Semiclassical limit of the Schrödinger-Poisson-Landau-Lifshitz-Gilbert system

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Joint work with
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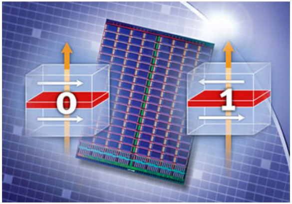
YRW 2016, Duke

Outline

Outline

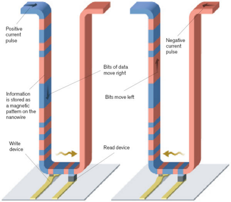
Magnetic devices

- Magnetic recording devices and computer storages
- Spinvalues[1]



Magnetic random access memory

- Domain walls [2]



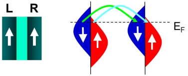
Racetrack memories

[1] Science@Berkeley Lab: The Current Spin on Spintronics

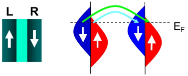
[2] http://www2.technologyreview.com/article/412189/tr10-racetrack-memory/

Methodology for detecting the orientation

- Tunnel magnetoresistance [3]



$G_P \propto n_L^{\uparrow} n_R^{\uparrow} + n_L^{\downarrow} n_R^{\downarrow}$



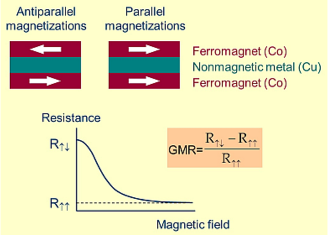
$G_{AP} \propto n_L^{\uparrow} n_R^{\downarrow} + n_L^{\downarrow} n_R^{\uparrow}$

Julliere's model:
Constant tunneling matrix

$$TMR \equiv \frac{G_{AP} - G_P}{G_{AP}} = \frac{2P_L P_R}{1 - P_L P_R}$$

$$P_L = \frac{n_L^{\uparrow} - n_L^{\downarrow}}{n_L^{\uparrow} + n_L^{\downarrow}} \qquad P_R = \frac{n_R^{\uparrow} - n_R^{\downarrow}}{n_R^{\uparrow} + n_R^{\downarrow}}$$

- Giant magnetoresistance [4]

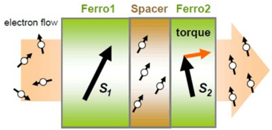


- Albert Fert & Peter Grünberg: 2007 Nobel Prize in Physics

[3] http://unlcms.unl.edu/cas/physics/tsymbal/reference/
[4] http://unlcms.unl.edu/cas/physics/tsymbal/reference/

Methodology for rotating the orientation

- Spin transfer torque (STT) [5]



- Two layers of different thickness: different switching fields
- The thin film is switched, and the resistance measured

[5] http://www.wpi-aimr.tohoku.ac.jp/mizukami_lab/

Micromagnetics: Landau-Lifshitz-Gilbert model

Basic quantity of interest:

$$\boldsymbol{m} : \Omega \longrightarrow \mathbb{R}^3; \quad |\boldsymbol{m}| = M_s$$

Landau-Lifshitz energy functional:

$$\begin{aligned} F[\boldsymbol{m}] &= \int_\Omega \phi\left(\frac{\boldsymbol{m}}{M_s}\right) dx + \frac{A}{M_s^2} \int_\Omega |\nabla \boldsymbol{m}|^2 \, dx \\ &- \frac{\mu_0}{2} \int_\Omega \mathbf{H}_s \cdot \boldsymbol{m} \, dx - \mu_0 \int_\Omega \mathbf{H}_e \cdot \boldsymbol{m} \, dx \end{aligned}$$

- $\phi\left(\frac{\boldsymbol{m}}{M_s}\right)$: Anisotropy Energy: Penalizes deviations from the easy directions. For uniaxial materials $\phi(\boldsymbol{m}) = K_u\left((m_2/M_s)^2 + (m_3/M_s)^2\right)$.

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- The stray field, $\boldsymbol{H}_s = -\nabla u$ is obtained by solving the magnetostatic equation:

$$\Delta u = \text{div } \boldsymbol{m}, \quad \mathbf{x} \in \Omega, \quad \Delta u = 0, \quad \mathbf{x} \in \overline{\Omega}^c,$$

with jump boundary conditions

$$[u]_{\partial\Omega} = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_{\partial\Omega} = -\boldsymbol{m} \cdot \nu.$$

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- With the injection of spin current
  - $$\frac{\partial \boldsymbol{m}}{\partial t} = -\gamma\, \boldsymbol{m} \times (\boldsymbol{H} + J\boldsymbol{s}) + \alpha \boldsymbol{m} \times \frac{\partial \boldsymbol{M}}{\partial t},$$
    where $\boldsymbol{s}$ is the spin procession and J is the coupling strength between spin and magnetization

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- Quantum (Schrödinger equation in spinor form)

$$i\hbar\frac{\partial \boldsymbol{\psi}}{\partial t} = \left(\left(-\frac{\hbar^2}{2m}\nabla_{\boldsymbol{x}}^2 + V(\boldsymbol{x})\right)\widehat{\boldsymbol{I}} - \frac{\mu_B\omega}{2}\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}(\boldsymbol{x},t)\right)\psi.$$

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- Kinetic (Boltzmann equation)

$$\partial_t\boldsymbol{W}(\boldsymbol{x},\boldsymbol{v},t) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{W}(\boldsymbol{x},\boldsymbol{v},t) - \frac{e}{m}\boldsymbol{E}\cdot\nabla_{\boldsymbol{v}}\boldsymbol{W}(\boldsymbol{x},\boldsymbol{v},t)$$
$$-\frac{\mathrm{i}}{2\hbar}[\mu_B\omega\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}(\boldsymbol{x},t),\boldsymbol{W}(\boldsymbol{x},\boldsymbol{v},t)] = -\frac{\boldsymbol{W}-\bar{\boldsymbol{W}}}{\tau} - \frac{2}{\tau_{sf}}(\bar{\boldsymbol{W}} - \frac{\widehat{\boldsymbol{I}}}{2}\mathrm{Tr}_{\mathbb{C}^2}\bar{\boldsymbol{W}})$$

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- Hydrodynamic (Diffusion equation)

$$\frac{\partial\boldsymbol{s}}{\partial t} = -\mathrm{div}\,\mathbf{J}_{\boldsymbol{s}} - 2D_0(\mathbf{x})\frac{\boldsymbol{s}}{\lambda_{sf}^2} - 2D_0(\mathbf{x})\frac{\boldsymbol{s}\times\boldsymbol{m}}{\lambda_J^2},$$
$$\mathbf{J}_{\boldsymbol{s}} = \frac{\beta\mu_B}{e}\mathbf{J}_n\otimes\boldsymbol{m} - 2D_0(\mathbf{x})\left[\nabla\boldsymbol{s} - \beta\beta'\left(\nabla\boldsymbol{s}\cdot\boldsymbol{m}\right)\otimes\boldsymbol{m}\right].$$

Outline

The SPLLG system

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- The Schrödigner-Poisson (SP) equation

$$\mathrm{i}\varepsilon\partial_t\psi_j^\varepsilon(\boldsymbol{x},t) = -\frac{\varepsilon^2}{2}\Delta\psi_j^\varepsilon(\boldsymbol{x},t) + V^\varepsilon\psi_j^\varepsilon(\boldsymbol{x},t) - \frac{\varepsilon}{2}\boldsymbol{m}^\varepsilon\cdot\widehat{\boldsymbol{\sigma}}\psi_j^\varepsilon(\boldsymbol{x},t),$$

$$-\Delta_{\boldsymbol{x}}V^\varepsilon = \rho^\varepsilon, \quad \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\ j\in\mathbb{N}$$

$$\psi_j^\varepsilon(\boldsymbol{x},t=0) = \varphi_j^\varepsilon(\boldsymbol{x})$$

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- The Landau-Lifshitz-Gilbert (LLG) equation

$$\partial_t\boldsymbol{m}^\varepsilon = -\boldsymbol{m}^\varepsilon\times\boldsymbol{H}_{\text{eff}}^\varepsilon + \alpha\boldsymbol{m}^\varepsilon\times\partial_t\boldsymbol{m}^\varepsilon,$$
$$|\boldsymbol{m}^\varepsilon(\boldsymbol{x},t)| = 1,\ \boldsymbol{x}\in\Omega,\ t>0$$
$$\frac{\partial\boldsymbol{m}^\varepsilon}{\partial\nu} = 0 \quad \text{on } \partial\Omega$$
$$\boldsymbol{m}^\varepsilon(\boldsymbol{x},t=0) = \boldsymbol{m}_0$$

- The occupation numbers $\lambda_j^\varepsilon > 0$ satisfy that there exist $C > 0$ such that
$$\sum_{j=1}^{\infty} \lambda_j^\varepsilon + \varepsilon^2 \sum_{j=1}^{\infty} \lambda_j^\varepsilon \|\nabla \varphi_j^\varepsilon\|_{L^2(\mathbb{R}^3)}^2 + \varepsilon^{-3} \sum_{j=1}^{\infty} (\lambda_j^\varepsilon)^2 \leq C$$

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- Physical observables

$$\rho^\varepsilon(\boldsymbol{x}, t) = \sum_{j=1}^{\infty} \lambda_j^\varepsilon \, |\psi_j^\varepsilon(\boldsymbol{x}, t)|^2,$$

$$\boldsymbol{j}^\varepsilon(\boldsymbol{x}, t) = \varepsilon \sum_{j=1}^{\infty} \lambda_j^\varepsilon \, \mathrm{Im}\big(\psi_j^{\varepsilon\,\dagger}(\boldsymbol{x}, t) \nabla_{\boldsymbol{x}} \psi_j^\varepsilon(\boldsymbol{x}, t)\big),$$

$$\boldsymbol{s}^\varepsilon(\boldsymbol{x}, t) = \sum_{j=1}^{\infty} \lambda_j^\varepsilon \mathrm{Tr}_{\mathbb{C}^2} \left(\widehat{\boldsymbol{\sigma}}\left(\psi_j^\varepsilon(\boldsymbol{x}, t) \psi_j^{\varepsilon\,\dagger}(\boldsymbol{x}, t)\right)\right)$$

- The Landau-Lifshitz-Gilbert energy

$$F_{\mathrm{LL}} = \int_\Omega \left(\frac{1}{2} |\nabla \boldsymbol{m}^\varepsilon|^2 - \frac{1}{2} \boldsymbol{H}_{\mathrm{s}}^\varepsilon \cdot \boldsymbol{m}^\varepsilon - \frac{\varepsilon}{2} \boldsymbol{s}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \right) \,\mathrm{d}\boldsymbol{x},$$

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$$\boldsymbol{H}_{\mathrm{eff}}^\varepsilon = -\frac{\delta F_{\mathrm{LL}}}{\delta \boldsymbol{m}^\varepsilon} = \Delta \boldsymbol{m}^\varepsilon + \boldsymbol{H}_{\mathrm{s}}^\varepsilon + \frac{\varepsilon}{2} \boldsymbol{s}^\varepsilon$$

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- The stray field

$$\boldsymbol{H}_{\mathrm{s}}^\varepsilon(\boldsymbol{x}) = -\nabla \int_\Omega \nabla \mathcal{N}(\boldsymbol{x} - \boldsymbol{y}) \cdot \boldsymbol{m}^\varepsilon(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y},$$

with $\mathcal{N}(\boldsymbol{x}) = -\frac{1}{4\pi \boldsymbol{x}}$.

The Wigner-Poisson equation

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- The Wigner transformation

$$W^{\varepsilon}(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3_{\boldsymbol{y}}}\sum_{j=1}^{\infty}\lambda_j^{\varepsilon}\psi_j^{\varepsilon}\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\psi_j^{\varepsilon\dagger}\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right)\mathrm{e}^{\mathrm{i}\boldsymbol{v}\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,.$$

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- The Wigner equation

$$\partial_t W^{\varepsilon}+\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}W^{\varepsilon}-\left(\Theta^{\varepsilon}[V^{\varepsilon}]+\frac{\mathrm{i}}{2}\Gamma^{\varepsilon}[\boldsymbol{m}^{\varepsilon}]\right)W^{\varepsilon}=0,$$

where the operator Θ^ε is given by

$$\Theta^\varepsilon[V^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \frac{1}{\mathrm{i}\varepsilon}\left[V^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right) - V^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right] W^\varepsilon(\boldsymbol{x},\boldsymbol{v}') \times \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}',$$

and the operator Γ^ε is given by

$$\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \left[M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right) W^\varepsilon(\boldsymbol{x},\boldsymbol{v}') - W^\varepsilon(\boldsymbol{x},\boldsymbol{v}')M^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right] \times \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}',$$

where the matrix $M^\varepsilon = \widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon$.

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- The Schrödinger equations

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$$

- The Poisson potential $V = -\mathcal{N} * \rho$
- The effective field $\boldsymbol{H} = \Delta\boldsymbol{m} + \boldsymbol{H}_s + \frac{1}{2}\boldsymbol{s}$, and $\boldsymbol{H}_s = -\nabla(\nabla\mathcal{N} * \cdot\boldsymbol{m})$,

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- Theorem:

  Given any initial conditions with $\boldsymbol{\Phi} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)$ and $\boldsymbol{m}_0 \in H^1(\Omega)$, $|\boldsymbol{m}_0| = 1$, *a.e.* . Then there exists $\boldsymbol{\Psi} \in L^\infty([0,\infty), \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3))$ and $\boldsymbol{m} \in C([0,\infty), H^1(\Omega))$, $|\boldsymbol{m}| = 1$, *a.e.* , such that the SPLLG system hold weakly.

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- Introduce the summation Hilbert space $\mathcal{H}^r_{\boldsymbol{\lambda}}(K)$ by defined $\|\boldsymbol{\Psi}\|^2_{\mathcal{H}^r_{\boldsymbol{\lambda}}(K)} := \sum_{j=1}^{\infty} \lambda_j \|\psi_j\|^2_{H^r(K)}$.
- Theorem:

  Given any initial conditions with $\boldsymbol{\Phi} \in \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3)$ and $\boldsymbol{m}_0 \in H^1(\Omega)$, $|\boldsymbol{m}_0| = 1$, *a.e.* . Then there exists $\boldsymbol{\Psi} \in L^\infty([0,\infty), \mathcal{H}^1_{\boldsymbol{\lambda}}(\mathbb{R}^3))$ and $\boldsymbol{m} \in C([0,\infty), H^1(\Omega))$, $|\boldsymbol{m}| = 1$, *a.e.* , such that the SPLLG system hold weakly.

- We prove this theorem first in a bounded domain K and then let $K \to \mathbb{R}^3$.

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  - We seek the approximate solutions
    $\psi_{j_N}(\boldsymbol{x}, t) = \sum_{n=1}^{N} \alpha_{jn}(t)\theta_n(\boldsymbol{x})$, and
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  - and satisfy the initial conditions
    $\psi_{jN}(\cdot, 0) = \Pi_N^K\varphi_j$, and $\boldsymbol{m}_N(\cdot, 0) = \Pi_N^\Omega \boldsymbol{m}_0$,
    where Π_N^K and Π_N^Ω is the orthogonal projections to $\mathrm{Span}\{\theta_n\}_{n=1}^N$ and $\mathrm{Span}\{\omega_n\}_{n=1}^N$, *resp.*.

Galerkin solutions

[6] F. Alouges & A. Soyeur 1992

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- Estimates then are based the following conservation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_K\sum_{j=1}^{\infty}\lambda_j|\nabla\psi_{jN}|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_K|\nabla V_N|^2$$
$$+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\boldsymbol{m}_N|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3}|\boldsymbol{H}_{sN}|^2$$
$$+\frac{k}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(|\boldsymbol{m}_N|^2-1\right)^2 + 2\alpha\int_\Omega|\partial_t\boldsymbol{m}_N|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\boldsymbol{s}_N\cdot\boldsymbol{m}_N.$$

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- $|\boldsymbol{m}_N|^2 - 1 \xrightarrow{N\to\infty} |\boldsymbol{m}^k|^2 - 1 \in L^\infty(\mathbb{R}^+, L^2(\Omega))$, weak* ,

Weak Solutions to the penalized problem

Lemma

For all $\chi \in H^1([0,T] \times \Omega)$ *and* $\eta \in C([0,T], H^1(K))$, *it holds that*

$$
\begin{aligned}
\mathrm{i} \int_0^T \int_K \partial_t {\psi_j}^k \eta &= \frac{1}{2} \int_0^T \int_K \nabla {\psi_j}^k \cdot \nabla \eta + \int_0^T \int_K V^k {\psi_j}^k \eta \\
&\quad - \frac{1}{2} \int_0^T \int_K \boldsymbol{m}^k \cdot \widehat{\boldsymbol{\sigma}} {\psi_j}^k \eta, \\
\int_0^T \int_\Omega \alpha \partial_t \boldsymbol{m}^k \chi &= - \int_0^T \int_\Omega \left(\boldsymbol{m}^k \times \partial_t \boldsymbol{m}^k - {\boldsymbol{H}_s}^k - \frac{1}{2} \boldsymbol{s}^k \right) \chi \\
&\quad + \int_0^T \int_\Omega k \left(|\boldsymbol{m}^k|^2 - 1 \right) \boldsymbol{m}^k \chi + \nabla \boldsymbol{m}^k \cdot \nabla \chi .
\end{aligned}
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- Since $|\boldsymbol{m}| = 1$ *a.e.* , by a density argument, we also obtain the above equation holds for all $\xi \in H^1([0,T]\times\Omega)$.

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[7] F. Brezzi & P.A. Markowich 1991

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Outline

The WPLLG system in the semiclassical regime.

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- The Wigner equation
  $\partial_t W^\varepsilon + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} W^\varepsilon - \left(\Theta^\varepsilon[V^\varepsilon] + \frac{i}{2}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]\right) W^\varepsilon = 0$

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  $\boldsymbol{H}_{\text{s}}^\varepsilon = -\nabla(\nabla \mathcal{N} * \cdot \boldsymbol{m}^\varepsilon)$
- The behavior of the solution $(W^\varepsilon, \boldsymbol{m}^\varepsilon)$ in the semiclassical limit $\varepsilon \to 0$.

Main theorem: The semiclassical limit

There exists a subsequence of solutions $(W^\varepsilon, \boldsymbol{m}^\varepsilon)$ to the WPLLG system such that

$W^\varepsilon \xrightarrow{\varepsilon\to 0} W$ in $L^\infty((0,T); L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}))$ weak $*$

$\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{m}$ in $L^\infty((0,T); H^1(\Omega))$ weak $*$

and $(W, \boldsymbol{m})$ is a weak solution of the following VPLLG system,

$$\partial_t W = -\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} W + \nabla_{\boldsymbol{x}} V \cdot \nabla_{\boldsymbol{v}} W + \frac{\mathrm{i}}{2}[\widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m},\, W],$$

$$\partial_t \boldsymbol{m} = -\boldsymbol{m}\times\boldsymbol{H}_{\text{eff}} + \alpha\boldsymbol{m}\times\partial_t\boldsymbol{m},$$

$$V(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \frac{W(\boldsymbol{y},\boldsymbol{v},t)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{y},$$

$$\boldsymbol{H}_{\text{eff}} = \Delta\boldsymbol{m} + \boldsymbol{H}_{\text{s}},$$

$$\boldsymbol{H}_{\text{s}}(\boldsymbol{x}) = -\nabla\left(\frac{1}{4\pi}\int_\Omega \frac{\nabla\cdot\boldsymbol{m}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}\right).$$

Conservation quantities.

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Conservation of the total mass.

$$\int_{\mathbb{R}^3_{\boldsymbol{x}}} \rho^\varepsilon(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x}, \boldsymbol{v}, t)\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

$$= \int_{\mathbb{R}^3_{\boldsymbol{x}}} \rho^\varepsilon(\boldsymbol{x}, 0)\,\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{\infty} \lambda_j^\varepsilon.$$

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Conservation of the L^2-norm of W^ε.

$$\| W^\varepsilon(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} := \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left\{\left[W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right]^2\right\}\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

$$= \| W^\varepsilon_I\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} = \frac{2}{(4\pi\varepsilon)^3}\sum_{j=1}^{\infty}(\lambda_j^\varepsilon)^2.$$

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Conservation of the total mass.

$$\int_{\mathbb{R}^3_{\boldsymbol{x}}} \rho^\varepsilon(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x} = \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left(W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right)\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

$$= \int_{\mathbb{R}^3_{\boldsymbol{x}}} \rho^\varepsilon(\boldsymbol{x},0)\,\mathrm{d}\boldsymbol{x} = \sum_{j=1}^{\infty}\lambda_j^\varepsilon.$$

Conservation of the L^2-norm of W^ε.

$$\|W^\varepsilon(t)\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} := \int_{\mathbb{R}^3_{\boldsymbol{x}}}\int_{\mathbb{R}^3_{\boldsymbol{v}}} \mathrm{Tr}_{\mathbb{C}^2}\left\{\left[W^\varepsilon(\boldsymbol{x},\boldsymbol{v},t)\right]^2\right\}\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{x}$$

$$= \|W_I^\varepsilon\|^2_{L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})} = \frac{2}{(4\pi\varepsilon)^3}\sum_{j=1}^{\infty}(\lambda_j^\varepsilon)^2.$$

Conservation in the magnetization.

$$|\boldsymbol{m}^\varepsilon(\boldsymbol{x},t)| = |\boldsymbol{m}_0(\boldsymbol{x})| \equiv 1;$$

$$\|\boldsymbol{m}^\varepsilon(\boldsymbol{x},t)\|_{L^2(\Omega)} = \|\boldsymbol{m}_0(\boldsymbol{x})\|_{L^2(\Omega)} = |\Omega|.$$

Energy dissipation.

Energy dissipation.

- Schrödinger-Poisson energy
  $F_{\mathrm{SP}} = E^{\varepsilon}_{\mathrm{kin}} + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}} |\nabla V^{\varepsilon}|^2 \,\mathrm{d}\boldsymbol{x}$

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$$F_{SP} = E_{kin}^{\varepsilon} + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}} |\nabla V^{\varepsilon}|^2 \,\mathrm{d}\boldsymbol{x}$$

- The Landau-Lifschitz energy

$$F_{LL} = \int_{\Omega} \left(\frac{1}{2}|\nabla \boldsymbol{m}^{\varepsilon}|^2 + \frac{1}{2}|\boldsymbol{H}_s^{\varepsilon}|^2 - \frac{\varepsilon}{2}\boldsymbol{s}^{\varepsilon}\cdot\boldsymbol{m}^{\varepsilon} \right) \mathrm{d}\boldsymbol{x}$$

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$$F_{\mathrm{SP}} = E_{\mathrm{kin}}^{\varepsilon} + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}} |\nabla V^{\varepsilon}|^2 \,\mathrm{d}\boldsymbol{x}$$
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- The energy dissipation

$$\frac{\mathrm{d}}{\mathrm{d}t}(F_{\mathrm{LL}} + F_{\mathrm{SP}}) + \alpha \int_{\mathbb{R}^3_{\boldsymbol{x}}} |\partial_t \boldsymbol{m}^{\varepsilon}(\boldsymbol{x}, t)|^2 \,\mathrm{d}\boldsymbol{x} \leq 0$$

Energy dissipation. Cont.

Using the estimate

$$\left|\int_\Omega \boldsymbol{s}^\varepsilon \cdot \boldsymbol{m}^\varepsilon \,\mathrm{d}\boldsymbol{x}\right| \leq \int_\Omega |\boldsymbol{s}^\varepsilon \cdot \boldsymbol{m}^\varepsilon| \,\mathrm{d}\boldsymbol{x} \leq \int_{\mathbb{R}^3_{\boldsymbol{x}}} |\boldsymbol{s}^\varepsilon| \,\mathrm{d}\boldsymbol{x} \leq C$$

we get there exists a constant C independent of ε, such that[8]

$$\int_\Omega |\nabla \boldsymbol{m}^\varepsilon(t)|^2 \,\mathrm{d}\boldsymbol{x} + E^\varepsilon_{\mathrm{kin}}(t) + \frac{1}{2}\int_{\mathbb{R}^3_{\boldsymbol{x}}} |\nabla V^\varepsilon(\boldsymbol{x},t)|^2 \,\mathrm{d}\boldsymbol{x} + \alpha \int_0^t \int_{\mathbb{R}^3_{\boldsymbol{x}}} |\partial_t \boldsymbol{m}|^2$$

$$\leq C + F_{\mathrm{LL}}(0) + E^\varepsilon_{\mathrm{kin}}(0) + \int_{\mathbb{R}^3_{\boldsymbol{x}}} |\nabla V^\varepsilon(\boldsymbol{x},0)|^2 \,\mathrm{d}\boldsymbol{x}$$

$$\leq C.$$

[8] Brezzi & Markowich 1991, Arnold 1996

Boundedness

From the conservation equations and the energy dissipation we get the following boundedness

$$E^{\varepsilon}_{\text{kin}}(t) = \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} |\boldsymbol{v}|^2 \mathrm{Tr}_{\mathbb{C}^2}\left(W^{\varepsilon}(\boldsymbol{x}, \boldsymbol{v}, t)\right) \mathrm{d}\boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \leq C,$$

$$\|V^{\varepsilon}\|_{L^{\infty}((0,\infty),L^6(\mathbb{R}^3_{\boldsymbol{x}}))} + \|\nabla V^{\varepsilon}\|_{L^{\infty}((0,\infty),L^2(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C.$$

$$\|W^{\varepsilon}\|_{L^2(\mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}})} + \|\boldsymbol{m}^{\varepsilon}(t)\|_{L^2(\Omega)} + \|\nabla \boldsymbol{m}^{\varepsilon}(t)\|_{L^2(\Omega)} \leq C,$$

and

$$\|\partial_t \boldsymbol{m}^{\varepsilon}\|_{L^2([0,T],L^2(\mathbb{R}^3))} \leq C.$$

From the interpolation lemma[9]

Lemma

Let $1 \leq p \leq \infty$, $q = (5p-3)/(3p-1)$, $s = (5p-3)/(4p-2)$, and $\theta = 2p/(5p-3)$. Then $\exists C > 0$ s.t.
$\|\rho^\varepsilon\|_{L^q} \leq C|\lambda^\varepsilon|_p^\theta \left(\varepsilon^{-2} E_{\text{kin}}^\varepsilon\right)^{1-\theta}$, $\|\boldsymbol{j}^\varepsilon\|_{L^s} \leq C|\lambda^\varepsilon|_p^\theta \left(\varepsilon^{-2} E_{\text{kin}}^\varepsilon\right)^{1-\theta}$,
where $|\lambda^\varepsilon|_p = \left(\sum_{j=1}^\infty |\lambda_j^\varepsilon|^p\right)^{1/p}$

under the assumption $|\lambda^\varepsilon|_2 \leq C$ we get the estimats

$$\|\rho^\varepsilon\|_{L^\infty((0,\infty),L^q(\mathbb{R}^3_{\boldsymbol{x}}))} + \|\boldsymbol{s}^\varepsilon\|_{L^\infty((0,\infty),L^q(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad q \in [1, 6/5],$$

$$\|\boldsymbol{j}^\varepsilon\|_{L^\infty((0,\infty),L^s(\mathbb{R}^3_{\boldsymbol{x}}))} + \|J_{\mathrm{s}}^\varepsilon\|_{L^\infty((0,\infty),L^s(\mathbb{R}^3_{\boldsymbol{x}}))} \leq C, \quad s \in [1, 7/6].$$

[9] Arnold 1996

Convergence subsequences.

$$W^\varepsilon \xrightarrow{\varepsilon\to 0} W \text{ in } L^\infty((0,\infty);L^2(\mathbb{R}^3_{\boldsymbol{x}}\times\mathbb{R}^3_{\boldsymbol{v}})) \text{ weak} * ,$$

$$\rho^\varepsilon \xrightarrow{\varepsilon\to 0} \rho \text{ in } L^\infty((0,\infty);L^q(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak} * , q\in[1,6/5]$$

$$\boldsymbol{s}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{s} \text{ in } L^\infty((0,\infty);L^q(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak} * , q\in[1,6/5]$$

$$\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{m} \text{ in } L^\infty((0,\infty);H^1(\Omega)) \text{ weak} * ,$$

$$\boldsymbol{m}^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{m} \text{ in } L^2([0,T],L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ strongly.}$$

$$\boldsymbol{H}_s^\varepsilon \xrightarrow{\varepsilon\to 0} \boldsymbol{H} \text{ in } L^\infty((0,\infty);L^2(\Omega)) \text{ weak} * ,$$

$$V^\varepsilon \xrightarrow{\varepsilon\to 0} V \text{ in } L^\infty((0,\infty);L^6(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak} * ,$$

$$\nabla V^\varepsilon \xrightarrow{\varepsilon\to 0} \nabla V \text{ in } L^\infty((0,\infty);L^2(\mathbb{R}^3_{\boldsymbol{x}})) \text{ weak } * \ .$$

Passing to the limit of the Wigner equation

We study the weak formulation of the Wigner equation

$$\iiint \left[W^\varepsilon(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi) + \left(\Theta^\varepsilon[V^\varepsilon] + \frac{\mathrm{i}}{2}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] \right) W^\varepsilon \phi \right] = 0.$$

[10] Markowich & Mauser 1993

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- $\lim_{\varepsilon \to 0} \iiint W^\varepsilon(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi) = \iiint W(\partial_t \phi + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \phi).$

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- [10] $\lim_{\varepsilon\to 0}\iiint \Theta^\varepsilon[V^\varepsilon]W^\varepsilon\phi = -\iiint W\nabla_{\boldsymbol{x}}V\cdot\nabla_{\boldsymbol{v}}\phi$

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- [10] $\lim_{\varepsilon\to 0} \iiint \Theta^\varepsilon[V^\varepsilon]W^\varepsilon\phi = -\iiint W\nabla_{\boldsymbol{x}}V\cdot\nabla_{\boldsymbol{v}}\phi$
- $\lim_{\varepsilon\to 0} \iiint \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon\phi \overset{?}{=} \iiint [\boldsymbol{m}\cdot\widehat{\boldsymbol{\sigma}}, W]\phi$

[10] Markowich & Mauser 1993

Recall that the operator Θ^ε is given by

$$\Theta^\varepsilon[V^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \frac{1}{\mathrm{i}\varepsilon}\left[V^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right) - V^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right] W^\varepsilon(\boldsymbol{x},\boldsymbol{v}') \times \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}',$$

and the operator Γ^ε is given by

$$\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon(\boldsymbol{x},\boldsymbol{v}) = \frac{1}{(2\pi)^3}\iint \left[M^\varepsilon\left(\boldsymbol{x}-\frac{\varepsilon\boldsymbol{y}}{2}\right) W^\varepsilon(\boldsymbol{x},\boldsymbol{v}') - W^\varepsilon(\boldsymbol{x},\boldsymbol{v}')M^\varepsilon\left(\boldsymbol{x}+\frac{\varepsilon\boldsymbol{y}}{2}\right)\right] \times \mathrm{e}^{\mathrm{i}(\boldsymbol{v}-\boldsymbol{v}')\cdot\boldsymbol{y}}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{v}',$$

where the matrix $M^\varepsilon = \widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}^\varepsilon$.

$\lim_{\varepsilon \to 0} \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] W^\varepsilon$

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- If $\boldsymbol{m}^\varepsilon \in H^1(\mathbb{R}^3)$, we can prove (by Taylor's theorem)
$$\lim_{\varepsilon\to 0} \iiint \Gamma^\varepsilon[\boldsymbol{m}^\varepsilon] W^\varepsilon \phi = \iiint [\boldsymbol{m}\cdot\widehat{\boldsymbol{\sigma}}, W]\phi$$

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- Let $\boldsymbol{m}^{\varepsilon,\beta} = \boldsymbol{m}^\varepsilon *_{\boldsymbol{x}} \varphi^\beta$, and then $\boldsymbol{m}^\varepsilon = (\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{\varepsilon,\beta}) + \boldsymbol{m}^{\varepsilon,\beta}$, where $\varphi^\beta(\boldsymbol{x}) = \varphi(\boldsymbol{x}/\beta)$ and φ is a positive mollifier.

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- $$\begin{aligned}&\left|\iiint \left(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon - [M, W]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\\ \leq &\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{\varepsilon,\beta}\right] W^\varepsilon\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\\ &+\left|\iiint \left(\Gamma^\varepsilon\left[\boldsymbol{m}^{\varepsilon,\beta}\right] W^\varepsilon - \left[M^\beta, W\right]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\\ &+\left|\iiint \left[M^\beta - M, W\right]\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|,\end{aligned}$$

  where $M = \widehat{\boldsymbol{\sigma}}\cdot\boldsymbol{m}$ and $M^\beta = M *_{\boldsymbol{x}} \varphi^\beta$.

$\lim_{\varepsilon\to 0} \Gamma^{\varepsilon}[\boldsymbol{m}^{\varepsilon}] W^{\varepsilon}$ Cont.

$\lim_{\varepsilon\to 0}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon$ Cont.

- For the second integral, since $\boldsymbol{m}^{\varepsilon,\beta}\in H^1(\mathbb{R}^3)$ and $\boldsymbol{m}^{\varepsilon,\beta}\to\boldsymbol{m}*_{\boldsymbol{x}}\varphi^\beta$ strongly in $L^2([0,T]\times\mathbb{R}^3)$, we have $\lim_{\varepsilon\to 0}\iiint\left(\Gamma^\varepsilon[\boldsymbol{m}^{\varepsilon,\beta}]W^\varepsilon-[M^\beta,W]\right)\phi=0$.

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- For the third integral we have $$\left|\iiint \left[M^\beta - M, W\right]\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \le C_\beta \to 0, \text{ as } \beta \to 0.$$

$\lim_{\varepsilon\to 0}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon$ Cont.

- For the second integral, since $\boldsymbol{m}^{\varepsilon,\beta}\in H^1(\mathbb{R}^3)$ and $\boldsymbol{m}^{\varepsilon,\beta}\to\boldsymbol{m}*_{\boldsymbol{x}}\varphi^\beta$ strongly in $L^2([0,T]\times\mathbb{R}^3)$, we have $\lim_{\varepsilon\to 0}\iiint\left(\Gamma^\varepsilon[\boldsymbol{m}^{\varepsilon,\beta}]W^\varepsilon-[M^\beta,W]\right)\phi=0$.
- For the third integral we have $\left|\iiint\left[M^\beta-M,W\right]\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\le C_\beta\to 0$, as $\beta\to 0$.
- For the first integral, we use triangle inequality to get

$$
\begin{aligned}
&\left|\iiint\Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon-\boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\\
&\le C\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}+C\|\boldsymbol{m}^\beta-\boldsymbol{m}^{\varepsilon,\beta}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\\
&\quad+C\|\boldsymbol{m}-\boldsymbol{m}^\beta\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}\\
&\le C\|\boldsymbol{m}^\varepsilon-\boldsymbol{m}\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})}+C\|\boldsymbol{m}-\boldsymbol{m}^\beta\|_{L^2([0,T]\times\mathbb{R}^3_{\boldsymbol{x}})},
\end{aligned}
$$

$\lim_{\varepsilon\to 0}\Gamma^{\varepsilon}[\boldsymbol{m}^{\varepsilon}]W^{\varepsilon}$ Cont..

$\lim_{\varepsilon\to 0}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon$ Cont..

- Then

$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon-\boldsymbol{m}^{\varepsilon,\beta}\right]W^\varepsilon\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\leq C_\beta.$$

$\lim_{\varepsilon\to 0}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon$ Cont..

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- And Then

$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint \left(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon-[M,W]\right)\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right|\le C_\beta.$$

$\lim_{\varepsilon\to 0}\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon$ Cont..

- Then

$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint \Gamma^\varepsilon\left[\boldsymbol{m}^\varepsilon - \boldsymbol{m}^{\varepsilon,\beta}\right] W^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \le C_\beta.$$

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$$\overline{\lim_{\varepsilon\to 0}}\left|\iiint \left(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon - [M, W]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| \le C_\beta.$$

- But the left hand side of above inequality is independent of β, we then have

$$\lim_{\varepsilon\to 0}\left|\iiint \left(\Gamma^\varepsilon[\boldsymbol{m}^\varepsilon]W^\varepsilon - [M, W]\right)\phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{v}\,\mathrm{d}t\right| = 0.$$

Passing to the limit of the LLG equation

The weak formulation of the LLG equation

$$\iint \boldsymbol{m}^\varepsilon \, \partial_t \phi = \iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}^\varepsilon_{\text{eff}} \, \phi - \alpha \iint \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \, \phi.$$

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- Since $\partial_t \boldsymbol{m}^\varepsilon \to \partial_t \boldsymbol{m}^\varepsilon$ in $L^2([0,T], L^2(\Omega))$ weakly, we have
  $$\lim_{\varepsilon\to 0} \iint \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \partial_t \phi = \iint \boldsymbol{m} \times \partial_t \boldsymbol{m} \partial_t \phi.$$

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$$\lim_{\varepsilon \to 0} \iint \boldsymbol{m}^\varepsilon \times \partial_t \boldsymbol{m}^\varepsilon \partial_t \phi = \iint \boldsymbol{m} \times \partial_t \boldsymbol{m} \partial_t \phi.$$
-
$$\begin{aligned} \iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}_{\text{eff}}^\varepsilon\, \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = &- \iint \boldsymbol{m}^\varepsilon \times \nabla \boldsymbol{m}^\varepsilon \cdot \nabla \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ \iint \boldsymbol{m}^\varepsilon \times \boldsymbol{H}_{\text{s}}^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ &+ \frac{\varepsilon}{2} \iint \boldsymbol{m}^\varepsilon \times \boldsymbol{s}^\varepsilon \phi \,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t. \end{aligned}$$

Passing to the limit of the LLG equation

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- Thus we can take the limit

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^{\varepsilon}\times\boldsymbol{H}_{\text{eff}}^{\varepsilon}\,\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = &-\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^{\varepsilon}\times\nabla\boldsymbol{m}^{\varepsilon}\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+\lim_{\varepsilon\to 0}\iint \boldsymbol{m}^{\varepsilon}\times\boldsymbol{H}_{\text{s}}^{\varepsilon}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+\lim_{\varepsilon\to 0}\frac{\varepsilon}{2}\iint \boldsymbol{m}^{\varepsilon}\times\boldsymbol{s}^{\varepsilon}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
= &-\iint \boldsymbol{m}\times\nabla\boldsymbol{m}\cdot\nabla\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\
&+\iint \boldsymbol{m}\times\boldsymbol{H}_{\text{s}}\phi\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.
\end{aligned}
$$

The limit of the WPLLG system

$(W, \boldsymbol{m})$ is a weak solution of the following VPLLG system,

$$\begin{aligned}
\partial_t W &= -\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} W + \nabla_{\boldsymbol{x}} V \cdot \nabla_{\boldsymbol{v}} W + \frac{\mathrm{i}}{2}[\widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{m},\ W], \\
\partial_t \boldsymbol{m} &= -\boldsymbol{m} \times \boldsymbol{H}_{\text{eff}} + \alpha \boldsymbol{m} \times \partial_t \boldsymbol{m}, \\
V(\boldsymbol{x}) &= \frac{1}{4\pi} \int_{\mathbb{R}^3_{\boldsymbol{x}}} \int_{\mathbb{R}^3_{\boldsymbol{v}}} \frac{W(\boldsymbol{y}, \boldsymbol{v}, t)}{|\boldsymbol{x} - \boldsymbol{y}|} \,\mathrm{d}\boldsymbol{v}\,\mathrm{d}\boldsymbol{y}, \\
\boldsymbol{H}_{\text{eff}} &= \Delta \boldsymbol{m} + \boldsymbol{H}_{\text{s}}, \\
\boldsymbol{H}_{\text{s}}(\boldsymbol{x}) &= -\nabla \left(\frac{1}{4\pi} \int_{\Omega} \frac{\nabla \cdot \boldsymbol{m}(\boldsymbol{y})}{|\boldsymbol{x} - \boldsymbol{y}|} \,\mathrm{d}\boldsymbol{y} \right).
\end{aligned}$$

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- Prove the existence of H^1 solutions.
- Use Wigner transformation to get the kinetic description.
- In the semiclassical limit, the spin-magnetization coupling dynamics can be described by a Vlasov-Poisson-Landau-Lifshitz system.

THANKS FOR YOUR ATTENTION!