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Some convergence results for the compressible Navier-Stokes equations

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Compressible isentropic Navier-Stokes equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u})$$

for $x \in \mathbb{T}^3$, where $\mathbb{T}^3 = \left([0,1]|_{\{0,1\}}\right)^3$ is a torus, $t \in (0, T)$

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$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu\left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{3}\mathrm{div}_x \boldsymbol{u}\mathbb{I}\right) + \eta\, \mathrm{div}_x \boldsymbol{u}\mathbb{I},\ \mu > 0,\ \eta \geq 0,$$

- barotropic flow: $p = C\rho^\gamma$, $\gamma \geq 1$

$$p \in C^2(0,\infty) \cap C^1[0,\infty),\ p(0) = 0,\ p'(\varrho) > 0 \text{ for all } \varrho > 0,$$

$$\lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0,\ \ \lim_{\varrho \to 0+} \frac{p'(\varrho)}{\varrho^\alpha} = p_0 > 0, \alpha \leq 1$$

- initial data $\varrho(0,\cdot) = \varrho_0,\ \boldsymbol{u}(0,\cdot) = \boldsymbol{u}_0,\ \varrho_0 > 0$ in $\mathbb{T}^3$

- global in time weak solutions exists for $\gamma > 3/2$ ... Feireisl, Novotný, Petzoltová ('11)
- $\gamma > 3$ constructive existence proof ... convergence of a suitable numerical scheme: Karper ('13), Feireisl, Karper, Michálek ('16)
- $\gamma > 3$ if ρ_h is bdd., then numerical solution converges ... Feireisl, Hošek, Maltese, Novotný ('17)
- $\gamma > 3/2$ error estimates, if the strong solution exists ... Gallouët, Herbin, Maltese, Novotný ('16)

- $\gamma > 3/2$ **asymptotic preserving error estimates** w.r.t. Mach number in singular limit . . . Feireisl, M.L., Nečasová, Novotný, She ('16)
- $\gamma \leq 3/2$ **convergence of a numerical scheme to the dissipative MVS** . . . Feireisl, M.L. ('17)

Numerical scheme

- combined finite volume-finite element method
- upwinding for convective terms
- fully implicit in time

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• **Notation:**

- $K \in \mathcal{T}$ triangular element, $\mathcal{T}$ regular triangulation, allowing inverse inequality
- $\mathcal{E} = \{\sigma, \sigma$ is an edge of $\mathcal{T}\}$
- V_h Crouzeix-Raviart piecewise linear space of discrete velocities

$$V_h(\mathbb{T}^3) = \{v \in L^2(\mathbb{T}^3),\ \forall K \in \mathcal{T},\ v_{|K} \in \mathbb{P}_1(K),\ \forall \sigma \in \mathcal{E},$$
$$\sigma = K|L,\ \int_\sigma v_{|K}\,\mathrm{d}S = \int_\sigma v_{|L}\,\mathrm{d}S\},$$

- Q_h piecewise constant space of discrete densities

$$Q_h(\mathbb{T}^3) = \{q \in L^2(\mathbb{T}^3)\,|\ \forall K \in \mathcal{T},\ q_{|K} \in R\}$$

- for v in $C(\mathbb{T}^3)$, we set

$$v_K := \frac{1}{|K|}\int_K v\,\mathrm{d}x \text{ for } K \in \mathcal{T} \qquad v_\sigma := \frac{1}{|\sigma|}\int_\sigma v ds \text{ for } \sigma \in \mathcal{E}$$

$$\hat{v} \equiv \Pi_h^Q v(x) = \sum_{K\in\mathcal{T}} v_K 1_K(x); x \in \mathbb{T}^3,$$

-

$$\nabla_h v(x) = \sum_{K\in\mathcal{T}} \nabla_x v(x) 1_K(x), \;\; \mathrm{div}_h \boldsymbol{v}(x) = \sum_{K\in\mathcal{T}} \mathrm{div}_x \boldsymbol{v}(x) 1_K(x).$$

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- $v \in V_h(\mathbb{T}^3)$ then

$$\sum_{\sigma\in\mathcal{E}} \frac{1}{h}\int_\sigma [v]^2_{\sigma,\boldsymbol{n}_\sigma}\,\mathrm{d}S \lesssim \sum_{K\in\mathcal{T}}\int_K |\nabla_x v|^2 \mathrm{d}x,$$

where $[v]_{\sigma,\boldsymbol{n}_\sigma}$ is a jump of v

$$\forall x \in \sigma = K|L \in \mathcal{E}, \quad [v]_{\sigma,\boldsymbol{n}_\sigma}(x) = \begin{cases} v|_K(x) - v|_L(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,K} \\ v|_L(x) - v|_K(x) \text{ if } \boldsymbol{n}_\sigma = \boldsymbol{n}_{\sigma,L} \end{cases}$$

- upwinding

$$q_\sigma^{\mathrm{up}} = -\frac{h^\alpha}{2}[q]_{\sigma,\boldsymbol{n}_\sigma}\chi\left(\frac{\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K}}{h^\alpha}\right) + \begin{cases} q_K \text{ if } \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} > 0 \\ q_L \text{ if } \boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{\sigma,K} \le 0. \end{cases}$$

$$\chi(z) = \begin{cases} 0 \text{ for } z < -1, \\ z+1 \text{ if } -1 \le z \le 0, \\ 1-z \text{ if } 0 < z \le 1, \\ 0 \text{ for } z > 1. \end{cases}$$

$q_\sigma^{\mathrm{up}} \approx q_{K|L} - h^\alpha \Delta q$ if $\alpha > 0$ **dissipative upwinding**

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$q_\sigma^{\rm up} \approx q_{K|L} - h^\alpha \Delta q$ if $\alpha > 0$ **dissipative upwinding**

- $\Delta t = \mathcal{O}(h)$

Numerical scheme

$$\varrho^n \in Q_h(\mathbb{T}^3),\ \varrho^n > 0,\ \boldsymbol{u}^n \in V_h(\mathbb{T}^3; R^3), \quad n = 0, 1, \ldots, N$$

$$\sum_{K \in \mathcal{T}} |K| \frac{\varrho_K^n - \varrho_K^{n-1}}{\Delta t} \phi_K + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \varrho_\sigma^{n,\mathrm{up}} (\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}) \phi_K = 0$$

$$\forall \phi \in Q_h$$

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$$\forall \phi \in Q_h$$

$$\sum_{K\in\mathcal{T}} \frac{|K|}{\Delta t}\Big(\varrho_K^n \hat{\boldsymbol{u}}_K^n - \varrho_K^{n-1}\hat{\boldsymbol{u}}_K^{n-1}\Big) \cdot \boldsymbol{v}_K$$

$$+ \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\varrho_\sigma^{n,\mathrm{up}}\hat{\boldsymbol{u}}_\sigma^{n,\mathrm{up}}[\boldsymbol{u}_\sigma^n \cdot \boldsymbol{n}_{\sigma,K}] \cdot \boldsymbol{v}_K$$

$$-\frac{1}{\varepsilon^2}\sum_{K\in\mathcal{T}} p(\varrho_K^n) \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\boldsymbol{v}_\sigma \cdot \boldsymbol{n}_{\sigma,K} + \mu \sum_{K\in\mathcal{T}} \int_K \nabla \boldsymbol{u}^n : \nabla \boldsymbol{v}\ \mathrm{d}x$$

$$+\Big(\frac{\mu}{3} + \eta\Big) \sum_{K\in\mathcal{T}} \int_K \mathrm{div}\boldsymbol{u}^n \mathrm{div}\boldsymbol{v}\ \mathrm{d}x = 0 \ \ \forall\, \boldsymbol{v} \in V_h$$

Singular limit as $\varepsilon \to 0$

• Klainerman, Majda (1980): convergence of classical solution of fully compressible Euler/Navier-Stokes equations to the classical solution of incompressible equations as $\varepsilon \to 0$

• Feireisl, Novotný (2000): convergence of weak solutions of fully compressible Navier-Stokes equations to the weak solution of incompressible Navier-Stokes equations as $\varepsilon \to 0$

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Can we show such properties for numerical solutions ?

Limiting system

- incompressible Navier-Stokes equations

$$\overline{\varrho}\Big(\partial_t \boldsymbol{V} + \boldsymbol{V} \cdot \nabla_x \boldsymbol{V}\Big) + \nabla_x \Pi = \mu \Delta \boldsymbol{V},\ \mathrm{div}_x \boldsymbol{V} = 0,\ \overline{\varrho} > 0$$

for suitable initial data

$$\boldsymbol{V}(0) = \boldsymbol{V}_0, \qquad \boldsymbol{V}_0 \in W^{k,2}(\mathbb{T}^3; R^3),\ \mathrm{div}_x \boldsymbol{V}_0 = 0.$$

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Klainerman, Majda $\Longrightarrow$ solution is regular on some $[0, T), \ T \equiv T_{max}$

$$\partial_t^l \boldsymbol{V} \in C^l([0,T]; W^{k-l,2}(\mathbb{T}^3; R^3), \ l = 0, 1, 2,$$
$$\partial_t^j \Pi \in C^j([0,T]; W^{k-1-j,2}(\mathbb{T}^3)), \ j = 0, 1, \ k \geq 4$$

Relative energy functional

$$\mathcal{E}_\varepsilon(\varrho, \boldsymbol{u}|z, \boldsymbol{v}) = \int_{\mathbb{T}^3} \left(\varrho|\boldsymbol{u} - \boldsymbol{v}|^2 + \frac{1}{\varepsilon^2} E(\varrho|z) \right) \mathrm{d}x,$$

where

$$E(\varrho|z) = P(\varrho) - P'(z)(\varrho - z) - P(z), \;\; P(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s.$$

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where

$$E(\varrho|z) = P(\varrho) - P'(z)(\varrho - z) - P(z), \; P(\varrho) = \varrho \int_1^\varrho \frac{p(s)}{s^2} \mathrm{d}s.$$

- since $p'(\varrho) > 0 \implies \varrho \mapsto P(\varrho)$ is strictly convex in $(0, \infty)$

$$E(\varrho|z) \geq 0 \quad \text{and} \quad E(\varrho|z) = 0 \; \Leftrightarrow \; \varrho = z.$$

Theorem (Uniform error estimates)

Let the pressure coefficient $\gamma \geq 3/2$. *Let* $(\varrho^0, \boldsymbol{u}^0)$ *satisfy*

$$\mathcal{E}_\varepsilon(\varrho^0_\varepsilon, \boldsymbol{u}^0_\varepsilon \Big| \overline{\varrho}, \boldsymbol{V}_0) \leq E_0 < \infty, \;\; M_0/2 \leq \int_{\mathbb{T}^3} \varrho^0_\varepsilon \mathrm{d}x \leq 2M_0, \;\; M_0 = \overline{\varrho}|\mathbb{T}^3|$$

with $M_0 > 0$, $E_0 > 0$ *and* $\overline{\varrho} > 0$ *independent of* $\varepsilon, h, \Delta t$.
Let $[\Pi, \boldsymbol{V}]$ *be the classical solution of the incompressible N.-S. eqs.,* $\alpha = 0$ *(classical upwinding). Then there exists* $c > 0$,

$c = c\Big(M_0, E_0, \overline{\varrho}, |p'|_{C^1[\overline{\varrho}/2, 2\overline{\varrho}]}, \| \boldsymbol{V}, \Pi \|_{\mathcal{X}^k_{T,\mathbb{T}^3}}\Big)$, *s.t.*

$$\sup_{1 \leq n \leq N} \mathcal{E}_\varepsilon\Big(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \overline{\varrho}, \boldsymbol{V}(t_n, \cdot)\Big) + \Delta t \sum_{1 \leq n \leq N} \int_{\mathbb{T}^3} |\nabla_h \boldsymbol{u}^n - \nabla_x \boldsymbol{V}(t_n, \cdot)|^2$$
$$\leq c\left(\sqrt{\Delta t} + h^a + \varepsilon + \mathcal{E}_\varepsilon\Big(\varrho^0_\varepsilon, \hat{\boldsymbol{u}}^0_\varepsilon \Big| \overline{\varrho}, \boldsymbol{V}_0\Big)\right),$$

$$a = \min\left\{\frac{2\gamma - 3}{\gamma}, 1\right\}.$$

- well-prepared initial data:

$$E(\varrho_\varepsilon^0|\overline{\varrho}) \lesssim \varepsilon^{2+\xi}, \quad \int_{\mathbb{T}^3} \varrho_\varepsilon^0 |\hat{\boldsymbol{u}}_\varepsilon^0 - \boldsymbol{V}_0|^2 \ \mathrm{d}x \lesssim \varepsilon^{\xi}, \ \xi > 0$$

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  - unconditional uniform convergence as $(h, \Delta t, \varepsilon) \to 0$ to the strong solution of the incompressible N.-S.eqs
  - rate of convergence

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  - unconditional uniform convergence as $(h, \Delta t, \varepsilon) \to 0$ to the strong solution of the incompressible N.-S.eqs
  - rate of convergence
  - the distance is measured via the relative energy functional

$$\mathcal{E}_\varepsilon\Big(\varrho^n, \hat{\boldsymbol{u}}^n \Big| \overline{\varrho}, \boldsymbol{V}^n\Big) \approx \int_{\mathbb{T}^3} \varrho^n |\hat{\boldsymbol{u}}^n - \boldsymbol{V}(t_n)|^2 \, \mathrm{d}x + \left\| \frac{\varrho^n - \overline{\varrho}}{\varepsilon} \right\|^2_{L^q(\mathbb{T}^3)},$$

  where $q = \min\{2, \gamma\}$

  - in 2D pressure exponent can be $\gamma > 1$

Main ingredients of the proof

I. Discrete energy inequality

$$\int_{\mathbb{T}^3} \varrho^n \,\mathrm{d}x = \int_{\mathbb{T}^3} \varrho^0 \,\mathrm{d}x \quad \forall n = 1, ...N$$

Main ingredients of the proof

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$$\sum_{K\in\mathcal{T}} |K| \Big(\frac{1}{2} \varrho_K^m |\boldsymbol{u}_K^m|^2 + \frac{1}{\varepsilon^2} E(\varrho_K^m|\overline{\varrho}) \Big) - \sum_{K\in\mathcal{T}} |K| \Big(\frac{1}{2} \varrho_K^0 |\boldsymbol{u}_K^0|^2 + \frac{1}{\varepsilon^2} E(\varrho_K^0|\overline{\varrho}) \Big)$$

$$+ \Delta t \sum_{n=1}^{m} \sum_{K\in\mathcal{T}} \Big(\mu \int_K |\nabla_x \boldsymbol{u}^n|^2 \,\mathrm{d}x + (\frac{\mu}{3} + \eta) \int_K |\mathrm{div} \boldsymbol{u}^n|^2 \,\mathrm{d}x \Big)$$

$$+ [D^m] = 0 \qquad \text{m=1,...,N}$$

$[D^m] \geq 0$ sum of all dissipation defects

Main ingredients of the proof (cont.)

II. Uniform a priori estimates

$$\Delta t \sum_{n=1}^{N} \int_K |\nabla_x \boldsymbol{u}^n|^2 \, \mathrm{d}x \leq c,$$

$$\Delta t \sum_{n=1}^{N} \|\boldsymbol{u}^n\|^2_{L^6(\mathbb{T}^3;R^3)} \leq c,$$

$$\sup_{n=0,\dots N} \|\varrho^n |\hat{\boldsymbol{u}}^n|^2\|_{L^1(\mathbb{T}^3)} \leq c.$$

$$\sup_{n=0,\dots N} \|\varrho^n\|_{L^\gamma(\mathbb{T}^3)} \leq c,$$

$$\sup_{n=0,\dots N} \int_{\mathbb{T}^3} E(\varrho^n|\overline{\varrho}) \leq c\varepsilon^2.$$

Main ingredients of the proof (cont.)

III. Discrete relative energy inequality

$$\sum_{K\in\mathcal{T}} \frac{1}{2}|K|\Big(\varrho_K^m|\boldsymbol{u}_K^m - \boldsymbol{U}_K^m|^2 - \varrho_K^0|\boldsymbol{u}_K^0 - \boldsymbol{U}_K^0|^2\Big)$$

$$+ \frac{1}{\varepsilon^2}\sum_{K\in\mathcal{T}} |K|\Big(E(\varrho_K^m|r_K^m) - E(\varrho_K^0|r_K^0)\Big)$$

$$+ \Delta t \sum_{n=1}^{m}\sum_{K\in\mathcal{T}} \Big(\mu\int_K |\nabla_x(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x$$

$$+ (\frac{\mu}{3} + \eta)\int_K |\mathrm{div}(\boldsymbol{u}^n - \boldsymbol{U}^n)|^2 \,\mathrm{d}x\Big) \le \sum_{i=1}^{6} T_i,$$

T_i are the errors terms, controlled if test functions $(\varrho^n, \boldsymbol{U}^n) = (\overline{\varrho}, \boldsymbol{V}_h^n)$ are the limiting solution of the incompressible N.-S. eqs. $\boldsymbol{V}_h^n = \Pi_h^V[\boldsymbol{V}(t_n)]$, $\overline{\varrho} = \mathrm{const.}$

Main ingredients of the proof (cont.)

IV. Gronwall-type inequality

There exists

$$c = c\Big(M_0, E_0, \overline{\varrho}, |p'|_{C^1[\overline{\varrho}/2, 2\overline{\varrho}]}, \| \boldsymbol{V}, \Pi \|_{\mathcal{X}^k_{T,\mathbb{T}^3}}\Big) > 0$$

s.t.

$$\mathcal{E}_\varepsilon(\varrho^m, \boldsymbol{u}^m | \overline{\varrho}, \boldsymbol{V}_h^m) + \Delta t \frac{\mu}{2} \sum_{n=1}^{m} \sum_{K \in \mathcal{T}} \int_K |\nabla_x (\boldsymbol{u}^n - \boldsymbol{V}_h^n)|^2 \mathrm{dx}$$

$$\leq c\Big[h^a + \sqrt{\Delta t} + \varepsilon + \mathcal{E}_\varepsilon(\varrho^0, \boldsymbol{u}^0 | \overline{\varrho}, \hat{\boldsymbol{V}}_h(0))\Big] + c\, \Delta t \sum_{n=1}^{m} \mathcal{E}_\varepsilon(\varrho^n, \boldsymbol{u}^n | \overline{\varrho}, \boldsymbol{V}_h^n),$$

where $a = 1$ of $\gamma \geq 3$

Uniform error estimates

$$\sup_{1\le n\le N} \mathcal{E}_\varepsilon\left(\varrho^n, \hat{\boldsymbol{u}}^n \middle| \overline{\varrho}, \boldsymbol{V}(t_n,\cdot)\right) + \Delta t \sum_{1\le n\le N} \int_{\mathbb{T}^3} |\nabla_h \boldsymbol{u}^n - \nabla_x \boldsymbol{V}(t_n,\cdot)|^2$$

$$\le c\left(\sqrt{\Delta t} + h^a + \varepsilon + \mathcal{E}_\varepsilon\left(\varrho^0_\varepsilon, \hat{\boldsymbol{u}}^0_\varepsilon \middle| \overline{\varrho}, \boldsymbol{V}_0\right)\right),$$

$$a = \min\left\{\frac{2\gamma-3}{\gamma}, 1\right\}.$$

Numerical Experiments

Experiment: vortex in a box

Haack, Jin, Liu (2012)

$$u_1(x,y,0) = \sin^2(\pi x)\sin(2\pi y)$$
$$u_2(x,y,0) = -\sin(2\pi x)\sin^2(\pi y)$$
$$\rho(x,y,0) = 1 - \frac{\varepsilon^2}{2}\tanh(y-0.5)$$

$\Omega = [-1,1]^2$

- $\varrho, p = \varrho^\gamma, \boldsymbol{u}$ ... compressible N.-S. eqs., $\gamma = 1.4$
- Π, $\boldsymbol{V}$ ... incompressible N.-S. eqs.

$$z = (1+\varepsilon^2\Pi)^{1/\gamma}.$$

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- $\varrho, p = \varrho^\gamma, \boldsymbol{u}$ ... compressible N.-S. eqs., $\gamma = 1.4$
- $\Pi, \boldsymbol{V}$ ... incompressible N.-S. eqs.

$$z = (1+\varepsilon^2\Pi)^{1/\gamma}.$$

- error is measured by

$$e_{\mathcal{E}} = \sup_{1\le n\le N} \mathcal{E}_\varepsilon(\rho^n, \hat{\boldsymbol{u}}^n | z(t^n,\cdot), \boldsymbol{V}(t^n,\cdot)),\ e_{\nabla_x \boldsymbol{u}} = \|\nabla(\boldsymbol{u} - \boldsymbol{V})\|_{L^2(0,T;L^2(\Omega))}$$
$$e_{\boldsymbol{u}} = \|\boldsymbol{u} - \boldsymbol{V}\|_{L^2(0,T;L^2(\Omega))},\ e_\rho = \|\rho - z\|_{L^2(0,T;L^2(\Omega))}$$

Table: Convergence of the compressible to incompressible solution, $t = 0.01$ $\varepsilon = h$, $\mu = 0.01$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_p	EOC
1/8	1.12e-03	–	4.91e-01	–	1.82e-03	–	5.11e-04	–
1/16	3.74e-04	1.58	2.55e-01	0.95	1.18e-03	0.63	1.11e-04	2.20
1/32	1.09e-04	1.78	2.29e-01	0.16	7.87e-04	0.58	2.09e-05	2.41
1/64	1.91e-05	2.51	1.26e-01	0.86	3.24e-04	1.28	4.64e-06	2.17
1/128	4.50e-06	2.09	4.41e-02	1.51	1.41e-04	1.20	1.22e-06	1.93
1/256	1.14e-06	1.98	1.54e-02	1.52	6.32e-05	1.16	2.88e-07	2.08

: $\varepsilon = h, \mu = 1$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_p	EOC
1/8	5.42e-03	–	4.66e-01	–	2.99e-03	–	1.23e-03	–
1/16	1.34e-03	2.02	1.73e-01	1.43	2.17e-03	0.46	2.51e-04	2.29
1/32	3.66e-04	1.87	6.58e-02	1.39	1.17e-03	0.89	5.30e-05	2.24
1/64	1.06e-04	1.79	2.37e-02	1.47	6.13e-04	0.93	1.15e-05	2.20
1/128	2.96e-05	1.84	8.26e-03	1.52	2.78e-04	1.14	2.54e-06	2.18
1/256	7.96e-06	1.89	2.97e-03	1.48	1.31e-04	1.09	5.73e-07	2.15

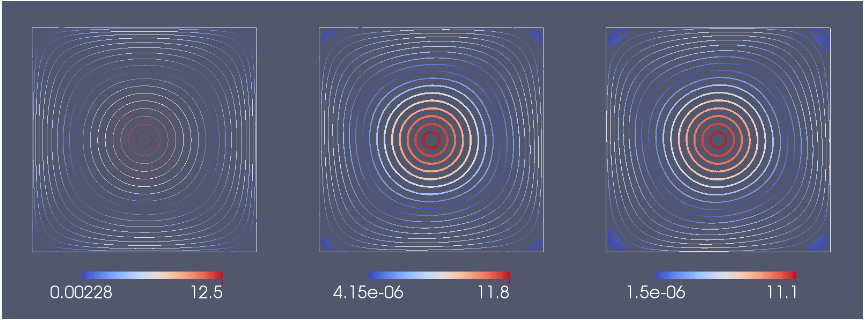

Figure: Streamlines with $\mu = 0.01$, $\varepsilon = 0.001$ at $t = 0,\ 0.1,\ 0.2$

Experiment: Taylor vortex flow

- exact incompressible solution (periodic BC)

$$V_1(x,y,t) = \sin(2\pi x)\cos(2\pi y)\, e^{-8\pi^2 \mu t}$$
$$V_2(x,y,t) = -\cos(2\pi x)\sin(2\pi y)\, e^{-8\pi^2 \mu t}$$
$$\Pi(x,y,t) = \frac{1}{4}\left(\cos(4\pi x) + \cos(4\pi y)\right) e^{-16\pi^2 \mu t}.$$

- initial data for the compressible N.-S. system

$$\rho(x,y,0) = 1 + \varepsilon^2 \Pi(x,y,0)$$
$$u_1(x,y,0) = V_1(x,y,0)$$
$$u_2(x,y,0) = V_2(x,y,0).$$

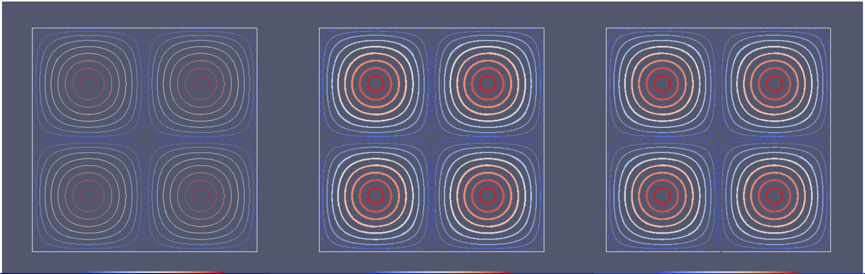

Table: Experimental order of convergence, $\mu = 0.1$.

: $\varepsilon = 0.8$

h	$\|\rho\|_{L^2}$	EOC	$\|\boldsymbol{u}\|_{L^2}$	EOC	$\|\nabla \boldsymbol{u}\|_{L^2}$	EOC
$1/8$	3.06e-02	–	4.04e-02	–	8.65e+00	–
$1/16$	1.56e-02	0.97	2.51e-02	0.69	7.39e+00	0.23
$1/32$	7.81e-03	1.00	9.23e-03	1.44	3.97e+00	0.90
$1/64$	3.87e-03	1.01	2.57e-03	1.84	1.50e+00	1.40
$1/128$	1.92e-03	1.01	6.64e-04	1.95	5.45e-01	1.46

: $\varepsilon = 0.001$

h	$\|\rho\|_{L^2}$	EOC	$\|\boldsymbol{u}\|_{L^2}$	EOC	$\|\nabla \boldsymbol{u}\|_{L^2}$	EOC
$1/8$	1.00e-07	–	3.13e-02	–	6.09e+00	–
$1/16$	4.07e-08	1.30	1.09e-02	1.52	2.70e+00	1.17
$1/32$	1.95e-08	1.06	4.55e-03	1.26	1.49e+00	0.86
$1/64$	8.34e-09	1.23	2.55e-03	0.84	1.13e+00	0.40
$1/128$	3.01e-09	1.47	7.49e-04	1.77	5.05e-01	1.16

Convergence of the relative energy

Table: Convergence $\mu = 0.01$, $\varepsilon = h$, $t = 0.01$, $\gamma = 1.4$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_p	EOC
$1/8$	1.22e-02	–	2.57e-01	–	5.54e-03	–	3.34e-04	–
$1/16$	1.09e-03	3.48	1.47e-01	0.81	1.92e-03	1.53	6.82e-05	2.29
$1/32$	2.02e-04	2.43	1.16e-01	0.34	9.09e-04	1.08	1.05e-05	2.70
$1/64$	2.63e-05	2.94	7.90e-02	0.55	3.20e-04	1.51	1.91e-06	2.46
$1/128$	4.45e-06	2.56	3.86e-02	1.03	8.79e-05	1.86	3.91e-07	2.29
$1/256$	9.86e-07	2.17	2.09e-02	0.89	2.84e-05	1.63	7.63e-08	2.36

: $\gamma = 3$

h	$e_{\mathcal{E}}$	EOC	$e_{\nabla_x \boldsymbol{u}}$	EOC	$e_{\boldsymbol{u}}$	EOC	e_p	EOC
$1/8$	3.60e-02	–	3.57e-01	–	7.56e-03	–	4.22e-04	–
$1/16$	3.04e-03	3.57	1.94e-01	0.88	2.35e-03	1.69	9.08e-05	2.22
$1/32$	2.98e-04	3.35	1.30e-01	0.58	9.44e-04	1.32	1.76e-05	2.37
$1/64$	5.26e-05	2.50	8.35e-02	0.64	3.25e-04	1.54	3.45e-06	2.35
$1/128$	1.46e-05	1.85	4.46e-02	0.90	1.11e-04	1.55	1.06e-06	1.70
$1/256$	3.88e-06	1.91	2.22e-02	1.01	4.05e-05	1.45	2.34e-07	2.18

What is the limit of numerical solutions for $\gamma < 3/2$?

Assumptions:

- no-slip B.C. $\boldsymbol{u} = 0$ on $\partial\Omega$, $\quad \Omega \in \mathbb{R}^3$
- $\varepsilon = 1,\ \gamma \in (1, 2)$ and $\alpha > 0$ (dissipative upwinding)

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Aim:

$$(\varrho_h, \boldsymbol{u}_h) \longrightarrow (\varrho, \boldsymbol{u}) \quad \text{as} \qquad h \to 0, \ \Delta t \approx h$$

- dissipative measure-valued sol.
- family of bdd. numerical solution generates Young measures
- dissipative MVS satisfy energy inequality and the concentration reminder of the momentum is dominated by the dissipation

I. Consistency: $(\varrho_h, \boldsymbol{u}_h)$ satisfies

$$-\int_{\Omega_h} \varrho_h^0 \varphi(0,\cdot) \ \mathrm{d}x = \int_0^T \int_{\Omega_h} \left[\varrho_h \partial_t \varphi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \varphi\right] \ \mathrm{d}x + \mathcal{O}(h^\beta), \ \beta > 0,$$

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$$-\int_{\Omega_h} \varrho_h^0 \boldsymbol{u}_h^0 \cdot \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_{\Omega_h} \Big[\varrho_h \boldsymbol{u}_h \partial_t \varphi + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \varphi + p(\varrho_h)\mathrm{div}_x \varphi$$

$$-\int_0^T \int_{\Omega_h} \Big[\mu \nabla_h \boldsymbol{u}_h : \nabla_x \varphi + (\mu/3 + \eta)\mathrm{div}_h \boldsymbol{u}_h \cdot \mathrm{div}_x \varphi\Big]\ \mathrm{d}x + \mathcal{O}(h^\beta),\ \beta > 0,$$

for any $\varphi \in C_c^\infty([0,\infty) \times \Omega_h; R^3)$.

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for any $\varphi \in C_c^\infty([0,\infty) \times \Omega_h; R^3)$.

energy inequality

$$\int_{\Omega_h} \left[\frac{1}{2}\varrho_h |\boldsymbol{u}_h|^2 + P(\varrho_h)\right](\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{\Omega_h} \mu |\nabla_h \boldsymbol{u}_h|^2 + (\mu/3+\eta)|\mathrm{div}_h \boldsymbol{u}_h|^2\ \mathrm{d}$$

$$\leq \int_{\Omega_h} \left[\frac{1}{2}\varrho_h^0 |\boldsymbol{u}_h^0|^2 + P(\varrho_h^0)\right]\ \mathrm{d}x$$

for a.e. $\tau \in [0, T]$.

II. Weak limits

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty(0,T;L^\gamma(\Omega)),\ \ \varrho \geq 0$$

$$< \boldsymbol{u}_h >,\ \boldsymbol{u}_h \to \boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega;R^3),$$

$$\nabla_h \boldsymbol{u}_h \to \nabla_x \boldsymbol{u} \text{ weakly in } L^2((0,T)\times\Omega;R^{3\times 3}),$$

$$\varrho_h < \boldsymbol{u}_h > \to \overline{\varrho \boldsymbol{u}} \text{ weakly-(*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^3)),$$

$$\varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h + p(\varrho_h)\mathbb{I} \to \{\varrho \boldsymbol{u} \otimes \boldsymbol{u} + p(\varrho)\mathbb{I}\} \text{weakly-(*) in } \left[L^\infty(0,T;\mathcal{M}(\Omega))\right]^{3\times 3}$$

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$$\varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h + p(\varrho_h)\mathbb{I} \to \{\varrho \boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)\mathbb{I}\} \text{ weakly-(*) in } \left[L^\infty(0,T;\mathcal{M}(\Omega))\right]^{3\times 3}$$

- the limit function satisfies the continuity eq.

$$-\int_{\mathbb{T}^3} \varrho_0 \varphi(0,\cdot)\ \mathrm{d}x = \int_0^T \int_{\mathbb{T}^3} \left[\varrho \partial_t \varphi + \overline{\varrho \boldsymbol{u}} \cdot \nabla_x \varphi\right]\ \mathrm{d}x \Delta t$$

for any test function $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$

III. Dissipative MVS

-bounded seq. of numerical solutions generates the Young measure

$$\nu_{t,x} \in L^\infty((0,T)\times\Omega;\mathcal{P}([0,\infty)\times R^3)) \text{ for a.a. } (t,x)\in(0,T)\times\Omega,$$

such that

$$\langle \nu_{t,x}, g(\varrho,\boldsymbol{u})\rangle = \overline{g(\varrho,\boldsymbol{u})}(t,x) \text{ for a.a. } (t,x)\in(0,T)\times\Omega,$$

for $g \in C([0,\infty)\times R^3)$, and

$$g(\varrho_h,\boldsymbol{u}_h) \to \overline{g(\varrho,\boldsymbol{u})} \text{ weakly in } L^1((0,T)\times\Omega)$$

- for the continuity eq.

$$\left[\int_{\mathbb{T}^3} \varrho\varphi(\tau,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^3} \left[\varrho\partial_t\varphi + \langle \nu_{t,x}, \varrho\boldsymbol{u}\rangle \cdot \nabla_x\varphi\right]\ \mathrm{d}x \tag{1}$$

- for the momentum eq.

$$\begin{aligned}
&\left[\int_{\mathbb{T}^3} \langle \nu_{t,x}; \varrho\boldsymbol{u}\rangle \cdot \varphi(0,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&= \int_0^\tau \int_{\mathbb{T}^3} \Big[\langle \nu_{t,x}; \varrho\boldsymbol{u}\rangle \cdot \partial_t\varphi + \{\varrho\boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)\mathbb{I}\} : \nabla_x\varphi\Big]\ \mathrm{d}x \\
&- \int_0^\tau \int_{\mathbb{T}^3} \Big[\mu\nabla\boldsymbol{u} : \nabla_x\varphi + (\mu/3+\eta)\mathrm{div}\boldsymbol{u}\cdot\mathrm{div}_x\varphi\Big]\ \mathrm{d}x
\end{aligned} \tag{2}$$

- for the continuity eq.

$$\left[\int_{\mathbb{T}^3} \varrho\varphi(\tau,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\mathbb{T}^3} \left[\varrho\partial_t\varphi + \langle\nu_{t,x}, \varrho\boldsymbol{u}\rangle \cdot \nabla_x\varphi\right]\ \mathrm{d}x \qquad (1)$$

- for the momentum eq.

$$\left[\int_{\mathbb{T}^3} \langle\nu_{t,x}; \varrho\boldsymbol{u}\rangle \cdot \varphi(0,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$= \int_0^\tau \int_{\mathbb{T}^3} \Big[\langle\nu_{t,x}; \varrho\boldsymbol{u}\rangle \cdot \partial_t\varphi + \{\varrho\boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)\mathbb{I}\} : \nabla_x\varphi\Big]\ \mathrm{d}x \qquad (2)$$
$$- \int_0^\tau \int_{\mathbb{T}^3} \Big[\mu\nabla\boldsymbol{u} : \nabla_x\varphi + (\mu/3+\eta)\mathrm{div}\boldsymbol{u}\cdot\mathrm{div}_x\varphi\Big]\ \mathrm{d}x$$

- concentration reminder:

$$\mathcal{R} = \{\varrho\boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)\mathbb{I}\} - \langle\nu_{t,x}; \varrho\boldsymbol{u}\otimes\boldsymbol{u} + p(\varrho)\mathbb{I}\rangle \in [L^\infty(0,T;\mathcal{M}(\Omega))]^{3\times 3}$$

$\Longrightarrow$ momentum eq.

$$\left[\int_{\mathbb{T}^3} \langle \nu_{t,x}; \varrho \boldsymbol{u} \rangle \cdot \varphi(0,\cdot) \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$= \int_0^\tau \int_{\mathbb{T}^3} \left[\langle \nu_{t,x}; \varrho \boldsymbol{u} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; \varrho \boldsymbol{u} \otimes \boldsymbol{u} \rangle : \nabla_x \varphi + \langle \nu_{t,x}, p(\varrho) \rangle \, \mathrm{div}_x \varphi \right] \ \mathrm{d}x$$

$$- \int_0^\tau \int_{\mathbb{T}^3} \left[\mu \nabla \boldsymbol{u} : \nabla_x \varphi + (\mu/3 + \eta) \mathrm{div} \boldsymbol{u} \cdot \mathrm{div}_x \varphi \right] \ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^3} \mathcal{R} : \nabla_x \varphi \ \mathrm{d}x$$

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$\Longrightarrow$ energy inequality

$$\left[\int_{\mathbb{T}^3} \left[\frac{1}{2} \left\langle \nu_{t,x}; \varrho |\boldsymbol{u}|^2 + P(\varrho) \right\rangle \right] \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_{\Omega_h} \mu |\nabla \boldsymbol{u}|^2 + (\mu/3 + \eta) |\mathrm{div} \boldsymbol{u}|^2 \ \mathrm{d}x$$
$$+ \mathcal{D}(\tau) \leq 0 \qquad (3)$$

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BUT:

$$\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)} \lesssim \int_0^\tau \mathcal{D}(t)$$
$$\mathcal{D}(\tau) \geq \liminf_{h \to \infty} \int_0^\tau \int_{\Omega_h} |\nabla_h \boldsymbol{u}_h|^2 \ \mathrm{d}x - \int_0^\tau \int_{\mathbb{T}^3} |\nabla_x \boldsymbol{u}|^2 \ \mathrm{d}x$$

Dissipative MVS

Definition: A parametrized measure $\{\nu_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$,
$\nu \in L^\infty(0,T)\times\Omega;\mathcal{P}([0,\infty)\times R^3)$ is a **dissipative MVS** to the compressible N.-S. eqs. with the initial condition

$$\nu_{0,x} = \delta_{[\varrho_0(x),\boldsymbol{u}_0(x)]}$$

and the dissipation defect $\mathcal{D}$; $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$,
iff
the continuity eq. (1), momentum eqs. (2), energy inequality (3),

$$\int_0^\tau \|\mathcal{R}\|_{\mathcal{M}(\Omega)} \lesssim \int_0^\tau \mathcal{D}(t)$$

and the *Poincaré inequality* holds

$$\lim_{h\to 0}\int_0^\tau\int_{\Omega_h} |\boldsymbol{u}_h - \boldsymbol{u}|^2 \, \mathrm{d}x \, \Delta t \leq$$

$$\liminf_{h\to\infty}\int_0^\tau\int_{\Omega_h} |\nabla_h \boldsymbol{u}_h|^2 \, \mathrm{d}x\mathrm{d}t - \int_0^\tau\int_{\mathbb{T}^3} |\nabla_x \boldsymbol{u}|^2 \, \mathrm{d}x\mathrm{d}t \leq \mathcal{D}(\tau)$$

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- dissipation defect controls oscillations and concentrations in $\nabla_x \boldsymbol{u}$
- needed to show weak-strong uniqueness between MVS & strong sol.

Theorem (Feireisl, M.L. (2017))

Let the pressure adiabatic exponent γ *satisfies* $1 < \gamma < 2$. *Let*

$$\Delta t \approx h, \; 0 < \alpha < 2(\gamma - 1).$$

Then any Young measure $\{\nu_{t,x}\}_{t,x \in (0,T) \times \Omega}$ *generated by* $[\varrho_h^k, \boldsymbol{u}_h^k]$ *for* $h \to 0$ *represents a dissipative MVS of the N.-S. eqs.*

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• the weak–strong uniqueness [Feireisl, Gwiazda, Swierczewska-Gwiazda, Wiedemann (2015)]
$\Longrightarrow$ for a regular sol.

$$\varrho, \ \nabla_x \varrho, \ \boldsymbol{u}, \nabla_x \boldsymbol{u} \in C([0,T] \times \overline{\Omega}), \ \partial_t \boldsymbol{u} \in L^2(0,T; C(\overline{\Omega}; R^3)), \ \varrho > 0, \ \boldsymbol{u}|_{\partial\Omega} = 0$$

it holds

$$\varrho_h \to \varrho \ \text{ in } L^\gamma((0,T) \times K), \ \boldsymbol{u}_h \to \boldsymbol{u} \ \text{ in } L^2((0,T) \times K; R^3)$$

for any compact $K \subset \Omega$

Concluding remarks

- multi-d Euler eqs. have infinitely many weak entropy solutions ...
  De Lelis, Székelyhidi ('10, '12), Chiordaroli, De Lelis, Kreml ('15)

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  De Lelis, Székelyhidi ('10, '12), Chiordaroli, De Lelis, Kreml ('15)
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  for multi-d **scalar** conservation laws ... existence & uniqueness of *entropy statistical solutions*, stability with respect to the 1-Wasserstein metric on probability measures
- if *vanishing viscosity* is assumed to be **the admissibility criterion** for the solution of the Euler eqs.
  $\Longrightarrow$ MVS may be the right concept for inviscid flows