Multi-agent System Learning and Control

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European Research Council

"Collective dynamics, control and imaging"
Institute for Theoretical Studies (ETHZ) – April 28, 2017

Joint work with M. Bongini, M. Caponigro, M. Hansen, D. Kalise, M. Maggioni, B. Piccoli, F. Rossi, F. Solombrino, E. Trelat

What is a self-organizing system?

Social dynamics

We consider large particle systems of form:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (S + K * \mu_N)(x_i, v_i), \\ i = 1, \dots N, \quad t \in [0, T], \\ \text{where, } \mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{(x_j, v_j)}, \end{cases}$$

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"Social forces" encoded in S and K :

- Repulsion-attraction
- Alignment
- Self-propulsion/friction
- ....

Possible noise/uncertainty by adding stochastic terms.

Mills in nature and in our simulations. J. A. Carrillo, M. Fornasier, G. Toscani, and F. Vecil, *Particle, kinetic, hydrodynamic models of swarming*, Birkhäuser 2010.

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Possible noise/uncertainty by adding stochastic terms.

Understanding how superposition of re-iterated binary "social forces" yields global self-organization.

Models for social dynamics - Cucker-Smale

For $S \equiv 0$ and $K(x, v) = a(\|x\|)(-v)$ we get the Cucker-Smale model

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{1}{N}\sum_{j=1}^{N} a(\|x_i - x_j\|)(v_j - v_i), \quad i = 1, \dots N, \end{cases}$$

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If $\beta < \frac{1}{2}$, alignment occurs everytime

If $\beta \geq \frac{1}{2}$, alignment occurs only for certain initial data

Models for social dynamics - Cucker-Dong

For $S(x,v) = -bv$ and $K(x,v) = a(\|x\|^2)(-x) - f(\|x\|^2)(-x)$ we get the Cucker-Dong model

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = -bv_i + \frac{1}{N}\sum_{j=1}^{N}(a-f)(\|x_i - x_j\|^2)(x_j - x_i), \quad i = 1, \dots N, \end{cases}$$

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Usually, $a(r) = \frac{1}{(1+r^2)^\beta}$ and $f(r) = \frac{1}{r^\delta}$. A sufficient condition for self-organization is given by the quantities

$$E(t) = \sum_{i=1}^N \|v_i(t)\|^2 + \frac{1}{2N}\sum_{i<j}^N \left(\int_0^{\|x_i(t)-x_j(t)\|^2} a(r)dr + \int_{\|x_i(t)-x_j(t)\|^2}^{+\infty} f(r)dr \right)$$

$$\theta = \frac{N-1}{2}\int_0^{+\infty} a(r)dr$$

Models for social dynamics - Cucker-Dong

If $E(0) \leq \theta$, cohesiveness occurs everytime

If $E(0) > \theta$, cohesiveness occurs only for certain initial data

Models for social dynamics - D'Orsogna-Bertozzi et al.

For $S(x,v) = (\alpha - \beta\|v\|^2)v$ and $K(x,v) = -\nabla U(\|x\|)\frac{x}{\|x\|}$ we get the D'Orsogna-Bertozzi et al. model

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (\alpha - \beta\|v_i\|^2)v_i - \frac{1}{N}\sum_{j=1}^{N} \nabla U(\|x_i - x_j\|)\frac{x_i - x_j}{\|x_i - x_j\|}, \quad i = 1, \dots N, \end{cases}$$

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Usually, $U(r) = -C_A e^{-r/\ell_A} + C_R e^{-r/\ell_R}$.

If $\frac{C_R}{C_A}\left(\frac{\ell_R}{\ell_A}\right)^d < 1$*, cristalline structures appear*

If $\frac{C_R}{C_A}\left(\frac{\ell_R}{\ell_A}\right)^d \geq 1$*, mill patterns arise*

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- However, it is common experience that coherence in a homophilious society can be lost, leading sometimes to dramatic consequences, questioning strongly the role and the effectiveness of governments.

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Question: can a government endowed with limited resources rescue/stabilize a society by minimal interventions? Which ones?

A parametric model of homo-to-hetero-philia

The Cucker-Smale model:

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where $a(t) := a_\beta(t) = \frac{1}{(1+t^2)^\beta}$, $\beta > 0$ governs the rate of communication.

[1] The *Laplacian* L of A is given by $L = D - A$, with $D = \mathrm{diag}(d_1, \dots, d_N)$ and $d_k = \sum_{j=1}^{N} a_{kj}$

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$$\begin{cases} \dot{x} = v \\ \dot{v} = -L_x v \end{cases}$$

where L_x is the Laplacian of the matrix[1] $(a(\|x_j - x_i\|)/N)_{i,j=1}^N$ and depends on x.

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- Mean-velocity conservation: $\frac{d}{dt}\bar{v}(t) = \frac{1}{N}\sum_{i=1}^N \dot{v}_i(t) = \frac{1}{N^2}\sum_{i=1}^N \sum_{j=1}^N \frac{v_j - v_i}{(1+\|x_j-x_i\|^2)^\beta} \equiv 0.$

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Conditional consensus emergence for a generic communication rate $a(\cdot)$

Consider the symmetric bilinear form

$$B(u, v) = \frac{1}{2N^2} \sum_{i,j} \langle u_i - u_j, v_i - v_j \rangle = \frac{1}{N} \sum_{i=1}^{N} \langle u_i, v_i \rangle - \langle \bar{u}, \bar{v} \rangle,$$

and

$$X(t) = B(x(t), x(t)), \quad V(t) = B(v(t), v(t)).$$

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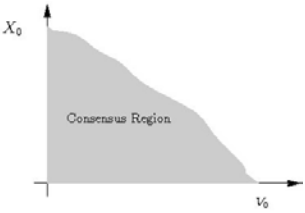

Theorem (Ha-Ha-Kim)

Let $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ be such that $X_0 = B(x_0, x_0)$ and $V_0 = B(v_0, v_0)$ satisfy

$$\gamma(X_0) := \sqrt{N}\int_{\sqrt{NX_0}}^{\infty} a(\sqrt{2}r)dr > \sqrt{V_0}\,.$$

Then the solution with initial data (x_0, v_0) tends to consensus.

Non-consensus events

Consider $\beta = 1$ and $x(t) = x_1(t) - x_2(t)$, $v(t) = v_1(t) - v_2(t)$ relative pos. and vel. of two agents on the line:

$$\begin{cases} \dot{x} = v \\ \dot{v} = -\dfrac{v}{1 + x^2} \end{cases}$$

with initial conditions $x(0) = x_0$ and $v(0) = v_0 > 0$.

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By direct integration

$$v(t) = -\arctan x(t) + \arctan x_0 + v_0.$$

Hence, if $\arctan x_0 + v_0 > \pi/2 + \varepsilon$ we have

$$v(t) > \pi/2 + \varepsilon - \arctan x(t) > \varepsilon, \quad \forall t \in \mathbb{R}_+.$$

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Admissible controls: measurable functions $u = (u_1, \ldots, u_N) : [0, +\infty) \to \mathbb{R}^N$ such that $\sum_{i=1}^N \|u_i(t)\| \leq M$ for every $t > 0$, for a given constant M:

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

for $i = 1, \ldots, N$, and $x_i \in \mathbb{R}^d$, $v_i \in \mathbb{R}^d$.

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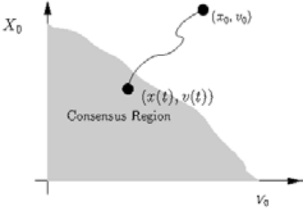

Our aim is then to find admissible controls steering the system to the consensus region.

Total control

Proposition (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and* $u : [0, T] \to (\mathbb{R}^d)^N$*, with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated solution reaches the consensus region.*

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Proof.

Consider a solution of system with initial data (x_0, v_0) associated with a feedback control $u = -\alpha(v - \bar{v})$, with $0 < \alpha \leq M/(N\sqrt{B(v_0, v_0)})$.

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\frac{d}{dt}V(t) &= \frac{d}{dt}B(v(t), v(t)) \\
&= -2B(L_x v(t), v(t)) + 2B(u(t), v(t)) \\
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Therefore $V(t) \leq e^{-2\alpha t} V(0)$ and $V(t)$ tends to 0 exponentially fast as $t \to \infty$. Moreover $\sum_{i=1}^N \|u_i\| \leq M$. $\square$

More economical choices?

We wish to make

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the smallest possible and use the minimal amount of intervention:

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the smallest possible and use the minimal amount of intervention: minimize $B(u(t), v(t))$ with additional sparsity constraints.

Greedy sparse control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every initial condition $(x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ *and* $M > 0$ *there exist* $T > 0$ *and a sparse control* $u : [0, T] \to (\mathbb{R}^d)^N$*, with* $\sum_{i=1}^N \|u_i(t)\| \leq M$ *for every* $t \in [0, T]$ *such that the associated AC solution reaches the consensus region.*

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$$\min B(v, u) + \frac{\gamma(x)}{N} \sum_{i=1}^N \|u_i\| \quad \textit{subject to } \sum_{i=1}^N \|u_i\| \leq M \,,$$

where $\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2}r) dr$ *threshold by Ha-Ha-Kim.*

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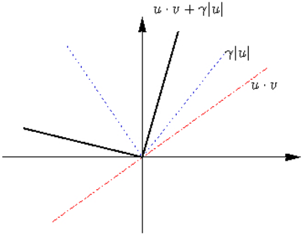
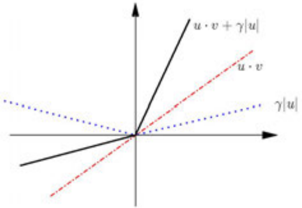
where $\gamma(x) = \sqrt{N} \int_{\sqrt{NB(x,x)}}^{\infty} a(\sqrt{2}r)dr$ *threshold by Ha-Ha-Kim.*
This choice of the control makes $V(t) = B(v(t), v(t))$ *vanishing in finite time, hence there exists* T *such that* $B(v(t), v(t)) \leq \gamma(x)^2$*,* $t \geq T$*.*

Geometrical interpretation in the scalar case



For $|v| \leq \gamma$ the minimal solution $u \in [-M, M]$ is zero.



For $|v| > \gamma$ the minimal solution $u \in [-M, M]$ is $|u| = M$.

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Otherwise there exists a "best index" $i \in \{1, \ldots, N\}$ such that

$$\|v_{\perp_i}\| > \gamma(x) \quad \text{and} \quad \|v_{\perp_i}\| \geq \|v_{\perp_j}\| \quad \text{for every } j = 1, \ldots, N.$$

Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

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Hence the control acts on the most "stubborn". We may call this control the "shepherd dog strategy".

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Therefore we can choose $i \in \{1, \ldots, N\}$ satisfying it, and a control law

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Hence the control acts on the most "stubborn". We may call this control the "shepherd dog strategy". This choice of the control makes $V(t) = B(v(t), v(t))$ vanishing in finite time, hence there exists T such that $B(v(t), v(t)) \leq \gamma(x)^2$, $t \geq T$.

Instantaneous optimality of the greedy strategy

Consider generic control u (solution of the variation problem) of components

$$u_i(x, v) = \begin{cases} 0 & \text{if } v_{\perp_i} = 0 \\ -\alpha_i \dfrac{v_{\perp_i}}{\|v_{\perp_i}\|} & \text{if } v_{\perp_i} \neq 0 \end{cases}$$

where $\alpha_i \geq 0$ such that $\sum_{i=1}^{N} \alpha_i \leq M$.

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Theorem (Caponigro-F.-Piccoli-Trélat)

The 1-sparse control is the minimizer of

$$\mathcal{R}(t, u) := \mathcal{R}(t) = \frac{d}{dt} V(t),$$

among all the control of the previous form.

∗ A policy maker, who is not allowed to have prediction on future developments, should always consider more favorable to intervene with stronger actions on the fewest possible instantaneous optimal leaders than trying to control more agents with minor strength.
∗ Homophilious society can be stabilized by parsiminious interventions!

Homophilious societies are sparsely stabilizable – 1

If we allow external intervention, the CS system in the homophilious regime ($\beta > \frac{1}{2}$)

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \frac{1}{N} \sum_{j=1}^{N} a(|x_i - x_j|)(x_j - x_i) + u_i \end{cases}$$

can be stabilized for any initial condition by using only sparse controls, i.e., zero for almost every agent [2].

[2] M. Caponigro, M. Fornasier, B. Piccoli, and E. Trélat, *Sparse stabilization and control of alignment models*, Math. Models Methods Applied Sciences, '14

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The sparse control acts on the most “stubborn” agent at every time, like the “shepherd dog strategy”.

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Homophilious societies are sparsely stabilizable – 2

For the CD system in the homophilious regime (that is $E(0) \geq \theta$), the shepard dog strategy stabilization of

$$\begin{cases} \dot{x}_i &= v_i, \\ \dot{v}_i &= -bv_i + \frac{1}{N}\sum_{j=1}^{N}(a-f)(|x_i - x_j|)(x_j - x_i) + u_i, \end{cases}$$

with $b \equiv 0$ occurs under the additional hypothesis

$$\theta > E(0) \exp\left(-\frac{2\sqrt{3}}{9}\frac{N\|\bar{v}(0)\|^3}{E(0)^{3/2}}\right)$$

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The shepherd dog strategy does not work for every initial condition!

Observing the future: sparse optimal control

The problem is to minimize, for a given $\gamma > 0$

$$\mathcal{J}(u) = \int_0^T \frac{1}{N} \sum_{i=1}^N \left(\left(v_i(t) - \frac{1}{N} \sum_{j=1}^N v_j(t) \right)^2 \quad + \quad \gamma \| u_i(t) \| \right) dt,$$

$$\text{s.t.} \ \sum_{i=1}^N \|u_i\| \leq M$$

where the state is a trajectory of the control system

$$\begin{cases} \dot{x}_i = v_i \\ \dot{v}_i = \dfrac{1}{N} \displaystyle\sum_{j=1}^N a(\|x_j - x_i\|)(v_j - v_i) + u_i \end{cases}$$

with initial constraint

$$(x(0), v(0)) = (x_0, v_0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N.$$

Beyond a greedy approach: sparse optimal control

Theorem (Caponigro-F.-Piccoli-Trélat)

For every (x_0, v_0) *in* $(\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$*, for every* $M > 0$*, and for every* $\gamma > 0$ *the optimal control problem has an optimal solution. The optimal control* $u(t)$ *is "usually" instantaneously a vector with at most one nonzero coordinate.*

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The PMP ensures the existence of $\lambda \geq 0$ and of a nontrivial covector $(p_x, p_v) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ satisfying the adjoint equations, for $i = 1, \ldots, N$,

$$\begin{cases} \dot{p}_{x_i} = \frac{1}{N} \sum_{j=1}^{N} \frac{a(\|x_j - x_i\|)}{\|x_j - x_i\|} \langle x_j - x_i, v_j - v_i \rangle (p_{v_j} - p_{v_i}) \\ \dot{p}_{v_i} = -p_{x_i} - \frac{1}{N} \sum_{j \neq i} a(\|x_j - x_i\|)(p_{v_j} - p_{v_i}) - 2\lambda v_i + \frac{2\lambda}{N} \sum_{j=1}^{N} v_j. \end{cases}$$

The application of the PMP leads to minimize

$$\min \sum_{i=1}^{N} \langle p_{v_i}, u_i \rangle + \lambda\gamma \sum_{i=1}^{N} \|u_i\|, \quad \text{subject to } \sum_{i=1}^{N} \|u_i\| \leq M.$$

What if the population is too large $N \approx \infty$?

Mean-field (sparse) optimal control?

What if $N \to \infty$?

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We consider a perhaps natural control problem:

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H * \mu_N)(x_i, v_i) + u_i, i = 1, \dots N, \quad t \in [0, T], \\ \text{where } , \mu_N = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j, v_j)}, \end{cases}$$

controlled by the minimizer of the cost functional

$$\mathcal{J}(u) := \int_0^T \int_{\mathbb{R}^{2d}} \left(L(x, v, \mu_N) d\mu_N(t, x, v) + \frac{1}{N} \sum_{i=1}^N \|u_i\| \right) dt,$$

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Which topology on $\mu_N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(x_j, v_j)}$? Which topology on $\nu_N = \frac{1}{N} \sum_{j=1}^{N} u_i \delta_{(x_j, v_j)}$?

Too weak convergence

The compactness of the problem is way too weak

$$\nu_N = \frac{1}{N}\sum_{j=1}^{N} u_i \delta_{(x_j,v_j)} \rightharpoonup \nu, \quad \mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{(x_j,v_j)} \rightharpoonup \mu,$$

as it can happen that $\nu \perp \mu$.

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$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot \left[\left(H * \mu + f\right)\mu\right],$$

but $f \in L^1_\mu(\mathbb{R}^{2d}, \mathbb{R}^d)$ only and no well-posedness can be expected!

Mean-field Sparse Optimal Control?

"Ultimately it would be good to have a theory that combined both the collective behaviour of a large number of "ordinary" agents with the decisions of a few key players of unusually large (relative) influence – some complicated combination of PDE and game theory, presumably – but our current mathematical technology is definitely insufficient for even a zeroth approximation to this task".

– Terry Tao, January 7, 2010

https://terrytao.wordpress.com/2010/01/07/mean-field-equations

A natural relaxation: smoother controls

Definition

For a horizon time $T > 0$, and an exponent $1 \leq q < +\infty$ we fix a control bound function $\ell \in L^q(0, T)$. The class of admissible control functions $\mathcal{F}_\ell([0, T])$ is so defined: $f \in \mathcal{F}_\ell([0, T])$ if and only if

(i) $f : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d$ is a Carathéodory function,

(ii) $f(t, \cdot) \in W^{1,\infty}_{loc}(\mathbb{R}^n, \mathbb{R}^d)$ for almost every $t \in [0, T]$, and

(iii) $\|f(t, 0)\| + \mathrm{Lip}(f(t, \cdot), \mathbb{R}^d) \leq \ell(t)$ for almost every $t \in [0, T]$.

Mean-field optimal control

Theorem (F. and Solombrino)

Assume that we are given maps H, L, and ψ as in assumptions (H), (L), and (Ψ). For $N \in \mathbb{N}$ and an initial datum $((x_N^0)_1, \dots, (x_N^0)_N, (v_N^0)_1, \dots, (v_N^0)_N) \in B(0, R_0) \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, for $R_0 > 0$ independent of N, we consider

$$\min_{f \in \mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} [L(x, v, \mu_N(t, x, v)) + \psi(f(t, x, v))] \, d\mu_N(t, x, v) dt,$$

where $\mu_N(t, x, v) = \frac{1}{N} \sum_{j=1}^N \delta_{(x_j(t), v_j(t))}(x, v)$, constrained by being the solution of

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = (H * \mu_N)(x_i, v_i) + f(t, x_i, v_i), \quad i = 1, \dots N, \quad t \in [0, T], \end{cases}$$

with initial datum $(x(0), v(0)) = (x_N^0, v_N^0)$ and, for consistency, we set

$$\mu_N^0 = \frac{1}{N} \sum_{i=1}^M \delta_{((x_N^0)_i, (v_N^0)_i)}(x, v).$$

For all $N \in \mathbb{N}$ let us denote the function $f_N \in \mathcal{F}_\ell$ as a solution of the finite dimensional optimal control problem.

Mean-field optimal control

If there exists a compactly supported $\mu_0 \in \mathcal{P}_1(\mathbb{R}^{2d})$ such that $\lim_{N\to\infty} \mathcal{W}_1(\mu_N^0, \mu^0) = 0$, then there exists a subsequence $(f_{N_k})_{k\in\mathbb{N}}$ and a function $f_\infty \in \mathcal{F}_\ell$ such that f_{N_k} converges to f_∞ and f_∞ is a solution of the infinite dimensional optimal control problem

$$\min_{f\in\mathcal{F}_\ell} \int_0^T \int_{\mathbb{R}^{2d}} \left[L(x, v, \mu(t, x, v)) + \psi(f(t, x, v))\right] d\mu(t, x, v)dt,$$

where $\mu : [0, T] \to \mathcal{P}_1(\mathbb{R}^{2d})$ is the unique weak solution of

$$\frac{\partial \mu}{\partial t} + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H * \mu + f)\, \mu\right],$$

with initial datum $\mu(0) := \mu^0$ and forcing term f.

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The proof is based on the simultaneous development of the mean-field limit for the equation and the Γ-limit for the optimization of the control.

Sparse optimal control? Mixing diffuse and granular

Let us now consider a controlled system with m leaders for $m \gg N$

$$\begin{cases} \dot{y}_k = w_k, & \\ \dot{w}_k = H * \mu_N(y_k, w_k) + H * \mu_m(y_k, w_k) + u_k & k = 1, \dots m, \\ \dot{x}_i = v_i, & \\ \dot{v}_i = H * \mu_N(x_i, v_i) + H * \mu_m(x_i, v_i) & i = 1, \dots N, \end{cases}$$

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For $N \to \infty$ the limit dynamics is

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, & k = 1, \dots m, \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H * (\mu + \mu_m))\, \mu \right], \end{cases}$$

where the weak solutions of the equations have to be interpreted in the Carathéodory sense.

Mixing diffuse and granular

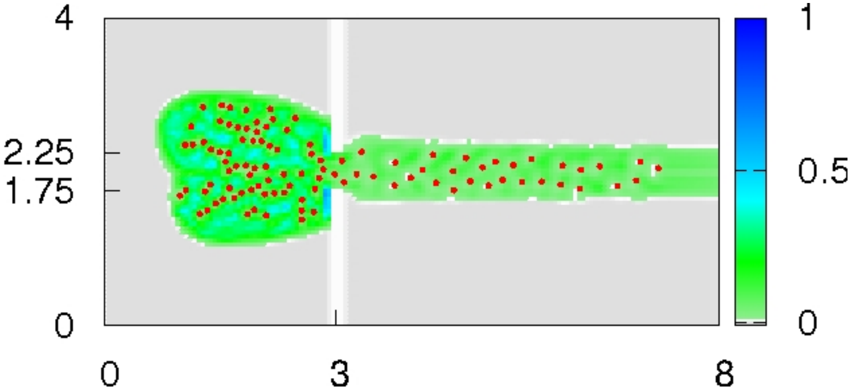

Figure : A mixed granular-diffuse crowd leaving a room through a door. This figure was kindly provided by the authors.

Main result on mean-field sparse control, I

Denote $\mathcal{X} := \mathbb{R}^{2d\times m} \times \mathcal{P}(\mathbb{R}^{2d})$.

Theorem (F., Piccoli, and Rossi)

Let H and L be maps satisfying conditions (H) and (L) respectively. Given an initial datum $(y^0, w^0, \mu^0) \in \mathcal{X}$, with μ^0 compactly supported, $\mathrm{supp}(\mu^0) \subset B(0,R)$, $R > 0$, the optimal control problem

$$\min_{u\in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y(t), w(t), \mu(t)) + \frac{1}{m}\sum_{k=1}^m \|u_k(t)\| \right\} dt,$$

has solutions, where the triplet (y, w, μ) defines the unique solution of

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H * (\mu + \mu_m)(y_k, w_k) + u_k, & k = 1, \dots m, \quad t \in [0,T] \\ \partial_t \mu + v \cdot \nabla_x \mu = \nabla_v \cdot \left[(H * (\mu + \mu_m))\, \mu \right], \end{cases}$$

with initial datum (y^0, w^0, μ^0) and control u, and

$$\mu_m(t) = \frac{1}{m}\sum_{k=1}^n \delta_{(y_k(t), w_k(t))}.$$

Main result on mean-field sparse control, II

Moreover, solutions to the problem can be constructed as weak limits u^* of sequences of optimal controls u_N^* of the finite dimensional problems

$$\min_{u\in L^1([0,T],\mathcal{U})} \int_0^T \left\{ L(y_N(t), w_N(t), \mu_N(t)) + \frac{1}{m}\sum_{k=1}^m \|u_k(t)\| \right\} dt,$$

where $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{(x_{i,N}(t), v_{i,N}(t))}$ and $\mu_{m,N}(t) = \frac{1}{m}\sum_{k=1}^m \delta_{(y_{k,N}(t), w_{k,N}(t))}$ are the time-dependent atomic measures supported on the trajectories defining the solution of the system

$$\begin{cases} \dot{y}_k = w_k, \\ \dot{w}_k = H * \mu_N(y_k, w_k) + H * \mu_{m,M}(y_k, w_k) + u_k & k = 1, \dots m, \quad t \in [0, T], \\ \dot{x}_i = v_i, \\ \dot{v}_i = H * \mu_N(x_i, v_i) + H * \mu_{m,M}(x_i, v_i) & i = 1, \dots N, \quad t \in [0, T], \end{cases}$$

with initial datum (y^0, w^0, x_N^0, v_N^0), control u, and $\mu_N^0 = \frac{1}{N}\sum_{i=1}^N \delta_{(x_i^0, v_i^0)}$ is such that $\mathcal{W}_1(\mu_N^0, \mu^0) \to 0$ for $N \to +\infty$.

Evacuating an unknown environment

Simulations I

Simulations II

Learning the dynamics

Consider the dynamics

$$\dot{x}_i(t) = \frac{1}{N} \sum_{j \neq i} a(\|x_i - x_j\|)(x_j - x_i), \quad i = 1, \dots, N.$$

with $a \in X = \left\{ b : \mathbb{R}_+ \to \mathbb{R} \,|\, b \in L^\infty(\mathbb{R}_+) \cap W^{1,\infty}_{loc}(\mathbb{R}_+) \right\}$.

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with $a \in X = \left\{b : \mathbb{R}_+ \to \mathbb{R} \mid b \in L^\infty(\mathbb{R}_+) \cap W^{1,\infty}_{loc}(\mathbb{R}_+)\right\}$. Can we "learn" the interaction function a from observations of the dynamics?

A least square functional

As an approximation to a we seek for a minimizer of the following *discrete error functional*

$$\mathcal{E}_N(\widehat{a}) = \frac{1}{T}\int_0^T \frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{N}\sum_{j=1}^{N}\left(\widehat{a}(\|x_i(t)-x_j(t)\|)(x_i(t)-x_j(t)) - \dot{x}_i(t)\right)\right\|^2 dt,$$

among all functions $\widehat{a} \in X$.

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among all functions $\widehat{a} \in X$. In particular, given a finite dimensional space $V \subset X$, we consider the minimizer:

$$\widehat{a}_{N,V} = \arg\min_{\widehat{a}\in V} \mathcal{E}_N(\widehat{a}).$$

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The fundamental question is

(Q) For which choice of the approximating spaces $V \in \Lambda$ (we assume here that Λ is a countable family of invading subspaces of X) does $\widehat{a}_{N,V} \to a$ for $N \to \infty$ and $V \to X$ and in which topology should this convergence hold?

Mean-field limit

The empirical measure $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ weakly converges for $N \to \infty$ to the probability measure valued trajectory $t \to \mu(t)$ satisfying weakly the equation

$$\partial_t \mu(t) = -\nabla \cdot ((H[a] * \mu(t))\mu(t)), \quad \mu(0) = \mu^0.$$

where $H[a](x) = -a(\|x\|)x$, for $x \in \mathbb{R}^d$.

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The empirical measure $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ weakly converges for $N \to \infty$ to the probability measure valued trajectory $t \to \mu(t)$ satisfying weakly the equation

$$\partial_t \mu(t) = -\nabla \cdot ((H[a] * \mu(t))\mu(t)), \quad \mu(0) = \mu^0.$$

where $H[a](x) = -a(\|x\|)x$, for $x \in \mathbb{R}^d$. We define

$$\mathcal{E}(\widehat{a}) = \frac{1}{T}\int_0^T \int_{\mathbb{R}^d} \left\| \big(H[\widehat{a}] - H[a]\big) * \mu(t) \right\|^2 d\mu(t)(x) dt,$$

Coercivity property I

By Jensen inequality

$$\begin{aligned}
\mathcal{E}(\widehat{a}) &\leq \frac{1}{T}\int_0^T \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |\widehat{a}(\|x-y\|) - a(\|x-y\|)|^2 \|x-y\|^2 d\mu(t)(x) d\mu(t)(y) dt \\
&= \frac{1}{T}\int_0^T \int_{\mathbb{R}_+} \left|\widehat{a}(s) - a(s)\right|^2 s^2 d\varrho(t)(s) dt \qquad (1)
\end{aligned}$$

where $\varrho(t) = (\|x-y\|_{\#}\mu_x(t) \otimes \mu_y(t))$.

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where $\varrho(t) = (\|x-y\|_{\#}\mu_x(t) \otimes \mu_y(t))$. We define the prob. measure

$$\bar{\rho} := \frac{1}{T}\int_0^T \varrho(t) dt.$$

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Then one can reformulate (1) in a very compact form as follows

$$\mathcal{E}(\widehat{a}) \leq \int_{\mathbb{R}_+} \left|\widehat{a}(s) - a(s)\right|^2 d\rho(s) = \|\widehat{a} - a\|^2_{L^2(\mathbb{R}_+,\rho)}.$$

Coercivity property II

To establish coercivity of the learning problem it is essential to assume that there exists $c_T > 0$ such that also the following additional lower bound holds

$$c_T \|\widehat{a} - a\|^2_{L_2(\mathbb{R}_+,\rho)} \leq \mathcal{E}(\widehat{a}),$$

for all relevant $\widehat{a} \in X \cap L^2(\mathbb{R}_+, \rho)$. This crucial assumption eventually determines also the natural space $X \cap L^2(\mathbb{R}_+, \rho)$ for the solutions.

Uniform approximation property

For $M > 0$ and an interval $K = [0, 2R]$ define the set

$$X_{M,K} = \left\{ b \in W^{1,\infty}(K) : \|b\|_{L^\infty(K)} + \|b'\|_{L^\infty(K)} \leq M \right\}.$$

Additionally for every $N \in \mathbb{N}$, let V_N be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence on K with the following *uniform approximation property*: for all $b \in X_{M,K}$ there exists a sequence $(b_N)_{N \in \mathbb{N}}$ converging uniformly to b on K and such that $b_N \in V_N$ for every $N \in \mathbb{N}$.

Main learnability result I

Theorem (Bongini, F., Hansen, Maggioni, 2015)

Fix $M \geq \|a\|_{L^\infty(K)} + \|a'\|_{L^\infty(K)}$ for $K = [0, 2R]$, for $R > 0$ large enough. For every $N \in \mathbb{N}$, let V_N be a closed subset of $X_{M,K}$ w.r.t. the uniform convergence on K with the uniform approximation property. Then the minimizers

$$\widehat{a}_N \in \arg\min_{\widehat{a} \in V_N} \mathcal{E}_N(\widehat{a}).$$

converge uniformly up to subsequ. for $N \to \infty$ to a continuous function $\widehat{a} \in X_{M,K}$ such that $\mathcal{E}(\widehat{a}) = 0$.

Main learnability result II

If we additionally assume the coercivity condition, then $\widehat{a} = a$ *in* $L_2(\mathbb{R}_+, \rho)$*. Moreover, in this latter case, if there exist rates* $\alpha, \beta > 0$*, constants* $C_1, C_2 > 0$*, and a sequence* $(a_N)_{N \in \mathbb{N}}$ *of elements* $a_N \in V_N$ *such that*

$$\|a - a_N\|_{L^\infty(K)} \leq C_1 N^{-\alpha},$$

and

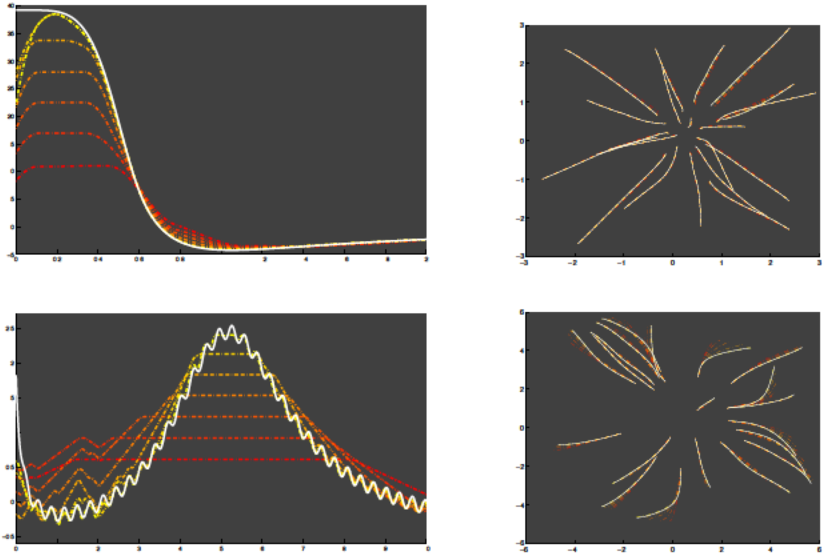
$$\mathcal{W}_1(\mu_0^N, \mu_0) \leq C_2 N^{-\beta},$$

then there exists a constant $C_3 > 0$ *such that*

$$\|a - \widehat{a}_N\|^2_{L^2(\mathbb{R}_+,\rho)} \leq C_3 N^{-\min\{\alpha,\beta\}},$$

for all $N \in \mathbb{N}$*. In particular, in this case, it is the entire sequence* $(\widehat{a}_N)_{N \in \mathbb{N}}$ *(and not only subsequences) to converge to* a *in* $L^2(\mathbb{R}_+, \rho)$*.*

Numerical experiments



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Conclusion

- We presented dynamical systems with self-organization features $\Rightarrow$ *organization by external intervention*.
- We proved that the most effective *greedy strategy* is by instantaneous 1-sparse controls.
- We presented relaxations of the (sparse) finite dimensional optimal control problems and a general technique to derive their mean-field limits.
- We showed recent results on learnability of systems modeling social dynamics.

A few info

- **WWW:** http://www-m15.ma.tum.de/
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