Exponential moments for the homogeneous Kac equation

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Outline

1. **Introduction**
2. **Previous results**
3. **Our contribution**

Introduction

Similarities and differences b/w Kac and Boltzmann equations

Similarities between the Kac equation and the Boltzmann equations:

- **Both model the evolution of rarefied gases** via a distribution function $f(t, x, v)$.
- **Binary interactions** between particles.
- **Quadratic** collision operator.
- **Our assumption: spatial homogeneity**, i.e. we assume the distribution function $f(t, v)$ does not depend on x.

Differences between the Kac equation and the Boltzmann equation:

- **The dimension** of spatial and velocity spaces is $d \geq 2$ for the Boltzmann equation, while for the Kac equation $d = 1$.
- **Conservation laws**: collisions modeled by the Boltzmann equation conserve mass, momentum and energy, while those for the Kac equation conserve only mass and energy (but not the momentum).
- **Pre-post collisional velocity laws** are be different.

The Kac equation

- **The Kac equation** (1959, Kac) models a **1-dimensional gas** in which collisions **conserve the mass and the energy, but not the momentum**. The spatially homogeneous Kac equation reads

$$\boxed{\partial_t f(t,v) = \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left(f' f'_* - f f_*\right) b_K(|\theta|) \, \mathrm{d}\theta \, \mathrm{d}v_*,}$$

with the standard abbreviations $f_* := f(t, v_*)$, $f' := f(t, v')$, $f'_* := f(t, v'_*)$.

- Conservation of energy $v'^2 + v'^2_* = v^2 + v^2_*$ implies that the pre and post-collisional velocities v', v'_* and v, v_* can be related by introducing a parameter $\theta \in [-\pi, \pi]$ as:

$$v' = v \cos\theta - v_* \sin\theta,$$
$$v'_* = v \sin\theta + v_* \cos\theta.$$

By a change of variables, one can also use the following relation:

$$v' = \sqrt{v^2 + v_*^2} \cos\theta,$$
$$v'_* = \sqrt{v^2 + v_*^2} \sin\theta.$$

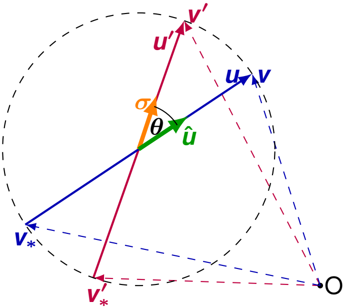
- **The Boltzmann equation** (late 1860s and 1870s, Maxwell and Boltzmann) models evolution of a d-dimensional gas ($d \geq 2$) in which particles interact via binary collisions. The spatially homogeneous Boltzmann equation reads:

$$\partial_t f = \int_{\mathbb{R}^d} \int_{S^{d-1}} (f' f'_* - f f_*) \, |v - v_*|^\gamma \, b_B\left(\frac{v-v_*}{|v-v_*|} \cdot \sigma\right) \, \mathrm{d}\sigma \, \mathrm{d}v_*,$$

- Conservation of momentum and energy implies the following relation between pre and post-collisional velocities (v', v'_*) and (v, v_*):

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|\sigma}{2}$$
$$v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|\sigma}{2}, \quad \sigma \in S^{d-1}.$$

Notation: $u' = v' - v'_*$, $u = v - v_*$,
$\gamma \in (0, 1]$: hard potentials,
$\gamma = 0$: **Maxwell molecules,**
$\gamma \in (-d, 0)$: soft potentials.



The angular cross-section

Angular kernels $b_K(|\theta|)$ and $b_B(\cos\theta)$ often have a non-integrable singularity at $\theta = 0$. However, such singularity is often cut-off ($b_K(|\theta|) \in L^1([0,\pi])$ or $b_B(\hat{u}\cdot\sigma) \in L^1(S^{d-1})$). **Cutoff simplifies the analysis of the collision operator** by enabling its splitting into the gain and loss terms $Q(f,f) = Q^+(f,f) - Q^-(f,f)$. It was believed that this removal of the non-integrability does not influence the equation significantly. However, it has been observed that **singularity carries regularizing effect**. This motivates further study of the non-cutoff regime:

- **Non-cutoff (Kac):**

$$\int_{-\pi}^{\pi} b_K(|\theta|)\, d\theta = \infty,$$

$$\int_{-\pi}^{\pi} b_K(|\theta|)\, \sin^\beta\theta\, d\theta < \infty, \quad \beta \in (0,2]$$

- **Non-cutoff (Boltzmann):**

$$\int_0^{\pi} b_B(\cos\theta)\ \sin^{d-2}\theta\, d\theta = \infty,$$

$$\int_0^{\pi} b_b(\cos\theta)\ \sin^\beta\theta\ \sin^{d-2}\theta\, d\theta < \infty, \quad \beta \in (0,2]$$

Kac equation:

$$\partial_t f = \int_{\mathbb{R}} \int_{-\pi}^{\pi} \left(f' \, f'_* - f \, f_*\right) b_K(|\theta|) \, \mathrm{d}\theta \, \mathrm{d}v_*,$$

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Boltzmann equation for Maxwell molecules:

$$\partial_t f = \int_{\mathbb{R}^d} \int_{S^{d-1}} \left(f' \, f'_* - f \, f_*\right) \, b_B\left(\frac{v - v_*}{|v - v_*|} \cdot \sigma\right) \, \mathrm{d}\sigma \, \mathrm{d}v_*,$$

$$\int_0^\pi b_B(\cos\theta) \; \sin^{d-2}\theta \, d\theta = \infty,$$

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What do we study?

We say that f has an exponential tail in the L^1 if, for some rate $\alpha > 0$ and some order $s > 0$, the following norm is finite:

$$\|f\|_{L^1_{\exp(\alpha \langle v \rangle^s)}}(t) := \int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \langle v \rangle^s}\, dv < \infty.$$

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We study whether these norms propagate in time:

- **propagation** of exponentially weighted L^1 norms means:

$$\int_{\mathbb{R}^d} f_0(v)\; e^{\alpha_0\,\langle v\rangle^s} dv < C_0, \quad \text{for some} \quad \alpha_0, s > 0$$

$$\Rightarrow \qquad \exists \alpha, C > 0,\;\; \forall t \geq 0 : \;\int_{\mathbb{R}^d} f(t,v)\; e^{\alpha\,\langle v\rangle^s}\, dv < C.$$

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- **generation** of exponentially weighted L^1 norms means

$$\int_{\mathbb{R}^d} f_0(v)\, \langle v\rangle^q\, dv < C_0, \quad \text{for some} \quad q \in \mathbb{N}$$

$$\Rightarrow \qquad \exists \alpha, s, C > 0, \;\; \forall t > 0: \int_{\mathbb{R}^d} f(t,v)\, e^{\alpha\,\langle v\rangle^s}\, dv < C.$$

Why exponential moments?

- They provide information about tail behavior of solutions (exponential decay of $f(t, v)$ for large velocities).
- First step towards pointwise exponential bounds L^∞_{exp}. In fact, in 1972 Arkeryd asked if the following is true for the Boltzmann equation:

$$f_0(v) \leq c\, e^{-\alpha_0 |v|^2} \quad \Rightarrow \quad f(t, v) \leq c_1\, e^{-\alpha_1 |v|^2}.$$

First results in this direction were in the L^1 setting by Bobylev in 1984 and 1997. Original question was addressed by Gamba, Panferov and Villani in 2009 in the cutoff case, by relating L^∞_{exp} norm with L^1_{exp} norms.

Previous results

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- 1984 Bobylev: Boltzmann equation for Maxwell molecules, $s = 2$
- 1993 Desvillettes: Kac equation, $s = 1$, $s = 2$
- 1997 Bobylev: Hard sphere, $s = 2$. This paper opened the doors for many extensions:
  - 2004 Bobylev-Gamba-Panferov
  - 2009 Gamba-Panferov-Villani
  - 2006 Mouhot 2006
  - 2013 Alonso-Canizo-amba-Mouhot
  - 2012 Lu-Mouhot 2012
  - 2016 T.-Alonso-Gamba-Pavlović

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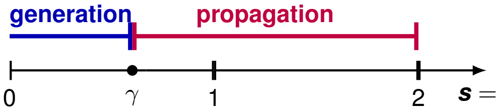
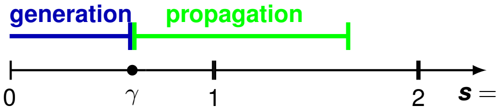
Cutoff case for $\gamma > 0$:



s = order of the exponential $e^{\alpha \langle v \rangle^s}$

Non-cutoff case for $\gamma > 0$:



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Desvillettes 1993

- For the Kac equation with constant angular kernel and for orders $s = 1$ and $s = 2$, Desvillettes 1993 proved propagation of exponential moments of integer orders $s = 1$ and $s = 2$.
- His elegant proof works directly with exponential moments. For example, for $s = 2$, let the exponential moment of order 2 be denoted by

$$\mathcal{M}(t,\alpha) = \int_{\mathbb{R}} f(t,v) e^{\alpha |v|^2} dv.$$

Multiply the Kac equation with the exponential weight and integrate in v:

$$\begin{aligned}
\partial_t \mathcal{M}(t,\alpha) &= \int_{\mathbb{R}} K(f,f) e^{\alpha |v|^2} dv \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f f_* \left(e^{\alpha |v'|^2} - e^{\alpha |v|^2} \right) \frac{d\theta}{2\pi} dv dv_* \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{-\pi}^{\pi} f f_* \left(e^{\alpha (|v|^2 + |v_*|^2) \cos^2 \theta} - e^{\alpha |v|^2} \right) \frac{d\theta}{2\pi} dv dv_* \\
&= \int_{-\pi}^{\pi} \mathcal{M}^2(t, \alpha \cos^2 \theta) - \mathcal{M}(t,\alpha) \mathcal{M}(0,0) \frac{d\theta}{2\pi}
\end{aligned}$$

Bobylev 1997

The first groundbreaking work on exponential moments for hard potentials and cutoff, was done by Bobylev in 1997. The key idea is to **Taylor expand exponential function** to reformulate (formally) the question to showing summability of polynomial moments renormalized by Gamma functions:

$$\int_{\mathbb{R}^d} f(t,v)\, e^{\alpha \langle v \rangle^s}\, dv = \int_{\mathbb{R}^d} f(t,v) \sum_{q=0}^{\infty} \frac{\langle v \rangle^{qs} \alpha^q}{q!} dv = \sum_{q=0}^{\infty} \frac{m_{qs}(t)\, \alpha^q}{\Gamma(q+1)}.$$

Because of the required summability, constants need to be sharper compared to results concerning polynomial moments. Estimates are developed by looking for ordinary differential inequalities for m_{qs}, and then deriving:

- term-by-term estimates
- partial sum estimates

Classical technique

Multiply the Boltzmann equation with $\langle v\rangle^{2q}$:

$$\partial_t f\ \langle v\rangle^{2q} = Q(f,f)\ \langle v\rangle^{2q}.$$

Integrate in velocity. **The weak form of the collision operator** yields

$$\begin{aligned} m_{2q}'(t) &= \int_{\mathbb{R}^d} Q(f,f)\ \langle v\rangle^{2q}\, dv \\ &= \frac{1}{2}\int_{\mathbb{R}^{2d}} f\, f_* \left(\int_{S^{d-1}} \left(\langle v'\rangle^{2q} + \langle v_*'\rangle^{2q} - \langle v\rangle^{2q} - \langle v_*\rangle^{2q} \right) |u|^\gamma\ b(\hat{u}\cdot\sigma)\, d\sigma \right) dv dv_* \end{aligned}$$

Then, look for a bound of the right-hand-side in terms of polynomial moments. For the bounds to be good enough it is important that the highest-order moment comes with a negative sign.

$$\begin{aligned} m_{2q}' &\leq -K_1\, m_{2q+\gamma} + K_2\, m_{2q} \\ &\quad + K_3\, \varepsilon_q \sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q}{k} \left(m_{2k+\gamma}\, m_{2(q-k)} + m_{2k}\, m_{2(q-k)+\gamma} \right) \end{aligned}$$

Our contribution

Our main result

Theorem (Pavic-Colic, T. 2016)

Suppose initial datum $f_0 \geq 0$ *has finite mass, energy and entropy. Let* $f(t, v)$ *be an associated weak solution to the Kac equation with the angular kernel satisfying:*

$$\int_{-\pi}^{\pi} b_K(|\theta|)\, d\theta = \infty,$$

$$\int_{-\pi}^{\pi} b_K(|\theta|) \sin^{\beta}\theta \, d\theta < \infty, \quad \beta \in (0, 2]$$

If

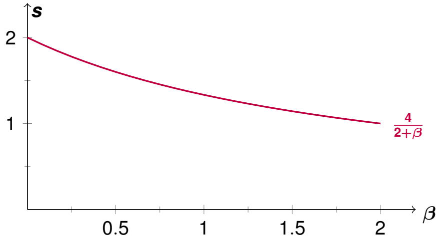
$$s \leq \frac{4}{2+\beta},$$

then for every $\alpha_0 > 0$ *there exists* $0 < \alpha \leq \alpha_0$ *and a constant* $C > 0$ *(depending only on the initial data and* κ*) so that*

$$\text{if } \int_{\mathbb{R}} f_0(v) e^{\alpha_0 \langle v \rangle^s} \mathrm{d}v \leq M_0 < \infty,$$

$$\text{then } \int_{\mathbb{R}} f(t, v) e^{\alpha \langle v \rangle^s} \mathrm{d}v \leq C, \quad \forall t \geq 0.$$

Interpretation



Strategy

We use recent techniques of T.-Alonso-Gamba-Pavlovic 2016, which consists of studying generalized exponential moments, so-called **Mittag-Leffler moments**. They are L^1 norms weighted with **Mittag-Leffler functions**:

$$\mathcal{E}_a(x) := \sum_{q=0}^{\infty} \frac{x^q}{\Gamma(aq+1)}$$

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They generalize exponentials and are known to **asymptotically behave like exp**:

$$\mathcal{E}_a(x) \sim e^{x^{1/a}}, \qquad \text{for } x \gg 1.$$

Because of this asymptotic behavior, finiteness of exponential moment is equivalent to the finiteness of the corresponding Mittag-Leffler moment. So, we work with:

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Definition (Mittag-Leffler moment)

The **Mittag-Leffler moment** of f of order s and rate $\alpha > 0$ is introduced via:

$$ML^{(\alpha,s)} = \sum_{q=0}^{\infty} \frac{m_{2q}\, \alpha^{2q/s}}{\Gamma(\frac{2}{s}q + 1)}.$$

Why Mittag-Leffler moments? part 1

We develop the partial sum technique for the non-cutoff case. The story begins as before

$$m_{2q}'(t) = \int_{\mathbb{R}} K(f,f)\,\langle v\rangle^{2q}\,dv$$

$$= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} f\,f_* \int_{-\pi}^{\pi}\left(\langle v'\rangle^{2q} + \langle v_*'\rangle^{2q} - \langle v\rangle^{2q} - \langle v_*\rangle^{2q}\right)\,b_K(|\theta|)\,\mathrm{d}\theta\,\mathrm{d}v_*\,\mathrm{d}v.$$

First challenge is angular singularity. Need to exploit certain cancellation properties that become visible after an application of the Taylor expansion to the test functions.

This leads to an ODI for polynomial moments with **another challenge**: compared to the Grad's cutoff, there are two extra powers of q in the last term of the inequality:

$$m_{2q}' \leq -K_1\,m_{2q} + K_2\,m_{2q-2} + K_3\,\varepsilon_{\beta,q}\,q\,(q-1)\sum_{k=1}^{\lfloor\frac{q+1}{2}\rfloor}\binom{q-2}{k-1} m_{2k}\,m_{2(q-k)}$$

Why Mittag-Leffler moments? part 2

Sequence ε_q depends on the angular kernel $b(\cos\theta)$. It has the following property:

$$\begin{aligned}
&\beta \in [0,2] && \boldsymbol{q^{1-\frac{\beta}{2}}\,\varepsilon_{\beta,q} \to 0}, \quad \text{as } q \to \infty,\\
&\beta = 0 \text{ (Grad's cutoff)} && \boldsymbol{q\,\varepsilon_{0,q} \to 0}, \quad \text{as } q \to \infty,\\
&\beta = 2 \text{ (Full non-cutoff)} && \boldsymbol{\varepsilon_{2,q} \to 0}, \quad \text{as } q \to \infty.
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Need to reduce the quadratic power of $q(q-1)$ in

$$m'_{2q} \leq \ldots + \boxed{\varepsilon_{\beta,q}\, q\,(q-1)} \sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} m_{2k}\, m_{2(q-k)}$$

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Going towards partial sums, divide all moments by appropriate Gamma functions:

$$\frac{m'_{2q}}{\Gamma(aq+1)} \leq \ldots + \frac{\varepsilon_{\beta,q}\, q\,(q-1)}{\Gamma(aq+1)} \sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} \times \frac{m_{2k}}{\Gamma(ak+1)}\, \frac{m_{2(q-k)}}{\Gamma(a(q-k)+1)}$$

$$\Gamma(aq+2) B(ak+1, a(q-k)+1)$$

Why Mittag-Leffler moments? part 3

Use

$$\sum_{k=1}^{\lfloor \frac{q+1}{2} \rfloor} \binom{q-2}{k-1} B(ak+1, a(q-k)+1) \leq C_a \frac{1}{q^{a+1}}$$

Last inequality holds **only if $a > 1$**, and hence the need for Mittag-Leffler moments. This yields $\boxed{\varepsilon_{\beta,q}\, q^{2-a}}$ in an ODE for partial sums. For this sequence to converge to zero as $q \to \infty$, we need:

$$2 - a = 2 - \frac{2}{s} \leq 1 - \frac{\beta}{2}$$

Hence the order of the exponential moment depends on the singularity rate of the angular kernel as:

$$s \leq \frac{4}{2+\beta}.$$

Possible further questions

1. L^{∞}_{exp} bounds for the Kac equation
2. Exponential moments for soft potentials $\gamma < 0$?
3. Lower bounds?
4. Relation with the rate of convergence to the equilibrium?

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Thank you!!!