Recent progress on the classical and quantum Kuramoto synchronization

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Outline

PROLOGUE

What is synchronization ?
Three main characters
Goal of this talk

INTERLOGUE

Kuramoto oscillators
Lohe oscillators
Schrödinger-Lohe oscillators

EPILOGUE

Take-Home message

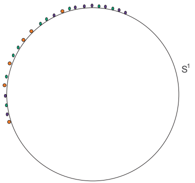
Outline

Synchronization

"Synchronization (=syn (same, common) + chronous (time))" is an adjustment of rhythms of oscillating objects due to their weak interaction.

A phase coupled model for synchronization

Consider a weakly coupled limit-cycle oscillators $\{x_i = e^{\sqrt{-1}\theta_i}\}$ rotating along S^1 with natural frequency Ω_i which is randomly drawn from some probability distribution $g(\Omega)$.



$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

This is the most well-known synchronization model like "the Burgers equation" for PDE

Outline

From SL oscillators to KM oscillators

◇ Coupled Stuart-Landau oscillators:

$z_i = z_i(t)$: state of the i-th LS oscillator at time t.

$$\dot{z}_i = (1 - |z_i|^2 + \mathrm{i}\Omega_i)z_i + \frac{K}{N}\sum_{j=1}^{N}(z_j - z_i).$$

For $K = 0$ and $z_i = r_i e^{\mathrm{i}\theta_i}$,

$$\dot{r}_i = r_i(1 - r_i^2), \quad \dot{\theta}_i = \Omega_i.$$

Unit circle $r_i = 1$ is the limit cycle for a decoupled system.

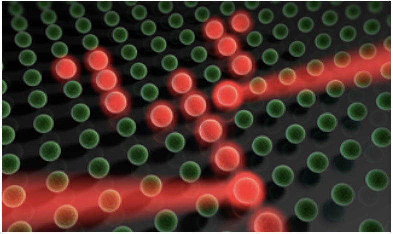
We set

$$z_j = e^{\mathrm{i}\theta_j}.$$

- The Kuramoto model (1975)

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

From Lohe oscillators to KM oscillators



- The Lohe model (2009):

$$i\dot{U}_i U_i^+ = H_i + \frac{iK}{2N}\sum_{j=1}^{N}\left(U_j U_i^+ - U_i U_j^+\right),$$

cf. Lohe, M. A.: Non-abelian Kuramoto model and synchronization. J. Phys. A: Math. Theor. 42, 395101-395126 (2009).

$U_i(t)$: $d \times d$ unitary matrix, H_i: $d \times d$ Hermitian matrix.

Why Lohe model ?

◇ (one-dimensional case $d = 1$),

$$U_i = e^{-i\theta_i}, \quad H_i = \Omega_i.$$

The Lohe model becomes the Kuramoto model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

◇ (zero coupling $K = 0$),

$$i\dot{U}_i U_i^{+} = H_i, \quad \text{or} \quad \dot{U}_i = -iH_i U_i.$$

This yields

$$U_i(t) = e^{-iH_i t}U_i(0), \quad t > 0.$$

From Schrödinger-Lohe to Kuramoto

- The Schrödinger-Lohe model (2010):

Choi-H '14, Cho-Choi-H '16, H-Huh '16, Antonelli-Marcati '17, $\cdots$

$$\mathrm{i}\partial_t \psi_i = -\Delta \psi_i + V_i(x)\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(\psi_k - \langle \psi_i, \psi_k\rangle \psi_i\right).$$

We set

$$V_i(x) = \Omega_i : \text{constant}, \qquad \psi_i(t,x) = e^{-\mathrm{i}\theta_i},$$

to obtain the Kuramoto model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k - \theta_i).$$

Outline

Goal of this talk

I will briefly discuss some emergent properties of aforementioned models, e.g., existence, size and structure of relative equilibria

- The Kuramoto model

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

- The Lohe model

$$\mathrm{i}\dot{U}_i U_i^{+} = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\Big(U_i U_j^{+} - U_j U_i^{+}\Big).$$

- The Schrödinger-Lohe model

$$\mathrm{i}\partial_t\psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(\psi_k - \langle\psi_i, \psi_k\rangle\psi_i\right).$$

Outline

Preliminary analysis

- Equilibrium equation:

$$\Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) = 0, \quad i = 1, \cdots, N.$$

- Simple observations:

1. If there exists an eqilibrium $\Theta = (\theta_1, \cdots, \theta_N)$,

$$\sum_i \Omega_i + \frac{K}{N}\sum_{i,j=1}^{N}\sin(\theta_j - \theta_i) = \sum_i \Omega_i = 0.$$

Thus, we need to consider "relative equilibrium", i.e., equilibrium in the rotating frame with velocity $\Omega_c = \frac{1}{N}\sum \Omega_i$.

2. Note that

$$\Big|\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)\Big| \le K.$$

3. For $K = 0$,

$$\theta_i(t) = \theta_i(0) + t\Omega_i, \quad i = 1, \cdots, N \quad \text{completely integrable}$$

Hence, to continue the story of equilibria, we need to consider

$$\sum_i \Omega_i = 0, \quad K \geq C|\Omega_i|, \quad i = 1, \cdots, N.$$

- **Definition.**

Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a (strong) phase-locked state(PLS) if and only if

$$|\theta_i(t) - \theta_j(t)| = \theta_{ij}^{\infty}, \quad t \geq 0.$$

Clearly, equilibrium solutions are phase-locked states.

$$\Omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i) = 0, \quad \sum \Omega_i = 0.$$

Emergent dynamics of KM osillators

- The Kuramoto model: Matlab simulations by Jinyeong Park

$$N = 50, \Omega_i \in [-1, 1], K = 0.8$$

$N = 50, \Omega_i \in [-1, 1], K = 2.2$

Numerical simulations seem to suggest that "phase-locked states" emerge from "all (generic) initial phase configurations" in a large coupling regime.

Complete synchronization problem(CSP)

- Problem Statement: For a given generic initial phase configuration Θ_0 and sufficiently large K, show that

$$\lim_{t\to\infty} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| = 0, \quad \forall\, i \neq j.$$

Order parameter approach

- Order parameters:

$$Re^{\mathrm{i}\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{\mathrm{i}\theta_j}.$$

Note that R is always bounded, i.e., $0 \le R \le 1$ and

$$\text{KM} \quad \Longleftrightarrow \quad \dot{\theta}_i = \Omega_i - KR\sin(\theta_i - \phi).$$

- Dynamics of order parameters: H-Slemrod '11

$$\begin{aligned} \dot{R} &= -\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)\Big(\Omega_j - KR\sin(\theta_j-\phi)\Big), \\ \dot{\phi} &= \frac{1}{RN}\sum_{j=1}^{N}\cos(\theta_j-\phi)\Big(\Omega_j - KR\sin(\theta_j-\phi)\Big). \end{aligned}$$

Final Resolution on CSP

- **Theorem** H-Kim-Ryoo '16

Suppose that the following conditions hold.

$$R^0 := \left|\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(0)}\right| > 0, \quad \theta_{i0} \neq \theta_{j0}, \quad 1 \leq i,j \leq N, \qquad \max_{1\leq i\leq N} |\Omega_i| \leq L < \infty.$$

Then, there exists a large coupling strength $K_\infty > 0$ such that, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ to the Kuramoto model with initial data Θ_0 and $K \geq K_\infty$, there exists a unique phase-locked state Θ^∞ such that

$$\lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_\infty = 0,$$

where the norm $||\cdot||_\infty$ is the standard ℓ^∞-norm in ^{N}.

Brief outline of the proof

- The Kuramoto model:

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad t > 0.$$

subject to following conditions:

$$\sum_{i=1}^{N}\Omega_i = 0, \qquad \sum_{i=1}^{N}\theta_{i0} = 0, \qquad \theta_{i0} \in [-\pi, \pi), \quad 1 \le i \le N.$$

- Scaled Kuramoto model:

$$\frac{d\theta_i}{d\tau} = \tilde{\Omega}_i + \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \quad \tau = Kt, \quad \tilde{\Omega}_i := \frac{\Omega_i}{K}.$$

As far as large-time dynamics is concerned,

$$\Omega_i : \text{ fixed } K \gg 1 \quad \Longleftrightarrow \quad K = 1, \ |\Omega_i| \ll 1.$$

Key ingredients for the proof

- The first ingredient van Hemmen-Wreszinski '93, Dong-Xue '13

KM is a gradient flow with analytical potential

$$\dot{\Theta} = -\nabla_{\Theta} V(\Theta), \quad t > 0,$$

where

$$V[\Theta] := -\sum_{k=1}^{N} \Omega_k \theta_k + \frac{K}{2N} \sum_{k,l=1}^{N} \big(1 - \cos(\theta_k - \theta_l)\big).$$

- **Theorem** Dong-Xue '13, H-Li-Xue '13

Let $\Theta = \Theta(t)$ be a uniformly bounded global solution to the KM in R^N:

$$\sup_{t>0} ||\Theta(t)||_\infty < \infty.$$

Then, there exists a phase locked state Θ^∞ such that

$$\lim_{t\to\infty} ||\Theta(t) - \Theta^\infty||_\infty = 0, \quad \lim_{t\to\infty} ||\dot{\Theta}(t)||_\infty = 0.$$

cf. Identical Kuramoto oscillators (Dong-Xue '13)

- Formation of a chain of black-holes

- **Lemma**

Suppose that the initial configuration Θ_0 satisfies

$$\theta_{i0} \in [-\pi, \pi), \quad 1 \le i \le N,$$

and let n_0, ℓ, and K satisfy

$$n_0 \in_+ \cap \Big(\frac{N}{2}, N\Big], \quad \ell \in \Big(0, 2\cos^{-1}\frac{N - n_0}{n_0}\Big),$$

$$\max_{1 \le i,j \le n_0} |\theta_{i0} - \theta_{j0}| < \ell, \quad K > \frac{D(\Omega)}{\frac{n_0}{N}\sin\ell - \frac{2(N-n_0)}{N}\sin\frac{\ell}{2}}.$$

Let Θ be a global solution to KM. Then, we have

$$\sup_{t \ge 0} D(\Theta(t)) \le 4\pi + \ell.$$

Slightly improved result

- **Theorem** H-Ryoo '17

Suppose that Θ^0 and K satisfy

$$R_0 := R(\Theta^0) > 0 \qquad K > \frac{96}{\sqrt{7}} \frac{D(\Omega)}{R_0^2} \approx 32.2846 \frac{D(\Omega)}{R_0^2}. \qquad (1)$$

Then, Kuramoto flow $\Theta = \Theta(t)$ issued from Θ^0 tends to the phase-locked state asymptotically.

Finiteness result

What is the cardinality of the set of all phase-locked states ?

- **Example** (Identical oscillators)

For $\mu \in [0, 2\pi]$ and $N \geq 4$, we define a state Θ^μ:

$$\theta_k^\mu := \begin{cases} \frac{2k\pi}{N-2} + \mu, & k = 1, \ldots, N-2, \\ 0, & k = N-1, \\ \pi, & k = N. \end{cases}$$

Note that distinct μ's clearly yield non-equivalent phase locked states, and

$$R = \frac{1}{N}\Big| \sum_{k=1}^{N-2} e^{\mathrm{i}\frac{2k\pi}{N-2}} + e^{\mathrm{i}0} + e^{\mathrm{i}\pi} \Big| = 0.$$

We have a continuum of phase-locked states for the KM for identical oscillators. Thus, for finiteness, we need to restrict $R > 0$.

- **Theorem**

  1. For identical Kuramoto oscillators with $K > 0$, every standard phase-locked state is a permutation of

  $$(0, \dots, 0, \pi, \dots, \pi),$$

  where the number of 0's is greater than $N/2$, and the π's do not necessarily exist. Thus, we have

  $$|\mathcal{P}| = \begin{cases} 2^{N-1}, & N \text{ odd}, \\ 2^{N-1} - \frac{1}{2}\binom{N}{N/2}, & N \text{ even}, \end{cases}$$

  2. There are at most 2^N phase locked states for the non-identical case, i.e.,

  $$|\mathcal{P}| \le 2^N.$$

Mathematical Challenge I

- The Kuramoto model with frustration

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_j - \theta_i + \alpha_{ij}), \quad i = 1, \cdots, N.$$

cf. No gradient flow structure !!!,

◇ Fact: For the case

$$\alpha_{ij} = \alpha_{ji}, \quad |\alpha_{ij}| < \frac{\pi}{2},$$

there exists a set of admissible initial data leading to the complete synchronization (H-Kim-Li ’14), but numerical simulations suggests

- Simulatiions by Dongnam Ko.

No proof in full generality so far !

Identical oscillators

- **Theorem** (H-Ko-Zhang '17)

Let Θ be a solution with initial data Θ^0 satisfying conditions:

$$|\alpha| < \arctan\left(\frac{1}{2\sqrt{N}}\right), \quad K > 0, \quad D(\Omega) = 0,$$
$$R^0 > \frac{\cos\alpha + \sin|\alpha|}{1 + \cos\alpha} + \frac{1}{N}, \quad \theta_j^0 \neq \theta_k^0 \text{ for all } j \neq k.$$

Then the following dichotomy holds.

1. Either there exists $T_0 \geq 0$ such that for all $t \geq T_0$
$$D(\Theta(t)) \leq D(\Theta(T_0))e^{-\Lambda(\alpha, T_0, \mathcal{N})t}.$$
2. or there exists $T_1 \geq 0$ and some $j_0 \in \{1, \cdots, N\}$ such that for all $t \geq T_1$
$$\left|(\theta_{j_0}(t) - \alpha) + (\theta_j(t) - \alpha) - \pi\right| \leq D_{\tilde{\mathcal{I}}}(\Theta(T_1))e^{-\Lambda(\alpha, T_1, \tilde{\mathcal{I}})t}, \quad \text{for all } j \neq j_0,$$
$$\left|\theta_j(t) - \theta_k(t)\right| \leq D_{\tilde{\mathcal{I}}}(\Theta(T_1))e^{-\Lambda(\alpha, T_1, \tilde{\mathcal{I}})t}, \quad \text{for all } j, k \neq j_0.$$

Outline

The Lohe model

- The Lohe model: Max. Lohe '09

$$\mathrm{i}\dot{U}_i U_i^+ = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(U_i U_j^+ - U_j U_i^+\right).$$

$\diamond$ For $d = 2$,

$$U_i := e^{-i\theta_i}\left(\sum_{k=1}^{3} x_i^k \sigma_k + x_i^4 I_2\right), \quad H_i = \sum_{k=1}^{3} \omega_i^k \sigma_k + \nu_i I_2,$$

where I_2 and σ_i are the identity matrix and Pauli matrices, respectively, defined by

$$I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

After some algebraic manipulations, we obtain $5N$ equations for the angles θ_i and the 4-vectors x_i:

$$
\begin{aligned}
||x_i||^2\dot{\theta}_i &= \nu_i + \frac{K}{N}\sum_{k=1}^{N}\sin(\theta_k-\theta_i)\langle x_i, x_k\rangle, \quad 1\le i\le N,\\
||x_i||^2\dot{x}_i &= \Omega_i x_i + \frac{K}{N}\sum_{k=1}^{N}\cos(\theta_k-\theta_i)(||x_i||^2 x_k - \langle x_i, x_k\rangle x_i),
\end{aligned}
$$

where Ω_i is a real 4×4 antisymmetric matrix given by

$$
\Omega_i := \begin{pmatrix} 0 & -\omega_i^3 & \omega_i^2 & -\omega_i^1 \\ \omega_i^3 & 0 & -\omega_i^1 & -\omega_i^2 \\ -\omega_i^2 & \omega_i^1 & 0 & -\omega_i^3 \\ \omega_i^1 & \omega_i^2 & \omega_i^3 & 0 \end{pmatrix}, \quad 1\le i\le N.
$$

cf. Choi-H: sufficient conditions for CPS

Beauty is a Truth

$$i\dot{U}_i U_i^{+} = H_i - \frac{iK}{2N}\sum_{j=1}^{N}\left(U_i U_j^{+} - U_j U_i^{+}\right),$$

- Invariance under Lohe flow

1. Invariance of $U_i U_i^{\dagger}$:

$$\frac{d}{dt}(U_i U_i^{\dagger}) = 0, \quad t > 0.$$

2. Invariance under the right-multiplication by a unitary matrix: if $L \in U(d)$ and $V_i = U_i L$, then V_i satisfies

$$\mathrm{i}\dot{V}_i V_i^{\dagger} = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\left(V_i V_j^{\dagger} - V_j V_i^{\dagger}\right), \quad V_i(0) = U_{i0}L.$$

- **Definition**: (Phase-locked states)
  1. $\{U_i(t)\}$ is a *phase-locked state* if and only if $U_i(t)U_j(t)^\dagger$ is constant for all i, j and $t \geq 0$.
  2. The Lohe flow $\{U_i(t)\}$ achieves *asymptotic phase-locking* if and only if the limit of $U_i U_j^\dagger$ as $t \to \infty$ exists.

In classical Kuramoto oscillators, phase-locked states takes the following form:

$$\theta_i(t) = \theta_i^\infty + \Omega_c t, \qquad \Omega_c := \frac{1}{N}\sum_{j=1}^{N} \Omega_i.$$

- **Proposition**: The phase-locked states(PLS) $\{U_i\}$ for the Lohe flow take the form of

$$U_i = U_i^\infty e^{-\mathrm{i}\Lambda t},$$

where $U_i^\infty \in \mathcal{U}(d)$ and Λ is the constant $d \times d$ Hermitian matrix satisfying the relation

$$U_i^\infty \Lambda U_i^{\infty\dagger} = H_i - \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(U_i^\infty U_k^{\infty\dagger} - U_k^\infty U_i^{\infty\dagger}\right).$$

Main strategy

1. Step A: Introduce an ensemble diameters:

$$D(U) := \max_{1 \le i,j \le N} \|U_i - U_j\|, \qquad D(H) := \max_{1 \le i,j \le N} \|H_i - H_j\|.$$

2. Step B: Derive a Gronwall type differential inequality for $D(U)$:

$$\left| \frac{d}{dt} D(U)^2 + 2K D(U)^2 \right| \le 2D(H)D(U) + K D(U)^4 \quad \text{a.e. } t \in (0, \infty).$$

3. Step C: Establish the existence of PLSs. For example, for identical oscillators with $D(H) = 0$,

$$\lim_{t \to \infty} D(U(t)) = 0.$$

- **Theorem**: (H-Ryoo '16, J. Stat. Phys.) Suppose that K and U^0 satisfy

$$K > \frac{54}{17} D(\Omega), \quad U^0 \in B(0, \alpha) \quad \text{for some } \alpha > 0$$

Then, we have

1. $\{U_i\}$ achieves asymptotic phase-locking:

$$\lim_{t \to \infty} U_i U_j^\dagger$$

   converges exponentially fast, with exponential rate bounded above by $-K(1 - 3\alpha_1)$.

2. There exists a phase-locked state $\{V_i\}$ and $L \in \mathcal{U}(d)$ such that

$$\lim_{t \to \infty} \|U_i - V_i L\| = 0$$

   converges exponentially fast.

3. All phase-locked states $\{V_i\}$ with diameter $D(V) < \alpha_2$ are right multiplication of this phase-locked state by a unitary matrix.

cf. For $n = 2$, there were some preliminary works by Choi and H.

Mathematical Challenges II

- The Lohe model

$$\mathrm{i}\dot{U}_i U_i^{+} = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\Big(U_i U_j^{+} - U_j U_i^{+}\Big).$$

1. Is the Lohe model gradient flow for suitable potential ?
2. Can the Lohe model derivable from more higher-dimensional dynamical systems ?
3. Can we introduce a Lohe like system for a set of tensor networks ?
4. What is the structure of the emergent phase-locked states ?

Outline

The Schrödinger-Lohe oscillators

- The Schrödinger-Lohe model

$$\mathrm{i}\partial_t \psi_i = -\Delta \psi_i + V_i(x)\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N} a_{ik}\left(\psi_k - \langle \psi_i, \psi_k \rangle \psi_i\right).$$

- Main question

Find sufficient conditions leading to the synchronization of wave functions, i.e.,

$$\exists \;\; d_{ij} := \lim_{t\to\infty} ||\psi_i(\cdot,t) - \psi_j(\cdot,t)||_{L^2}, \quad 1 \le i,j \le N.$$

Lyapunov functional approach

$$a_{ik} = 1, \quad D(\Psi) = \max_{i,j} ||\psi_i - \psi_j||_{L^2}, \quad D(\mathcal{V}) := \max_{i,j} ||V_i - V_j||_{L^\infty}.$$

- **Theorem**: Choi-H '14, Cho-Choi-H '16

1. Suppose that the coupling strength and initial data satisfy

$$K > 0, \quad V_j = V, \quad ||\psi_j^0||_2 = 1, \quad 1 \le j \le N, \quad D(\Psi^0) < \frac{1}{2}.$$

Then, the diameter $D(\Psi)$ satisfies

$$D(\Psi(t)) \le \frac{D(\Psi^0)}{D(\Psi^0) + (1 - 2D(\Psi^0))e^{Kt}}, \quad t \ge 0.$$

2. Suppose that the coupling strength and initial data satisfy

$$K > 54 D(\mathcal{V}), \quad ||\psi_{i0}|| = 1, \quad j = 1, \cdots, N, \qquad D(\Psi_0) \ll 1.$$

Then, we can achieve practical synchronization:

$$\lim_{K\to\infty} \limsup_{t\to\infty} D(\Psi(t)) = 0.$$

Dynamical systems approach

• Finite-dimensional reduction from the SL model Define correlation functions

$$h_{ij}(t) := \langle \psi_i(t), \psi_j(t) \rangle = \int \psi_i(x)\overline{\psi_j(x)}dx, \quad 1 \leq i, j \leq N.$$

Then, for congruent one-body potentials

$$V_i = V + \Omega_i, \quad 1 \leq i \leq N, \quad \Omega_i \in R$$

h_{ij} satisfies

$$\dot{h}_{ij} = -\mathrm{i}\Omega_{ij} h_{ij} + \frac{\kappa}{2N} \sum_{k=1}^{N} \Big[a_{ik}(h_{kj} - h_{ik} h_{ij}) + a_{jk}(h_{ik} - h_{kj} h_{ij}) \Big].$$

cf. $a_{ik} = 1$: H-Huh '16, Antonelli-Marcati '17

A two-oscillator system

We set

$$h := h_{12} \quad \text{and} \quad \Omega := \Omega_{12}.$$

Then h satisfies

$$\frac{dh}{dt} = -K\left[\left(h + \mathrm{i}\frac{\Omega}{2K}\right)^2 + \frac{\omega^2}{4K^2} - 1\right], \quad t > 0.$$

We need to consider the following three cases:

$$K > \frac{\Omega}{2}, \qquad K = \frac{\Omega}{2}, \qquad K < \frac{\Omega}{2}.$$

- Case 1 ($K > \frac{\Omega}{2}$): Two equilibria

$$h_{\infty,-} := -\frac{1}{2}\sqrt{4 - \left(\frac{\Omega}{K}\right)^2} - \mathrm{i}\frac{\Omega}{2K} \quad \text{and} \quad h_{\infty,+} := \frac{1}{2}\sqrt{4 - \left(\frac{\Omega}{K}\right)^2} - \mathrm{i}\frac{\Omega}{2K}.$$

$$h(t) = \frac{h_{\infty,+}(h^0 - h_{\infty,-}) + h_{\infty,-}(h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\Omega^2}t}}{h^0 - h_{\infty,-} - (h^0 - h_{\infty,+})e^{-\sqrt{4K^2-\Omega^2}t}}.$$

Thus, it is easy to see that for any initial data $h^0 \neq h_{\infty,-}$, we have

$$h(t) \to h_{\infty,+} \;\; \text{as} \;\; t \to \infty.$$

- Case 2 ($K = \frac{\Omega}{2}$): the unique equilibrium:

$$h_\infty := -\mathrm{i},$$

and explicit solution:

$$h(t) = \frac{h^0 - \mathrm{i}(h^0 + \mathrm{i})Kt}{1 + (h^0 + \mathrm{i})Kt}.$$

Thus, we have

$$h(t) \to h_\infty \quad \text{as} \quad t \to \pm\infty.$$

- Case 3 ($K < \frac{\Omega}{2}$): the equilibria:

$$h_{\infty,-} = \mathrm{i}\Big(-\frac{\Omega}{2K} - \frac{1}{2}\sqrt{\Big(\frac{\Omega}{K}\Big)^2 - 4}\Big), \quad h_{\infty,+} = \mathrm{i}\Big(-\frac{\Omega}{2K} + \frac{1}{2}\sqrt{\Big(\frac{\Omega}{K}\Big)^2 - 4}\Big).$$

Explicit periodic solution:

$$h(t) = \frac{h^0 \cos\left(\frac{t\sqrt{\Omega^2-4K^2}}{2}\right) - \frac{K}{\sqrt{\Omega^2-4K^2}}\left(\frac{\mathrm{i}\Omega h^0}{K} - 2\right)\sin\left(\frac{t\sqrt{\Omega^2-4K^2}}{2}\right)}{\cos\left(\frac{t\sqrt{\Omega^2-4K^2}}{2}\right) + \frac{K}{\sqrt{\Omega^2-4K^2}}\left(2h^0 + \frac{\mathrm{i}\Omega}{K}\right)\sin\left(\frac{t\sqrt{\Omega^2-4K^2}}{2}\right)}.$$

Bifurcation at $K = \frac{\Omega}{2}$.

Mathematical Challenges III

Establish the bifurcation (phase-transition phenomenon) for many-body system

Identical potentials $V_i = V$

Define a functional:

$$\mathcal{H}(t) := \max_{1 \le i \le N} \sum_{k=1}^{N} |1 - h_{ik}(t)|.$$

- **Lemma**: Assume that $a_{ik} > 0$. Then, the functional $\mathcal{H}$ satisfies

$$\frac{d}{dt}\mathcal{H}(t) \le -2K\left(A_m - D(\mathcal{A})\right)\mathcal{H}(t) + \frac{2Ka_\infty}{N}\mathcal{H}(t)^2,$$

where

$$A_m : \min_i \frac{1}{N}\sum_{k=1}^{N} a_{ik}, \qquad D(\mathcal{A}) := \max_{i,j}\max_k |a_{ik} - a_{jk}|, \quad a_\infty := \max_{i,k} a_{ik} > 0.$$

- **Theorem**: H-Huh-Kim '17 Suppose that initial data h_{ij}^0 satisfies

$$\max_{1\le i\le N} \frac{1}{N}\sum_{j=1}^{N} |1-h_{ij}^0| \le \frac{A_m - D(\mathcal{A}))}{a_\infty}.$$

Then, we have

$$|1-h_{ij}(t)| \le e^{-Ct}.$$

Outline

Take-Home message

So far, I have discussed some emergent properties of

- The Kuramoto model

$$\frac{d\theta_i}{dt} = \Omega_i + \frac{K}{N}\sum_{i=1}^{N}\sin(\theta_j - \theta_i), \quad i = 1, \cdots, N.$$

- The Lohe model

$$\mathrm{i}\dot{U}_i U_i^{+} = H_i - \frac{\mathrm{i}K}{2N}\sum_{j=1}^{N}\Big(U_i U_j^{+} - U_j U_i^{+}\Big).$$

- The Schrodiger-Lohe model

$$\mathrm{i}\partial_t\psi_i = -\Delta\psi_i + V_i(x)\psi_i + \frac{\mathrm{i}K}{2N}\sum_{k=1}^{N}\left(\psi_k - \langle\psi_i, \psi_k\rangle\psi_i\right), \quad (x,t) \in R^d \times R_+.$$

Good News "Never ending story"

There are still lots of "Beauty and Mystery" to be explored in the synchronization models

Ha, S.-Y. et al "Collective synchronization of classical and quantum oscillators ". EMS Surveys in Mathematical Sciences.**3**, 2016, pp. 209-267.

Thank you for your attention !!!