Blow-up conditions for two dimensional modified Euler-Poisson equations

Yongki Lee
yongki.lee@ucr.edu

Department of Mathematics
University of California, Riverside

KI-Net Conference
Kinetic Descriptions of Chemical and Biological Systems:
Models, Analysis and Numerics
Iowa State University
March 23-25, 2017

Table of contents

Riccati equation

Jacopo Riccati (1676-1754)

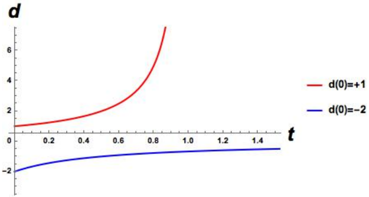
Riccati equation

$$\begin{cases} d'(t) = d^2(t), \quad t \in [0, ?) \\ d(0) = d_0. \end{cases}$$

- Integration gives

$$d(t) = \frac{d_0}{1 - td_0}, \ \textit{for} \ t \in [0, ?)$$

Critical Threshold in Riccati equation



The graph of $d(t)$

Riccati equation

$$\begin{cases} d'(t) = d^2(t), & t \in [0, ?) \\ d(0) = d_0. \end{cases}$$

- $d(t) = \frac{d_0}{1 - td_0}, \ \ for \ \ t \in [0, ?)$
- If $d_0 > 0$, then

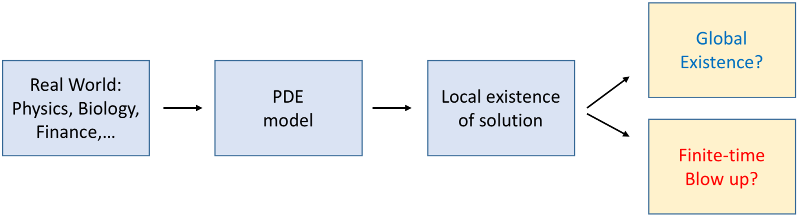
$$d(t) \to \infty \ as \ t \to 1/d_0$$

- If $d_0 \leq 0$, then

$$d(t) \ continuous \ for \ t \in [0, \infty)$$

Critical Threshold

Global regularity questions in PDEs



• Physical fluids : velocity can't actually go to infinity → Finite time blow-up scenario does not occur.

• PDE models for physical fluids(e.g. Euler Equation, Navier-Stokes Equations, Euler-Poisson Equations: $d := \nabla \cdot \vec{u}$, where $\vec{u} =$ velocity field):
If an answer to the global regularity problem is negative → For certain choice of initial data, finite time blow-up may occur → the equations will at some point be an inaccurate model for a physical fluid.

The problems we are considering

We are concerned with the threshold phenomenon in multi-dimensional Euler-Poission equations.

Multi-D Euler-Poisson equations

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\ x\in\mathbb{R}^n,\ t\in\mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho,$$

where $\mathbf{u}(t,x)$ =velocity and $\rho(t,x)$ =density. Here k is a physical constant which parameterizes the repulsive $k>0$ or attractive $k<0$ forcing.

This hyperbolic system with non-local forcing describes the dynamic behavior of many important physical flows, including plasma with collision, cosmological waves, charge transport, and the collapse of stars due to self gravitation.

- We are concerned with the questions of the persistence of the C^1 solution regularity for the conservation laws and Euler-Poisson equations.
- The $\underbrace{\textit{natural question}}_{why?}$ is whether there is a critical threshold for the initial data such that the persistence of the C^1 solution regularity depends only on crossing such a critical threshold.

Threshold configuration for initial data

- This concept of critical threshold and associated methodology is originated and developed in a series of paper by Engelberg, Liu, and Tadmor.

The $\underbrace{\textit{natural question}}_{why?}$...

Let us temporarily ignore the role of incompressibility and the pressure in the NS equations:

Navier-Stokes equations, heuristic view

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = \nu \Delta u - \cancel{\nabla p}$$

- One can view this equation as a *contest* between $(\mathbf{u} \cdot \nabla_x)\mathbf{u}$ and $\nu \Delta u$.
- If $(\mathbf{u} \cdot \nabla_x)\mathbf{u} >> \nu \Delta u$,

  We expect the solution to the NS equations to behave like $\partial_t u \approx (\mathbf{u} \cdot \nabla_x)\mathbf{u}$ $\rightarrow$ expect finite time blow-up(Burgers equation)
- If $(\mathbf{u} \cdot \nabla_x)\mathbf{u} << \nu \Delta u$,

  We expect the solution to the NS equations to behave like $\partial_t u \approx \nu \Delta u$ $\rightarrow$ expect global smooth solution(heat equation)

Euler-Poisson equations

Problem description

We are concerned with the threshold phenomenon in two dimensional Euler-Poisson equations.

Multi-D Euler-Poisson equations

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \; x \in \mathbb{R}^n, \; t \in \mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = k \nabla \Delta^{-1} \rho,$$

where $\mathbf{u}(t,x)$ =velocity and $\rho(t,x)$ =density. Here k is a physical constant which parameterizes the repulsive $k > 0$ or attractive $k < 0$ forcing.

We consider a gradient flow $M(t,x) := \nabla \mathbf{u}$ governed by Euler-Poisson equations, subject to initial data

$$(M,\rho)(0,\cdot) = (M_0, \rho_0).$$

Dynamics of u

Multi-D Euler-Poisson equations

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0,\ x \in \mathbb{R}^n,\ t \in \mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = k \nabla \Delta^{-1} \rho,$$

Expanding the second equation(in $2D$):

$$u^i_t + \left(u^1 \frac{\partial}{\partial x} + u^2 \frac{\partial}{\partial y}\right) u^i = k \frac{\partial}{\partial x_i} \Delta^{-1} \rho, \quad i = 1, 2.$$

Here,

$$k \frac{\partial}{\partial x_i} \Delta^{-1} \rho(t, \vec{x}) = k \cdot pv \int_{\mathbb{R}^2} \frac{\partial}{\partial y_i} G(\vec{y}) \rho(t, \vec{x} - \vec{y})\, d\vec{y}, \quad G : \textit{Poisson kernel in } 2D$$
$$= k \cdot pv \int_{\mathbb{R}^2} \frac{1}{2\pi} \cdot \frac{y_i}{y_1^2 + y_2^2} \rho(t, \vec{x} - \vec{y})\, d\vec{y}$$

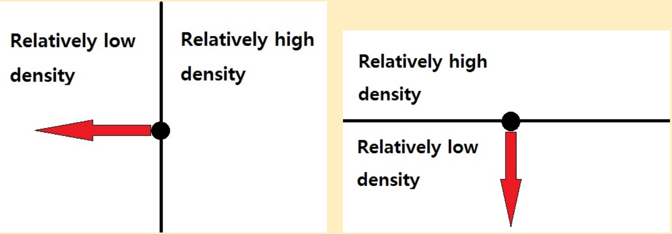
Dynamics of u (cont'd)

Thefore, re-writing the second equation gives

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho$$

$$\Rightarrow \frac{D}{Dt}u^i = k \cdot pv \int_{\mathbb{R}^2} \frac{1}{2\pi} \cdot \frac{y_i}{y_1^2 + y_2^2} \rho(t, \vec{x} - \vec{y})\, d\vec{y}$$

The role of $k\nabla\Delta^{-1}\rho$ when $k > 0$(repuslive case)



$\frac{D}{Dt}u^1 < 0$ $\qquad$ $\frac{D}{Dt}u^2 < 0$

Ultimate Goal

Multi-D Euler-Poisson equations

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\ x\in\mathbb{R}^n,\ t\in\mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho,$$

where $\mathbf{u}(t,x)$ =velocity and $\rho(t,x)$ =density. Here k is a physical constant which parameterizes the repulsive $k>0$ or attractive $k<0$ forcing.

- We are concerned with the questions of global regularity vs finite-time breakdown of Eulerian flows.

  Q: whether the smooth solution develops singularity in finite time?

Main obstacle

Multi-D Euler-Poisson equations

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \; x \in \mathbb{R}^n, \; t \in \mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho.$$

Let $M := \nabla u$ and apply ∇ to the second equation,

$$\Rightarrow \partial_t M + u \cdot \nabla M + M^2 = k\nabla \otimes \nabla \Delta^{-1}[\rho].$$

$$\Rightarrow M' + M^2 = kR[\rho],$$

where $' := \partial_t + u \cdot \nabla$ and $R = \{R_{ij}\} = \{\partial_{x_i x_j}\Delta^{-1}\}$.

Difficulty: There is no clear idea on how strong $kR[\rho]$ is compare to M^2. More precisely, it is the global forcing, $R[\rho]$, which presents the main obstacle to studying the CT phenomenon of the multi-dimensional Euler-Poisson setting.

Highly cited works

Multi-D Euler-Poisson equations

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0,\ x\in\mathbb{R}^n,\ t\in\mathbb{R}^+,$$
$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho.$$

- 1 dimension: Critical Threshold [Liu-Tadmor 2002]
- Blow-up of a spherically symmetric solution[B. Perthame 1990]
- Construction of a global smooth solution

  3D irrotational solution [Y. Guo 1998]: Let $n(x), v(x)\in C_c^\infty(\mathbb{R}^3)$. Suppose $\nabla\times u = 0$. Then there exists $\varepsilon_0 > 0$ such that for $0<\varepsilon<\varepsilon_0$, there exist unique smooth solutions $(\rho^\varepsilon(t,x), u^\varepsilon(t,x))$ to the Euler-Poisson equations for $0\le t<\infty$ with initial data $(\varepsilon n(x), \varepsilon v(x))$.
- Analogous theorem in 2D is open.

  2D radial symmetric solution [J. Jang 2014]

My works in Restricted type Euler-Poisson Equations

(the original) Euler-Poisson equations

$$\rho_t + \nabla\cdot(\rho \mathbf{u}) = 0,\ x\in\mathbb{R}^n,\ t\in\mathbb{R}^+,$$
$$M' + M^2 = kR[\rho].$$

Restricted EP(Local)

$$M' + M^2 = \frac{k}{n}\rho I_{n\times n}$$

Weakly Restricted EP

$$M' + M^2 = \frac{k}{n}\rho I_{n\times n} + R^{off}_{dig}$$

Modified EP(Global)

$$M' + M^2 = kR^{\nu}[\rho]$$

- 2D Critical Threshold Liu-Tadmor 2002
- nD Gloabl Existence, Blow-up Lee-Liu 2013

- R^{off}_{dig} :=off-diagonal elements matrix of $kR[\rho]$
- 2D Blow-up, Lee 2017

- $R^{\nu}[\rho]$:=modified Riesz transform where the singularity at the origin is removed
- 2D Blow-up, Lee 2016

$d = \nabla \cdot \mathbf{u}$ dynamics equation in 2D

(the original) Euler-Poisson equations

$$\rho_t + \nabla \cdot (\rho \mathbf{u}) = 0, \ x \in \mathbb{R}^n, \ t \in \mathbb{R}^+,$$
$$M' + M^2 = kR[\rho].$$

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\omega^2 - \frac{1}{2}\xi^2 + k\rho$$

Restricted EP(Local)

$$M' + M^2 = \frac{k}{n}\rho I_{2\times 2}$$

Weakly Restricted EP

$$M' + M^2 = \frac{k}{n}\rho I_{2\times 2} + R^{off}_{dig}$$

Modified EP(Global)

$$M' + M^2 = kR^{\nu}[\rho]$$

- $d = \nabla \cdot \mathbf{u}$, $\omega := \nabla \times \mathbf{u}$
- All restricted type EPs and the original EP share the same d dynamics equation. However, the evolutions of η and ξ are differ by models.

Derivation of the d dynamics equation

Expanding $M' + M^2 = kR[\rho]$, we obtain

Euler-Poisson system

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}' + \begin{bmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{bmatrix} = k \begin{bmatrix} R_{11}[\rho] & R_{12}[\rho] \\ R_{21}[\rho] & R_{22}[\rho] \end{bmatrix},$$

$\rho' + \rho \operatorname{tr} M = 0.$

We let

$$\begin{aligned} d &:= \operatorname{tr} M = \nabla \cdot \mathbf{u} (\textit{divergence}) \\ \omega &:= \nabla \times \mathbf{u} = M_{21} - M_{12} (\textit{vorticity}) \\ \eta &:= M_{11} - M_{22} \\ \xi &:= M_{12} + M_{21} \end{aligned}$$

Derivation of the d dynamics equation(cont'd)

Taking the trace, one obtain

$$
\begin{aligned}
d' &= -(M_{11}^2 + M_{22}^2) - 2M_{12}M_{21} + k(R_{11}[\rho] + R_{22}[\rho]) \\
&= -\left\{ \frac{(M_{11}+M_{22})^2}{2} + \frac{(M_{11}-M_{22})^2}{2} \right\} + \frac{(M_{21}-M_{12})^2}{2} - \frac{(M_{12}+M_{21})^2}{2} + k\rho \\
&= -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\omega^2 - \frac{1}{2}\xi^2 + k\rho
\end{aligned}
$$

(*Riccati type Ordinary Differential Equation*).

d-dynamics equation

d-dynamics equation of Euler-Poisson system

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\omega^2 - \frac{1}{2}\xi^2 + k\rho.$$

One can view the dynamics of d as the result of a

contest between negative and positive terms

in the d-dynamics equation. For example, one might think bigger $|\omega|$(correspond to the size of vorticity) prevents the finite time blow-up as opposed to the bigger η, ξ help the finite time blow-up.

d-dynamics equation(cont'd)

d-dynamics equation of Euler-Poisson system

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\omega^2 - \frac{1}{2}\xi^2 + k\rho.$$

From the matrix equaion we obtain

$$\eta' + \eta d = k(R_{11}[\rho] - R_{22}[\rho]), \tag{1a}$$
$$\omega' + \omega d = k(R_{21}[\rho] - R_{12}[\rho]) = 0, \tag{1b}$$
$$\xi' + \xi d = k(R_{12}[\rho] + R_{21}[\rho]), \tag{1c}$$
$$\rho' + \rho d = 0. \tag{1d}$$

From (1b) and (1d), we derive

$$\frac{\omega}{\omega_0} = \frac{\rho}{\rho_0}.$$

This allows us to rewrite the system,

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\left(\frac{\omega_0}{\rho_0}\right)^2\rho^2 - \frac{1}{2}\xi^2 + k\rho,$$
$$\rho' = -\rho d$$

Chae-Tadmor('08): Finite time blow-up; with no vorticity, attractive forcing

d and ρ-dynamics equations of Euler-Poisson system

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\left(\frac{\omega_0}{\rho_0}\right)^2\rho^2 - \frac{1}{2}\xi^2 + k\rho,$$
$$\rho' = -\rho d$$

- Chae-Tadmor(2008) : Assuming *vanishing initial vorticity*(i.e., $\omega_0 \equiv 0$), and dropping $-\eta^2$, $-\xi^2$ terms, the equation is reduced to simple Ricatti-type inequality
$$d' \leq -\frac{1}{2}d^2 + k\rho.$$
Using this argument, Chae and Tadmor proved the finite time blow-up for solutions of $k < 0$ case in arbitrary space dimension.

The restricted Euler-Poission system(REP) and a modified Euler-Poission system(MEP)

The restricted Euler-Poisson system(REP)

Motivation in REP:

$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = k\nabla\Delta^{-1}\rho$$
$$\Rightarrow \partial_t M + u\cdot\nabla M + M^2 = k\nabla\otimes\nabla\Delta^{-1}[\rho].$$
$$\Rightarrow M' + M^2 = kR[\rho]$$

There is no clear idea on how strong $kR[\rho]$ is compare to M^2. What we know is

$$tr(kR[\rho]) = k\rho.$$

REP is obtained from the full EP by restricting attention to the local isotropic trace $\frac{k}{2}\rho\cdot I_{2\times 2}$ of the global coupling term $kR[\rho]$.

The 2D restricted Euler-Poisson system(REP)[Liu-Tadmor (2002)]

$$M' + M^2 = \frac{k}{2}\rho\cdot\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},$$
$$\rho' + \rho\,\mathrm{tr}M = 0.$$

The restricted Euler-Poission system(REP) (cont'd)

The 2D restricted Euler-Poisson system(REP)[Liu-Tadmor (2002)]

$$M' + M^2 = \frac{k}{2}\rho \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\rho' + \rho \,\mathrm{tr} M = 0.$$

d-dynamics equation of 2D restricted Euler-Poisson system(REP)

$$d' = -\frac{d^2}{2} - \beta \cdot \frac{\rho^2}{2} + k\rho.$$

Liu-Tadmor(2003) studied the dynamics of (ρ, d) parametrized by β, and it was shown that in the repulsive case, the restricted two-dimensional REP system admits two-sided critical threshold.

For arbitrary $n \geq 3$ dimensional REP system, Lee-Liu(2014) identified both upper-thresholds for finite time blow-up of solutions and sub-thresholds for global existence of solutions.

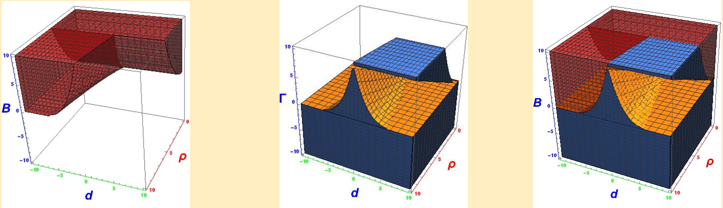
The weakly restricted Euler-Poisson system(WREP)

The 2D weakly restricted Euler-Poisson system(WREP)[Lee ('17)]

$$M' + M^2 = \begin{pmatrix} k\rho/2 & kR_{12}[\rho] \\ kR_{21}[\rho] & k\rho/2 \end{pmatrix},$$

$$\rho' + \rho \operatorname{tr} M = 0.$$

Comparison between WREP and REP



WREP blow-up   REP global-existence   Their union

The modified Euler-Poisson equations(MEP)

The modified Euler-Poisson equations(MEP)

(the original)Euler-Poisson system

$$\frac{D}{Dt}M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = k \begin{pmatrix} R_{11}[\rho] & R_{12}[\rho] \\ R_{21}[\rho] & R_{22}[\rho] \end{pmatrix},$$

$$\rho' + \rho \operatorname{tr} M = 0.$$

- Here,

$$R[\rho] := \nabla \otimes \nabla \Delta^{-1}[\rho] = \mathscr{F}^{-1}\left\{ \frac{\xi_i \xi_j}{|\xi|^2} \hat{\rho}(\xi) \right\}_{i,j=1,2}$$

-

$$(R_{ij}[\rho])(\vec{x}) := p.v. \int_{\mathbb{R}^2} \frac{\partial^2}{\partial y_i \partial y_j} G(\vec{y})\rho(\vec{x}-\vec{y})\, d\vec{y} + \frac{\rho(\vec{x})}{2\pi} \int_{|\vec{z}|=1} z_i z_j \, d\vec{z},$$

where $G(\vec{x}) = \frac{1}{2\pi} \log|\vec{x}|$ is the Green's function for the Poisson equation in two-dimensions.

The modified Euler-Poisson equations(cont'd)

Modified **Euler-Poisson system(MEP), Lee, '16**

$$\frac{D}{Dt}M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = k \begin{pmatrix} R_{11}^{\nu}[\rho] & R_{12}^{\nu}[\rho] \\ R_{21}^{\nu}[\rho] & R_{22}^{\nu}[\rho] \end{pmatrix},$$

$$\rho' + \rho \operatorname{tr} M = 0.$$

- Here,

$$(R_{ij}^{\nu}[\rho])(\vec{x}) := \underbrace{\int_{\mathbb{R}^2 \setminus B(0,\nu)} \frac{\partial^2}{\partial y_i \partial y_j} G(\vec{y})\rho(\vec{x}-\vec{y})\, d\vec{y}}_{truncated\ \ transform} + \frac{\rho(\vec{x})}{2\pi} \int_{|\vec{z}|=1} z_i z_j \, d\vec{z},$$

where $G(\vec{x}) = \frac{1}{2\pi}\log|\vec{x}|$ is the Green's function for the Poisson equation in two-dimensions.

The modified Riesz transform in the MEP system is intended to take into account the *global* forcing in the full Euler-Poisson equations, as opposed to the REP systems in [Liu-Tadmor] are *localized* Euler-Poisson equations.

Euler-Poisson

$$M' + M^2 = kR[\rho],$$
$$\rho' + \rho \operatorname{tr} M = 0.$$

Modified Euler-Poisson

$$M' + M^2 = kR^{\nu}[\rho],$$
$$\rho' + \rho \operatorname{tr} M = 0.$$

- $(R_{ij}[\rho])(\vec{x}) := pv \int_{\mathbb{R}^2} \cdots$
- $(R^{\nu}_{ij}[\rho])(\vec{x}) := \int_{\mathbb{R}^2 \setminus B(0,\nu)} \cdots$

$$(R_{ij}[\rho]) \xleftarrow{\nu \to 0} (R^{\nu}_{ij}[\rho])$$

- $R_{11}[\rho] + R_{22}[\rho] = \rho$
- $R^{\nu}_{11}[\rho] + R^{\nu}_{22}[\rho] = \rho$

- Lack of an accurate description for the propagation of $R[\rho]$
- We will later estimate $R^{\nu}_{ij}[\rho]$ using the L^1 norm of ρ

Statement of main theorems

Modified Euler-Poisson system(MEP)

$$\frac{D}{Dt}M + \begin{pmatrix} M_{11}^2 + M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21} + M_{22}^2 \end{pmatrix} = k \begin{pmatrix} R_{11}^{\nu}[\rho] & R_{12}^{\nu}[\rho] \\ R_{21}^{\nu}[\rho] & R_{22}^{\nu}[\rho] \end{pmatrix},$$

$$\rho' + \rho \operatorname{tr} M = 0.$$

Theorem 1(Lee, '16) : Blow-up for 2D MEP with attractive forcing ($k < 0$)

Consider the 2D attractive MEP system with $k < 0$. Suppose that $\rho(0,\cdot) \in L^1(\mathbb{R}^2)$, $d_0 < 0$ and $\rho_0 > 0$. If there exist a constant μ such that

$$\frac{|\omega_0|}{\rho_0} < \mu < \frac{\sqrt{\eta_0^2 + \xi_0^2}}{\rho_0},$$

and

$$F(\mu, d_0, \omega_0, \rho_0, \eta_0, \xi_0, \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}) \geq 0,$$

then $d(t)$ and $\rho(t)$ must blow-up at some finite time.

Statement of main theorems(cont'd)

Theorem 2(Lee, 16') : Blow-up for 2D MEP with repulsive forcing ($k > 0$)

Suppose that $\rho(0,\cdot) \in L^1(\mathbb{R}^2)$, $d_0 < 0$ and $\rho_0 > 0$. If there exist a constant μ such that

$$\sqrt{\left(\frac{\omega_0}{\rho_0}\right)^2 + \frac{2k}{\rho_0}} < \mu < \frac{\sqrt{\eta_0^2 + \xi_0^2}}{\rho_0},$$

and

$$F(\mu, d_0, \omega_0, \rho_0, \eta_0, \xi_0, \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}) \geq 0,$$

then $d(t)$ and $\rho(t)$ must blow-up at some finite time.

Here,

$$F(\mu, d, \omega, \rho, \eta, \xi, \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}) := \frac{\pi \nu^2}{\sqrt{2}|k| \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}} \left(\sqrt{\eta^2 + \xi^2} - \rho\mu\right) - \frac{\pi + 2\arctan(d/\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho})}{\sqrt{\mu^2\rho^2 - \omega^2 - 2k\rho}}.$$

Remarks on Theorems

- The critical threshold in 1D Euler-Poisson equations depends only on the relative size of the initial velocity gradient and initial density. In contrast to the one-dimensional Euler-Poisson equations, the threshold conditions in 2D MEP equations depend on several initial quantities: density ρ_0, divergence d_0, vorticity ω_0, gaps η_0, ξ_0 and even *total mass* $\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}$
- One can easily check that how F depends on those initial configurations:

$$\frac{\partial F}{\partial d} < 0, \ \frac{\partial F}{\partial(\omega^2)} < 0, \ \frac{\partial F}{\partial \rho} > 0, \ \frac{\partial F}{\partial \|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}} > 0, \ \frac{\partial F}{\partial \eta} > 0, \ \textit{and} \ \frac{\partial F}{\partial \xi} > 0.$$

For example, F is increasing in ρ, $\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}$ and $-d$. This is interpreted as if there is a point $\vec{x} \in \mathbb{R}^2$ with highly accumulated mass with low divergence, then there may be a finite time blow-up of the density.

Sketch of the proofs

Sketch of the proofs

d-dynamics equation: $d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\left(\frac{\omega_0}{\rho_0}\right)^2\rho^2 - \frac{1}{2}\xi^2 + k\rho \Rightarrow$

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\left[-\left(\frac{\omega_0}{\rho_0}\right)^2 + \left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)^2 + \left(\frac{\xi_0}{\rho_0} + \int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)^2\right]\rho^2 + k\rho$$

- Here, $f(t) := k(R^{\nu}_{11}[\rho] - R^{\nu}_{22}[\rho])$, $g(t) := k(R^{\nu}_{12}[\rho] - R^{\nu}_{21}[\rho])$.

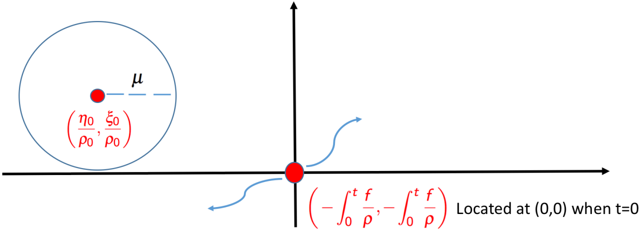
Sketch of the proofs

d-dynamics equation: $d' = -\frac{1}{2}d^2 - \frac{1}{2}\eta^2 + \frac{1}{2}\left(\frac{\omega_0}{\rho_0}\right)^2\rho^2 - \frac{1}{2}\xi^2 + k\rho \Rightarrow$

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\left[-\left(\frac{\omega_0}{\rho_0}\right)^2 + \left(\frac{\eta_0}{\rho_0} + \int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)^2 + \left(\frac{\xi_0}{\rho_0} + \int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)^2\right]\rho^2 + k\rho$$

- Here, $f(t) := k(R^{\nu}_{11}[\rho] - R^{\nu}_{22}[\rho])$, $g(t) := k(R^{\nu}_{12}[\rho] - R^{\nu}_{21}[\rho])$.
- For $t > 0$, it holds

$$\left|\int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right| \leq \frac{|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2} \cdot \int_0^t \frac{1}{\rho(\tau)}\,d\tau.$$



Sketch of the proofs

$$d' = -\frac{1}{2}d^2 - \frac{1}{2}\Bigg[-\left(\frac{\omega_0}{\rho_0}\right)^2 \underbrace{+\left(\frac{\eta_0}{\rho_0}+\int_0^t \frac{f(\tau)}{\rho(\tau)}\,d\tau\right)^2 + \left(\frac{\xi_0}{\rho_0}+\int_0^t \frac{g(\tau)}{\rho(\tau)}\,d\tau\right)^2}_{\textit{greater than } \mu^2 \textit{ for short initial period of time}}\Bigg]\rho^2 + k\rho$$

- For any $\mu \in (0, \frac{1}{\rho_0}\sqrt{\eta_0^2+\xi_0^2}]$, there exists $T > 0$ such that

$$\begin{aligned} d' &\leq -\frac{1}{2}d^2 + \frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2 - \mu^2\right\}\rho^2 + k\rho, \\ \rho' &= -d\rho, \end{aligned} \tag{4}$$

  for all $t \in [0, T]$. Furthermore, the lower bound $T^* > 0$ of T is obtained from

$$\sqrt{\left(\frac{\eta_0}{\rho_0}\right)^2 + \left(\frac{\xi_0}{\rho_0}\right)^2} - \mu = \frac{\sqrt{2}|k|\|\rho(0,\cdot)\|_{L^1(\mathbb{R}^2)}}{\pi\nu^2}\int_0^{t^*}\frac{1}{\rho(\tau)}\,d\tau.$$

- Find initial data such that $d \to -\infty$ at some time *before* T^*.

Thank you for your attention!