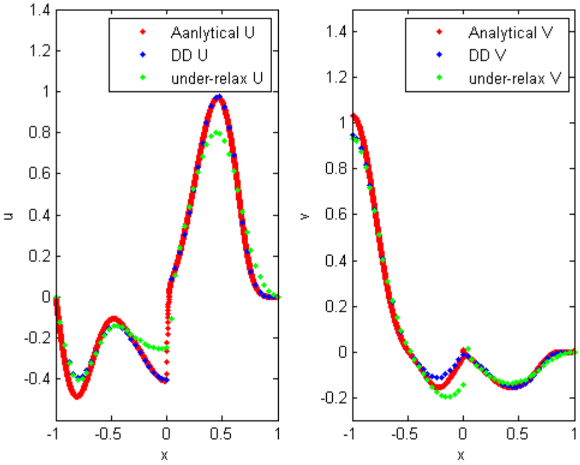


Figure: $T = 0.5, \ \Delta x = 0.04, \ \Delta t = 0.02.$

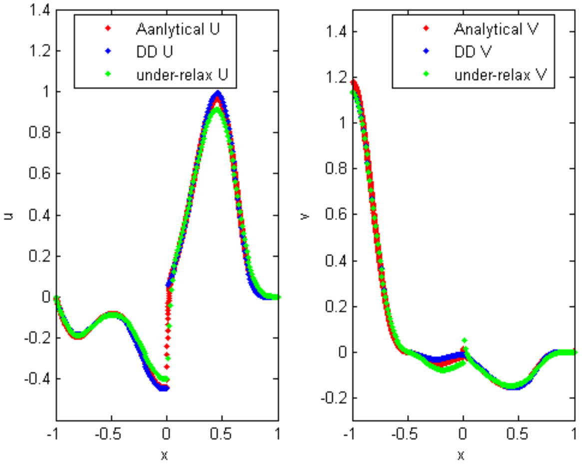


Figure: $T = 0.5$, $\Delta x = 0.01$, $\Delta t = 0.005$.

$$u^{\epsilon}(x,0) = \begin{cases} 1, & \text{if } -1 \le x \le 0.2; \\ -1, & \text{if } 0.2 < x \le 1. \end{cases}$$

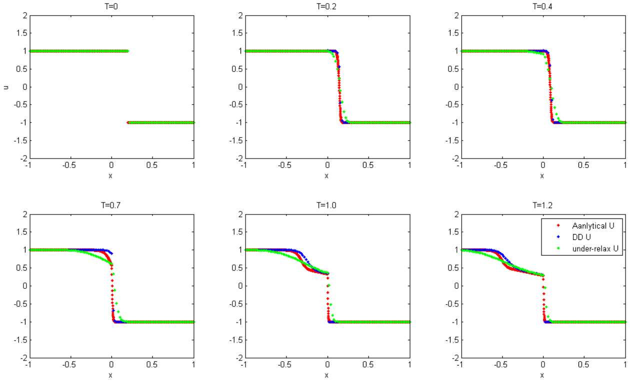


Figure: A shock from the right region passing through the interface.

$$u^{\epsilon}(x,0) = \begin{cases} -1, & \text{if } -1 \leq x \leq 0.2; \\ 1, & \text{if } 0.2 < x \leq 1. \end{cases}$$

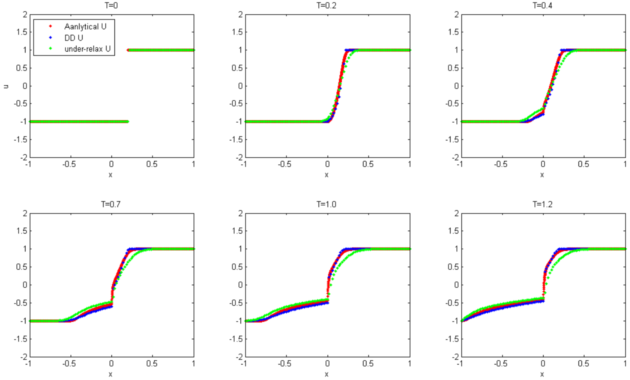
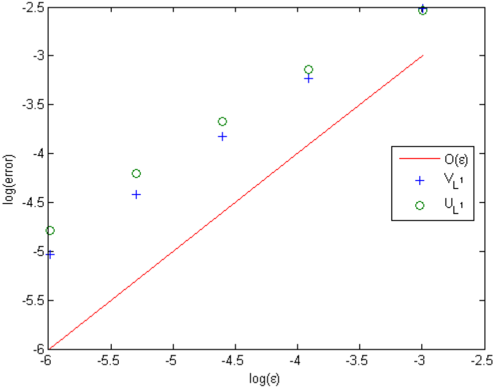


Figure: Rarefaction wave

Numerical example 2

Now we consider the case $f'(u) > 0$. Let $f(u^\epsilon) = \frac{1}{4}(e^{u^\epsilon} - 1)$, initial condition $u^\epsilon(x,0) = \sin(\pi x)^3$.



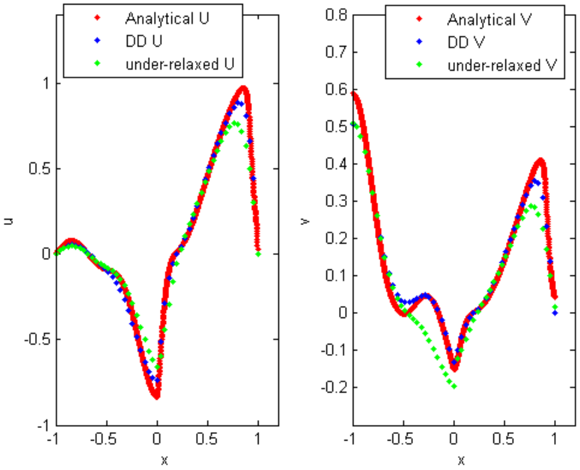


Figure: $T = 0.6$, $\Delta x = 0.04$, $\Delta t = 0.02$.

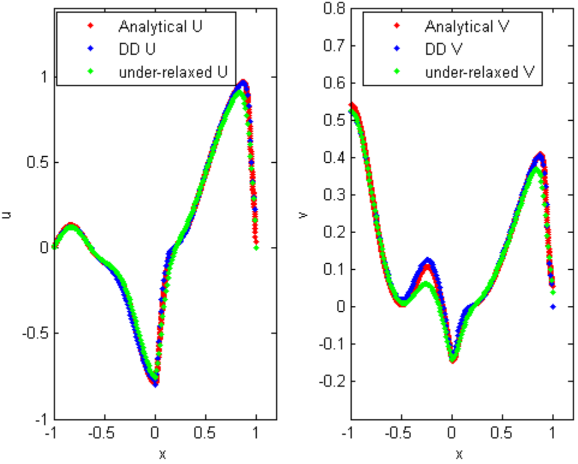


Figure: $T = 0.5$, $\Delta x = 0.01$, $\Delta t = 0.005$.

$$u^{\epsilon}(x,0) = \begin{cases} -1, & \text{if } -1 \le x \le -0.2; \\ 1, & \text{if } -0.2 < x \le 1. \end{cases}$$

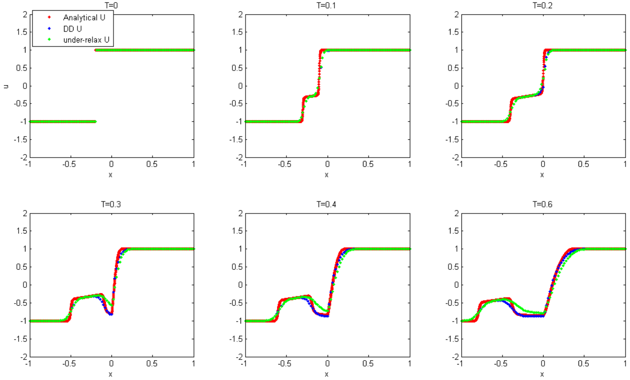


Figure: A contact discontinuity passing through the interface.

THANK YOU