Semi-implicit IMEX schemes for evolutionary non linear partial differential equations.

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Asymptotic preserving and multiscale methods
for kinetic and hyperbolic problems
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Outline

Introduction

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Chen, Levermore and Liu (CPAM, 1994) proved that $U \to \mathcal{E}(u)$, u satisfying the relaxed equation, provided a suitable *subcharacteristic condition* is satisfied.
The argument can be generalized to $N \times N$ system.
An Asymptotic Preserving scheme has to capture such a limiting behavior.

Additive and partitioned form

Here we shall use (mainly) finite difference discretization in space for simplicity, and concentrate on time discretization, so we can see the problem as a system of ODES:

$$y' = \underbrace{f(y)}_{\text{Explicit}} + \underbrace{\frac{1}{\varepsilon} g(y)}_{\text{Implicit}}, \tag{1}$$

The stiffness is associated to one of the terms on the RHS.
We say that in this case the stiffness is additive.

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In other cases the stiffness can be associated to a variable, e.g.

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A natural choice for all such cases is offered by IMEX methods.

General formulation

[Boscarino, Filbet, GR, J. Sci. Comput., submitted]
In many cases the separation of scales is not additive nor partitioned.
We may have a situation of the form

$$\begin{cases} \frac{du}{dt}(t) = \mathcal{H}_\varepsilon(t, u(t), u(t)), \quad \forall\, t \geq t_0, \\ \\ u(t_0) = u_0, \end{cases} \tag{3}$$

with $\mathcal{H}$: $\mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ sufficiently regular.
Dependence on the second argument of $\mathcal{H}$ is non stiff.
Dependence on the third argument is stiff.
This includes partitioned and additive as particular cases.
Strong relation with partitions systems: by setting $y = z/\varepsilon$, system (3) implies

$$\begin{cases} \frac{dy}{dt}(t) = \mathcal{H}(t, y(t), z(t)/\varepsilon), \\ \\ \varepsilon\,\frac{dz}{dt}(t) = \mathcal{H}(t, y(t), z(t)/\varepsilon), \end{cases}$$

Doubled system?

By doubling the variables, the systems takes a partitioned form.
Partitioned methods: apply two different R-K methods, *i.e.*

$$\begin{array}{c|c} \hat{c} & \hat{A} \\ \hline & \hat{b}^T \end{array} \qquad \begin{array}{c|c} c & A \\ \hline & b^T \end{array} \tag{4}$$

treat y with the method on the left, and z with the one on the right.
Then one has, for the stage fluxes:

$$k_i = \mathcal{H}(t^n + \hat{c}_i \Delta t, Y_i, Z_i), \quad \ell_i = \mathcal{H}(t^n + c_i \Delta t, Y_i, Z_i), \quad 1 \le i \le s,$$

with

$$Y_i = y^n + \Delta t \sum_{j=1}^{i-1} \hat{a}_{i,j}\, k_j, \quad Z_i = y^n + \Delta t \sum_{j=1}^{i} a_{ij}\, \ell_j, \quad 1 \le i \le s,$$

and the numerical solutions at the next time step are

$$y^{n+1} = y^n + \Delta t \sum_{i=1}^{s} \hat{b}_i\, k_i, \quad z^{n+1} = y^n + \Delta t \sum_{i=1}^{s} b_i\, \ell_i.$$

Avoid doubling

Remarks

- if $\hat{c} = c$ then $k = \ell \;\Rightarrow\; \mathcal{H}$ has to be computed only once per stage.
- furthermore if $\hat{b} = b \;\Rightarrow\; y^{n+1} = z^{n+1}$,
  if $\hat{b} \neq b$ and $y^n = z^n \;\Rightarrow\; y^{n+1} \neq z^{n+1}$,
  however if both schemes are consistent to order p one can choose any one of the two, say the one to compute y^{n+1}, and then set $n \leftarrow n+1$, and $z^n = y^n$
- finally, if $\hat{c} = c$ and the two schemes have different orders, then the difference $y^{n+1} - z^{n+1}$ can be used to estimate the local error
  $\Rightarrow$ time step control.

In all such cases, no duplication of variables is needed!

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Several numerical tests are presented (see talk by Francis Filbet)

Reaction diffusion problem

$\omega = (\omega_1, \omega_2) : \mathbb{R}^+ \times (0, 2\pi)^2 \mapsto \mathbb{R}^2$

$$\begin{cases} \dfrac{\partial \omega_1}{\partial t} = \Delta \omega_1 - \alpha_1(t)\,\omega_1^2 + \dfrac{9}{2}\,\omega_1 + \omega_2 + f(t), \\[2ex] \dfrac{\partial \omega_2}{\partial t} = \Delta \omega_2 + \dfrac{7}{2}\,\omega_2\,, \qquad t \geq 0, \end{cases}$$

with $\alpha(t) = 2\,e^{t/2}$ and $f(t) = -2e^{-t/2}$. Initial conditions compatible with exact solution

$$\begin{cases} \omega_1(t,x,y) = \exp(-0.5t)\,(1 + \cos(x)), \\[2ex] \omega_2(t,x,y) = \exp(-0.5t)\,\cos(2\,x). \end{cases}$$

Separate explicit variable $u = (u_1, u_2)$ from implicit $v = (v_1, v_2)$, according to:

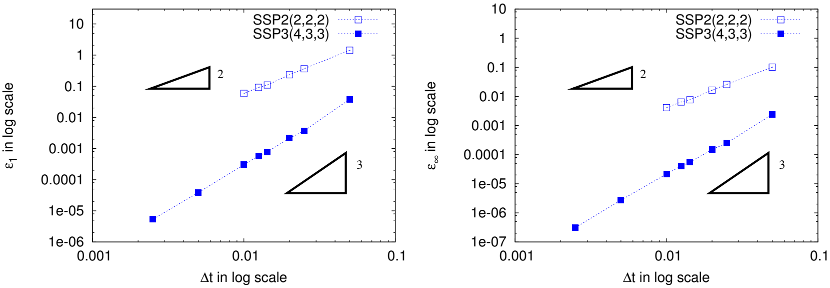
$$\mathcal{H}(t, u, v) = \begin{pmatrix} \Delta v_1 - \alpha(t) u_1\, v_1 + \dfrac{9 u_1}{2} + v_2 + f(t) \\[3ex] \Delta v_2 + \dfrac{7\, v_2}{2} \end{pmatrix}.$$

R-D equation: results

Fourth order accurate space discretization (error is mainly in time discretization).

Hyperbolic CFL condition $\Delta t = \Delta x/2$.

Schemes SSP2 and SSP3.



Nonlinear convection-diffusion equation

$$\begin{cases} \dfrac{\partial \omega}{\partial t} + [V + \mu \nabla \log(\omega)] \cdot \nabla \omega - \mu \, \Delta \omega = 0 \,, \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ \omega_0(t=0) = e^{-\|x\|^2/2}, \end{cases}$$

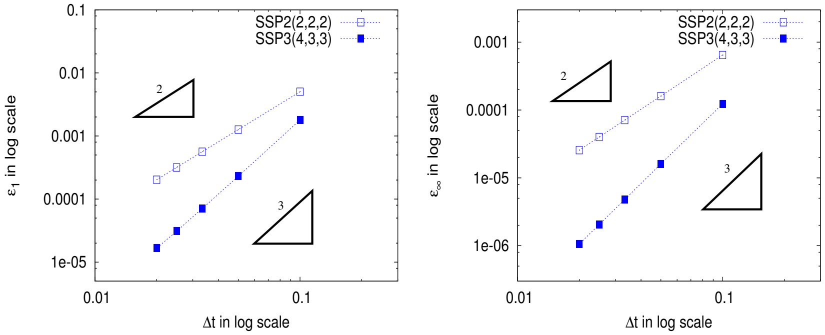
where $V = (1,1)^T$, $\mu = 0.5$. The exact solution is given by

$$\omega(t,x) = \frac{1}{\sqrt{4\,\mu\,t + 1}} \exp\left(-\frac{\|x - V\,t\|^2}{8\,\mu\,t + 2}\right), \quad t \geq 0, \quad x \in \mathbb{R}^2.$$

$$\mathcal{H}(t,u,v) = -(V + \mu \nabla \log(u)) \cdot \nabla v + \mu \Delta v.$$

Nonlinear c.-d. equation: results

$x \in (-10, 10)^2$. Final time $T = 0.5$.



Hele-Shaw flow

$$\frac{\partial \omega}{\partial t} + \frac{\partial}{\partial x}\left(\omega \frac{\partial^3 \omega}{\partial x^3}\right) = 0, \quad x \in \mathbb{R}, \quad t \geq 0, \tag{5}$$

with $\omega(x, t = 0) = \omega_0(x) \geq 0$.
Numerical results are compared with two solutions:

$$\omega(t, x) = \frac{1}{120(t + \tau)^{1/5}} \left[r^2 - \frac{x^2}{(t + \tau)^{2/5}}\right]_+^2,$$

where $[\cdot]_+$ denotes the positive part.
We have chosen $r = 2$, $\tau = 4^{-5}$ and $x \in (-2, 2)$.
and a smooth exact solution obtained by adding a suitable source term.
Equation preserves positivity. We found no high order discretization that preserves positivity.

Hele-Shaw flow

$$\mathcal{H}(t, u, v) = -\frac{\partial}{\partial x}\left(u \frac{\partial^3 v}{\partial x^3}\right) + f(t, x).$$

Concerning the space discretization, we apply a second order centred finite difference scheme for the space discretization

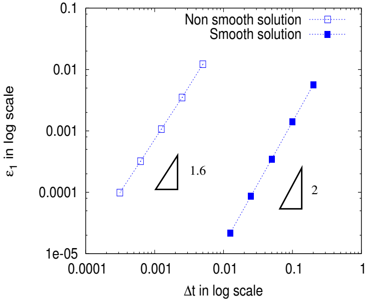
$$\mathcal{H}_\Delta(t, u_i, v_i) = -\frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x} + f(t+1, x_i),$$

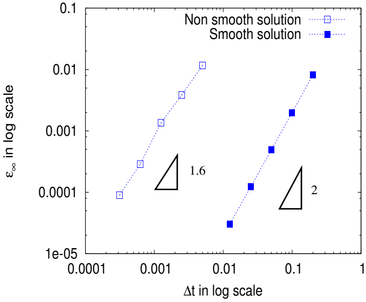
with

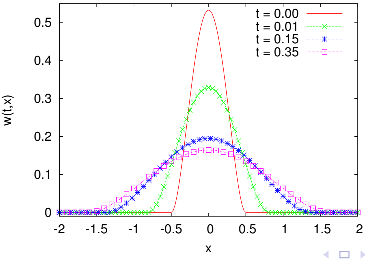
$$\mathcal{F}_{i+1/2} = u_{i+1/2} \frac{v_{i+2} - 3v_i + 3v_{i-1} - v_{i-2}}{\Delta x^3},$$

Hyperbolic CFL condition is used on the time step.

Hele-Shaw flow: results







A.K. test case

Consider

$$y' = 1 - k|y|y, \quad y(0) = 0$$

with a change of variables $y \to y/\sqrt{k}$ and $t \to t/\sqrt{k}$ the k disappears (equivalent to $k = 1$).
The exact solution to this problem is

$$y(t) = \tanh(t)$$

The stationary solution as $t \to \infty$ is $\bar{y} = 1$.
Now we consider different IMEX schemes applied to this problem, in the fashion illustrated before.

Two kinds of Implicit-Explicit Euler

Two tableaux are possible for IMEX Euler.
Simple splitting:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|c} 0 & 1 \\ \hline & 1 \end{array}$$

The application of the method gives

$$y_1 = y_0 + \Delta t(1 - |y_0| y_1)$$

which can be solved to

$$y_1 = \frac{y_0 + \Delta t}{1 + \Delta t |y_0|}$$

Stationary solution $\bar{y}$:

$$\bar{y} = \bar{y} + \Delta t(1 - |\bar{y}|\bar{y})$$

which can be uniquely solved giving $\bar{y} = 1 \Rightarrow$ Equilibria are maintained.
If $y_0 = 0$ then $y_1 = \Delta t$. If $\Delta t > 1$ the profile is not monotone.

Another form of Implicit-Explicit Euler [ARS(1,1,1)]:

$$
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 1 & 0 \\
\hline
 & 1 & 0
\end{array}
\qquad
\begin{array}{c|cc}
0 & 0 & 0 \\
1 & 0 & 1 \\
\hline
 & 0 & 1
\end{array}
$$

By applying this scheme to the problem one gets:

$$y_1 = \frac{y_0 + \Delta t}{1 + \Delta t(y_0 + \Delta t(1 - |y_0|y_0))}$$

If $y_0 = 0$ then

$$y_1 = \frac{\Delta t}{1 + \Delta t^2} < 1$$

therefore no overshoots are possible (at the first step).

Asymptotic behavior as $t \to \infty$:

$$\bar{y} = \frac{\bar{y} + \Delta t}{1 + \Delta t \bar{y} + \Delta t^2 - \Delta t^2 |\bar{y}| \bar{y}}$$

Look for positive $\bar{y} \Rightarrow |\bar{y}| = \bar{y}$. Simplifying:

$$(\Delta t \bar{y} - 1)(\bar{y}^2 - 1) = 0$$

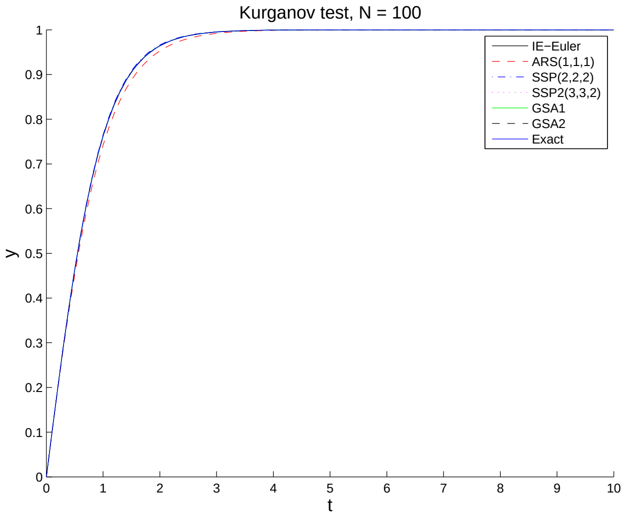
which gives two positive solutions:

$$\bar{y}_1 = 1, \quad \bar{y}_2 = 1/\Delta t$$

$\Rightarrow$ for $\Delta t > 1$ the asymptoric solution is wrong!

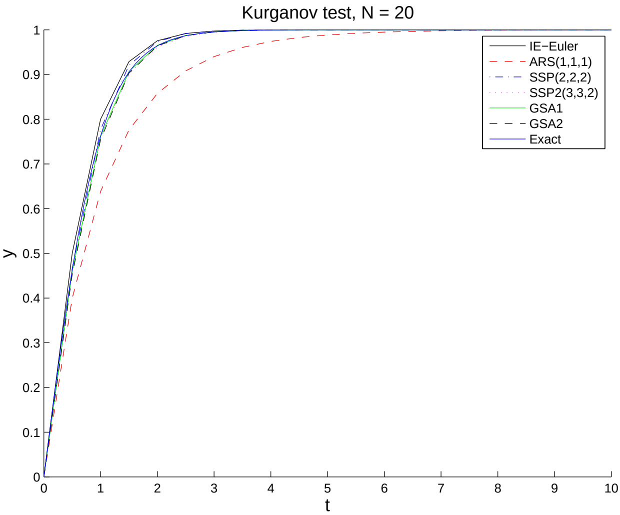
Comparison of various IMEX schemes

Start with $N = 100$, so $\Delta t = 1/10$.



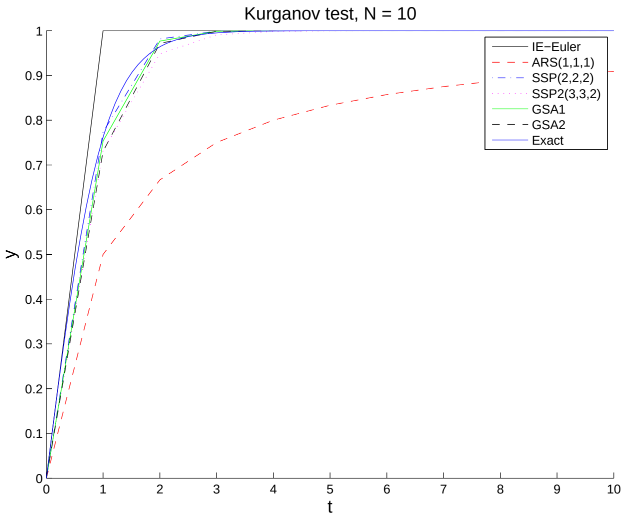
Comparison of various IMEX schemes

Start with $N = 20$, so $\Delta t = 1/2$.



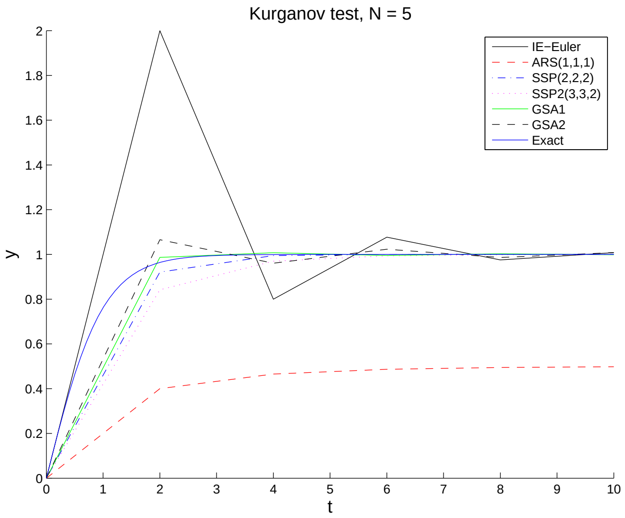
Comparison of various IMEX schemes

Start with $N = 10$, so $\Delta t = 1$.



Comparison of various IMEX schemes

Start with $N = 5$, so $\Delta t = 2$.



Some S-IMEX schemes appear to preserve both monotonicity of the solution and stationary solution. A deeper analysis is required.

Multispecies kinematic flow problems

[S. Boscarino, R. Bürger, P. Mulet, G. R., L. M. Villada, SISC (2015).]
We shall consider problems of the form

$$\partial_t \Phi + \partial_x \boldsymbol{f}(\Phi) = \partial_x \big(\boldsymbol{B}(\Phi)\partial_x \Phi\big), \tag{6}$$

where $\Phi(x,t) = (\phi_1, \dots, \phi_N)^{\mathrm{T}}$ is the unknown vector function,
$\boldsymbol{f}(\Phi) = (f_1(\Phi), \dots, f_N(\Phi))^{\mathrm{T}}$ is vector of flux density,
$\boldsymbol{B}(\Phi)$ is a given $N \times N$ matrix function expressing a diffusive correction.
$\boldsymbol{B}(\Phi) = \boldsymbol{0}$ on a set of nonzero measure, $\Rightarrow$ strongly degenerate problem.
Boundary conditions: either periodic or zero flux.
Two concrete applications: model of polydisperse sedimentation (diffusive correction describes sediment compressibility)[Burger et al, 2003], and multiclass Lighthill-Whitham-Richards (MCLWR) traffic model [Benzoni and Colombo 2003, Lighthill and Whitham 1955, Richards 1956, ...] the diffusive correction describes effects of reaction times and anticipation lengths.

Space discretization

Method of lines approach. Space is divided into M uniform cells. Set of ODEs:

$$\frac{\mathrm{d}\boldsymbol{\Phi}}{\mathrm{d}t} = -\frac{1}{\Delta x}(\Delta^- \boldsymbol{f})(\boldsymbol{\Phi}) + \frac{1}{\Delta x^2}\boldsymbol{\mathcal{B}}(\boldsymbol{\Phi})\boldsymbol{\Phi}, \tag{7}$$

where $\boldsymbol{\Phi} = (\Phi_1(t), \dots, \Phi_M(t))^{\mathrm{T}} \in \mathbb{R}^{NM}$: unknown solution,
$\Phi_j(t) \approx \Phi(x_j, t)$, $j = 1, \dots, M$,
$(\Delta^- \boldsymbol{f})(\boldsymbol{\Phi}) \in \mathbb{R}^{NM}$: numerical flux differences [discretization of $\partial_x \boldsymbol{f}(\Phi)$],
$\boldsymbol{\mathcal{B}}(\boldsymbol{\Phi}) \in \mathbb{R}^{(NM)\times(NM)}$: block tridiagonal matrix [discretization of $\partial_x(\boldsymbol{B}(\Phi)\partial_x \Phi)$].

Model 1: Poydisperse sedimentation

ϕ_i: volume fraction of specie i.
Initial-boundary value problem in a (vertical) interval $[0, L]$.
BC: zero flux at both ends, i.e.

$$\boldsymbol{f}(\Phi) - \boldsymbol{B}(\Phi)\partial_x \Phi = \mathbf{0} \quad \text{for} x = 0 \text{ and } x = L,$$

Flux density functions:

$$f_i(\Phi) = \mu \bar{\varrho}_{\mathrm{s}} \phi_i V(\phi)(1-\phi)(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi), \quad i = 1, \ldots, N, \tag{8}$$

$\mu > 0$: viscosity constant, $\bar{\varrho}_{\mathrm{s}} > 0$: solid mass density minus fluid density,
$\delta_i := d_i^2/d_1^2$, $\boldsymbol{\delta} := (\delta_1 = 1, \delta_2, \ldots, \delta_N)^{\mathrm{T}}$.
The expression for $V(\phi)$ is given by

$$V(\phi) = \begin{cases} (1-\phi)^{n_{\mathrm{RZ}}-2} & \text{for } 0 \le \phi \le \phi_{\max}, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

The diffusion matrix is given by

$$\boldsymbol{B}(\Phi) := (\alpha_{ij})_{1 \leq i,j \leq N},$$

with

$$\alpha_{ij} = \frac{\mu V(\phi)}{g\phi} \Big\{ (1-\phi)\phi_i(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi)\sigma_{\mathrm{e}}'(\phi) \\ - \left[\delta_i \delta_{ij} - \delta_j \phi_i - \frac{\phi_i}{\phi}(\delta_i - \boldsymbol{\delta}^{\mathrm{T}}\Phi) \right] \sigma_{\mathrm{e}}(\phi) \Big\}, \quad i,j = 1, \dots, N,$$

Model 2: Diffusively corrected MCLWR model

Describes a population of vehicles.
$v_i^{\max}$: preferred velocity of vehicle class i, where
$v_1^{\max} > v_2^{\max} > \cdots > v_N^{\max} > 0$.
This preferential velocity is multiplied by a *limitation function* $V = V(\phi)$, describes drivers' attitude to reduce velocity in presence of other cars.
Diffusion matrix:

$$\boldsymbol{B}(\Phi) := (\alpha_{ij})_{1 \leq i,j \leq N},$$

with

$$\alpha_{ij}(\Phi) = -V'(\phi)\big[L_i + \tau_i\big(V'(\phi)(\boldsymbol{v}^{\max})^{\mathrm{T}}\Phi + \big(v_j^{\max} - v_i^{\max}\big)V(\phi)\big)\big]\phi_i v_i^{\max}$$

Time discretization

IMEX schemes The convection term is explicit, and the degenerate diffusion term is fully implicit. The simplest prototype scheme IMEX-Euler

$$\mathbf{\Phi}^{n+1} = \mathbf{\Phi}^n - \frac{\Delta t}{\Delta x}(\Delta^- \boldsymbol{f})(\mathbf{\Phi}^n) + \frac{\Delta t}{\Delta x^2}\mathcal{B}(\mathbf{\Phi}^{n+1})\mathbf{\Phi}^{n+1}, \qquad (10)$$

Higher order schemes are obtained by higher order IMEX. The implementation requires some sort of Newton's iteration.

S-IMEX schemes Semi-IMplict-EXplicit schemes.

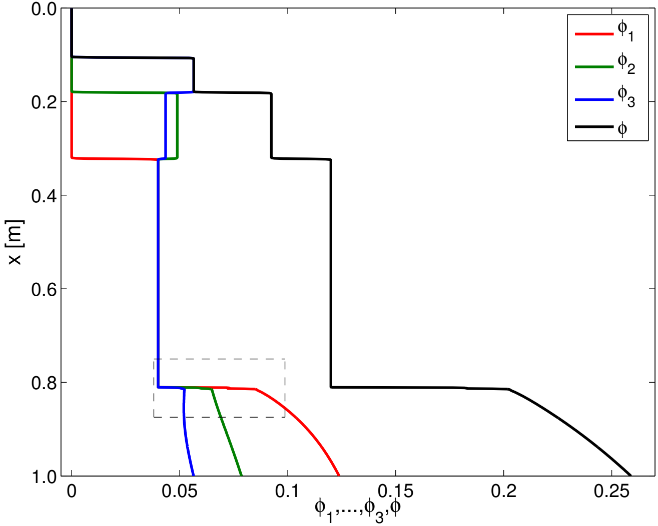
When we write the problem in the form

$$\frac{\mathrm{d}\mathbf{\Phi}^*}{\mathrm{d}t} = \mathcal{C}(\mathbf{\Phi}^*) + \mathcal{D}(\mathbf{\Phi}^*, \mathbf{\Phi})$$

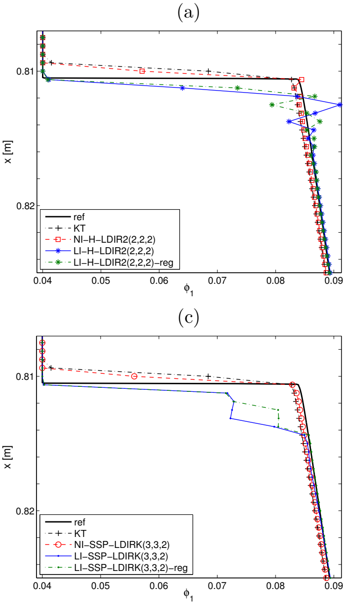
where we denote by a $*$ the variables that will be treated explicitly, and with no star the implicit variables.

Numerical results for model 1

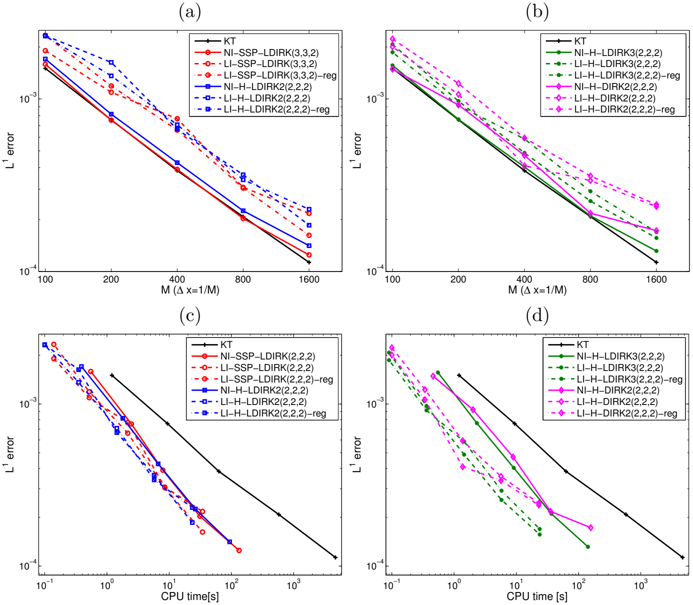
Typical profile obtained with $N = 4$



Enlargement of previous figure. Comparison among various (S)IMEX schemes.



Accuracy and cost comparison

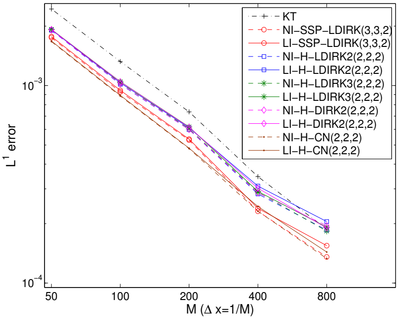


Linearly implicit schemes are generally less accurate but more cost effective

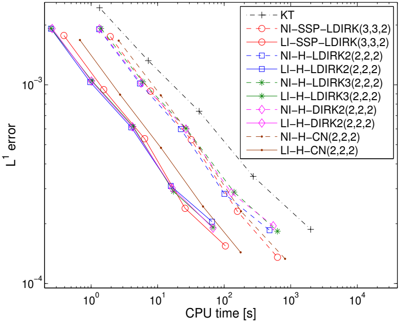
Numerical results for model 2

Several schemes have been compared on a typical test for model 2. Error vs M (left) and error vs CPU (right) are reported.
Linearly implicit schemes are those with continuous line.

(a)



(b)



Summarizing the comparison between the two approaches

IMEX approach:

- provides sharper and less oscillatory profiles for the same space and time discretization.
- requires iterative solution of non-linear equation

S-IMEX approach:

- slightly degraded solition, with small spurious oscillations
- no iterative solver $\Rightarrow$ it is in general more cost effective than IMEX-I.

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Error of S-IMEX over IMEX is mainly concentrated near discontinuities
$\Rightarrow$ hybrid scheme is desirable:
IMEX near discontinuities, S-IMEX elsewhere

All Mach number flow

When Mach number is very small, acoustic waves are much faster than material waves.

In many cases such acoustic waves possess very small energy, and one is not interested in resolving them.

Consider Euler equations of gas dynamics.

If explicit schemes are adopted, then the time CFL time restriction is dictated by the sound speed:

$$\Delta t < \Delta x / c_{\max}$$

where

$$c_{\max} = \max_{\Omega}(|u| + c_s)$$

with $c_s = (\partial p/\partial \rho)_s$ is the sound speed.

Incompressible vs compressible flow

For incompressible flow, in which the acoustic wave carry no energy and propagate at infinite speed, the pressure plays the role of a Lagrange multiplier, necessary to impose the incompressibility of the flow.
Incompressible Euler equations may be written in the form

$$\frac{\partial u}{\partial t} = -\mathbb{P} u \cdot \nabla u$$

where the operator $\mathbb{P}$ denotes the divergence-free projector.
The CFL time restriction of "explicit" schemes is therefore

$$\Delta t < \Delta x / u_{\max}(t^n)$$

where $u_{\max}(t^n) = \max_\Omega |u(x, t^n)|$

High Mach number:

- compressible flow: hyperbolic systems of conservation laws
- shock discontinuities are generic
- conservation schemes are necessary (at least near discontinuity) in order to guarantee consistency for weak solutions
- CFL restriction is *physiological* since one is in general interested in acoustic waves

Small mach number:

- quasi-incompressible flow
- often one is not interested in resolving acoustic waves
- material shocks do not form from smooth initial data and acoustic shocks have negligible amplitude
- classical CFL restriction is *pathological*: it is due to the stiffness of the problem and should be avoided.

Success and insuccess of semi-implicit

Here we consider a simple first order in time IMEX scheme appled to low Mach number Euler equation of gas dynamics.
We consider separately the case of

- isentropic gas dynamics
- complete Euler equations

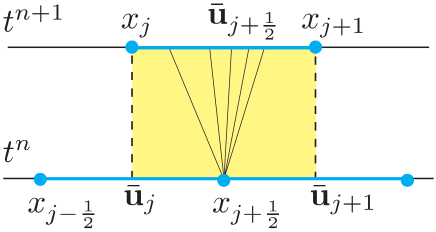
Furthermore, we consider both IMEX and S-IMEX approaches.

IMEX scheme for isentropic gas dynamics on staggered grid

Consider the equations for isentropic gas dynamics. The equations are rescaled so that the small Mach number ϵ explicitly appears in the equations:

$$\rho_t + m_x = 0$$
$$m_t + \left(\frac{m^2}{\rho} + \frac{p}{\epsilon^2}\right)_x = 0$$

The system is closed by $p = \rho^\gamma$.
Integrate the equation on a staggeerd grid, from time t^n to t^{n+1}



$$\bar{\rho}^{n+1}_{j+1/2} = \bar{\rho}^{n}_{j+1/2} - \frac{\Delta t}{\Delta x}(m^{m+1}_{j+1} - m^{n+1}_{j})$$

$$\bar{m}^{n+1}_{j+1/2} = \bar{m}^{n}_{j+1/2} - \frac{\Delta t}{\Delta x}(f^{n}_{j+1} - f^{n}_{j}) - \frac{\Delta t}{\epsilon^2 \Delta x}(\tilde{p}_{j+1} - \tilde{p}_{j})$$

where $f^n_j = (\bar{m}^n_j)^2/\bar{\rho}^n_j$. Second order in space is obtained by standard reconstruction adopted in Nessyahu-Tadmor scheme, es.

$$\bar{\rho}_{j+1/2} = \frac{\bar{\rho}_{j+1} + \bar{\rho}_j}{2} + \frac{1}{8}(\rho'_j - \rho'_{j+1})\Delta x$$

with ρ'_j a first order approximation of the first derivative on cell j (we use minmod in most cases).
A similar equation can be written for m^{n+1}_j. Using such equation, and plugging it in the equation for $\bar{\rho}^{n+1}_{j+1/2}$ one obtains an equation of the form:

$$\bar{\rho}^{n+1}_{j+1/2} = \rho^{*}_{j+1/2} + \frac{\Delta t^2}{\epsilon^2 \Delta x^2}(\tilde{p}_{j+3/2} - 2\tilde{p}_{j+1/2} + \tilde{p}_{j-1/2})$$

where by $\rho^{*}_{j+1/2}$ we denote quantities that can be computed explicitly (in a conservative way) and by $\tilde{p}$ we denote:

$$\tilde{p} = (\bar{\rho}^{n+1})^{\gamma} \qquad \text{in the IMEX case}$$

$$\tilde{p} = (\bar{\rho}^{n})^{\gamma-1}\bar{\rho}^{n+1} \qquad \text{in the S-IMEX case}$$

Notice that:

- in the IMEX case a non linear system has to be solved for each time step
- in practice it is simpler to use the p^{n+1} as unknown and consider $\rho = \rho(p)$, since in this case the nonlinearity is in the diagonal of the system.
- the S-IMEX scheme is only linearly implicit

isentropic gas dynamics

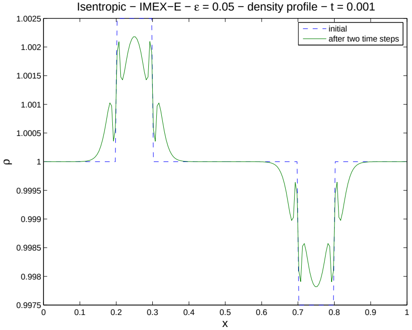
Consider the 2×2 system of Eueler equation, with

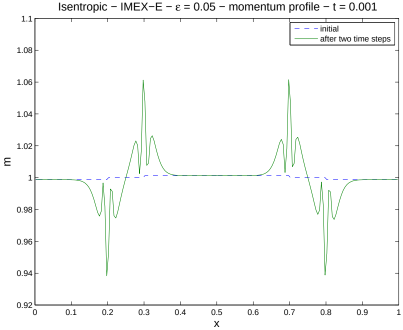
$$\begin{cases} p(\rho) = \rho^2 & \\ \rho(x,0) = 1.0, m(x,0) = 1 - \frac{\varepsilon^2}{2}, & x \in [0, 0.2] \bigcup [0.8, 1], \\ \rho(x,0) = 1 + \varepsilon^2, m(x,0) = 1, & x \in [0, 0.2], \\ \rho(x,0) = 1, m(x,0) = 1 + \frac{\varepsilon^2}{2}, & x \in [0.3, 0.7], \\ \rho(x,0) = 1 - \varepsilon^2, m(x,0) = 1, & x \in [0.7, 0.8], \end{cases} \tag{11}$$

This example consists of several Riemann problems, and has been used by Degond and Tang, 2011.

Numerical results for scheme S-IMEX

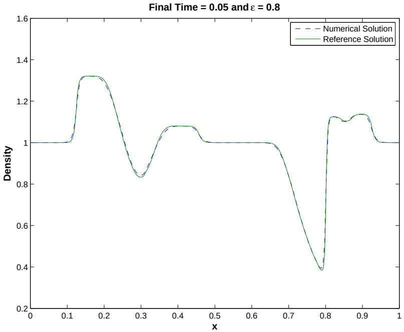
Here are the results for S-IMEX after one time step, for $\epsilon = 0.05$

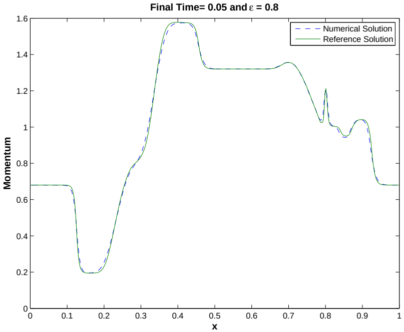




Spurious oscillations appear immediately.
Needless to say, the method does not converge.

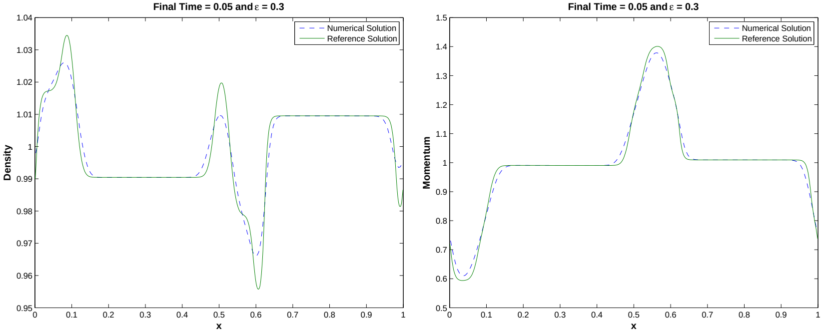
Numerical results for scheme IMEX





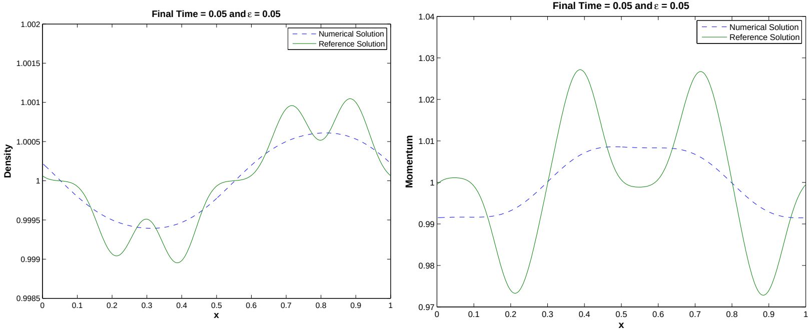
Numerical results at time $T = 0.05$ with $\Delta x = 1/200, \Delta t = 1/2000$, for the density (Left) and momentum (Right) for $\epsilon = 0.8$. The solid line is the reference solution calculated with $\Delta x = 1/500$ and $\Delta t = 1/20000$.

Numerical results for scheme IMEX



Same as before, with $\epsilon = 0.3$.

Numerical results for scheme IMEX-I



Same as before, with $\epsilon = 0.05$.
The results are very close to those obtained by Degond and Tang, except that our scheme is only first order in time.

Complete Euler equations

Consider the compressible Euler equations in 1D. We rescale the equations to emphasize the (possibly) small Mach number ϵ. For simplicity we assume a polytropic gas with constant γ.

$$\begin{aligned} \rho_t + m_x &= 0 \\ m_t + (\rho u^2 + p/\epsilon^2)_x &= 0 \\ E_t + ((E+p)u)_x &= 0 \end{aligned}$$

where $u = m/\rho$ is the fluid velocity.
Closure relation $p = (\gamma-1)E - (\gamma-1)\epsilon^2\rho u^2/2$.
The idea is now that as $\epsilon \ll 1$, the first equation becomes less and less relevant, while the total energy is essentially proportional to the pressure. Therefore we write an implicit system using the last two equations, and then compute ρ^{n+1} by post processing.

Discretize equations on a staggered grid, with a first order IMEX-E method in time, obtaining:

$$\begin{aligned}
\bar\rho^{n+1}_{j+1/2} &= \bar\rho^{n}_{j+1/2} && && -\frac{\Delta t}{\Delta x}\left(m^{n+1}_{j+1} - m^{n+1}_{j}\right)\\
\bar m^{n+1}_{j+1/2} &= \bar m^{n}_{j+1/2} && -\frac{\Delta t}{\Delta x}\frac{3-\gamma}{2}\left(f^{n}_{j+1} - f^{n}_{j}\right) && -\frac{\Delta t}{\Delta x}\frac{\gamma-1}{\epsilon^2}\left(E^{n+1}_{j+1} - E^{n+1}_{j}\right)\\
\bar E^{n+1}_{j+1/2} &= \bar E^{n}_{j+1/2} && -\frac{\Delta t}{\Delta x}\left(g^{n}_{j+1} - g^{n}_{j}\right) && -\gamma\frac{\Delta t}{\Delta x}\left(E^{n+1}_{j+1}u^n_j - E^{n+1}_{j}u^n_j\right)
\end{aligned}$$

Here for short we denoted $f = \rho u^2$ and $g = -(\gamma-1)\rho u^3/2$.
Just as in the case of isentropic gas dynamics, an equation for m^{n+1}_j is adopted and its expression is substituted in the equation for the energy obtaining:

$$E^{n+1}_{j+1/2} = E^{*}_{j+1/2} + \gamma\frac{\Delta t^2}{\epsilon^2\Delta x^2}\left(u^n_{j+3/2}E^{n+1}_{j+3/2} - 2u^n_{j+1/2}E^{n+1}_{j+1/2} + u^n_{j-1/2}E^{n+1}_{j-1/2}\right)$$

where, as usual, by $E^{*}_{j+1/2}$ denotes something that can be computed explicitly.
At variance with the case of the isentropic case, the milder nonlinearity allows use of S-IMEX scheme, providing effective solution with no iterative solver.

Two colliding acoustic pulses problem

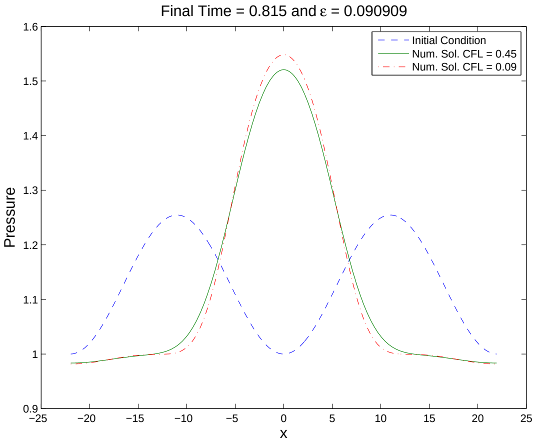
We consider a weakly compressible test problem. The setup consists of two colliding acoustic pulses in a weakly compressible regime. The domain is $-L \leqslant x \leqslant L = 2/\varepsilon$ and the initial data are given by

$$\begin{aligned} \rho(x,0) &= \rho_0 + \tfrac{1}{2}\varepsilon\rho_1\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right) \quad , \quad \rho_0 = 0.955, \quad \rho_1 = 2.0, \\ u(x,0) &= \tfrac{1}{2}u_0\,\mathrm{sign}(x)\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right) \ , \quad u_0 = 2\sqrt{\gamma}, \\ p(x,0) &= p_0 + \tfrac{1}{2}\varepsilon p_1\left(1-\cos\left(\tfrac{2\pi x}{L}\right)\right) \quad , \quad p_0 = 1.0, \quad p_1 = 2\gamma, \end{aligned} \tag{12}$$

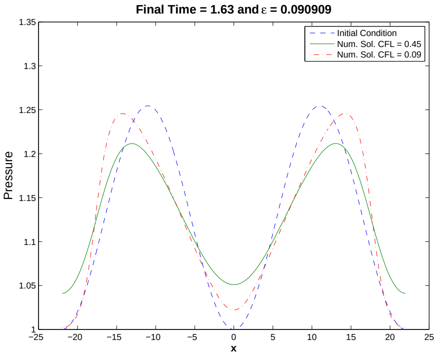
We set $\varepsilon = 1/11$. Number of cells $N = 440$.

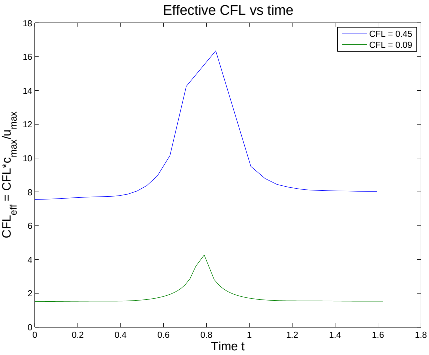
Complete Euler: results

Second order in space, first order in time.
Number of cells $N = 440$. Time step set by (material) CFL:
$\Delta t = \mathrm{CFL} * \Delta x / u_{\max}$ Periodic BC.



Left: same as before, at time $T = 1.63$.
Right: Effective CFL number $\mathrm{CFL} * c_{\max}/u_{\max}$ vs time.





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- A very simple semi-implicit scheme for the Euler equations of gas dynamics is proposed and tested on insentropic and full Euler equations in 1 space dimension.
- S-IMEX approach works for the full Euler equations, while isentropic case (with non-linear $p(\rho)$) requires an IMEX approach.

Work in progress

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Thank you!