Computing measure-valued solutions of hyperbolic conservation laws

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Hyperbolic conservation laws

- We consider nonlinear hyperbolic systems of conservation laws

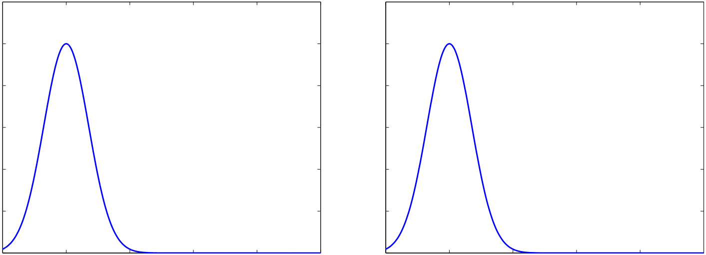
$$\begin{aligned} \partial_t u + \nabla \cdot f(u) = 0 \\ u(x,0) = u_0(x). \end{aligned} \tag{1}$$

  for $u = u(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$.
- Hyperbolic conservation laws model quantities which are *conserved over time*: mass, momentum, energy, number of particles, magnetic fields, etc.
- Hyperbolic conservation laws are used in modeling:
  - Flow in porous media (the Buckley Leverett equation)
  - Tsunamis, storm surges, tidal waves (the shallow water equations)
  - Gas dynamics (the Euler equations)
  - Flow of plasmas, solar physics (the magnetohydrodynamic equations)
  - +++

Weak solutions

$$\partial_t u + \nabla \cdot f(u) = 0 \qquad (1)$$

If $f(u)$ is nonlinear then the wave speed $f'(u)$ depends on the solution itself:



(a) Linear ($\partial_t u + \partial_x u = 0$) (b) Nonlinear ($\partial_t u + \left(u^2/2\right)_x = 0$)

Discontinuities (**shocks**) appear, and (1) cannot be interpreted in the classical sense.

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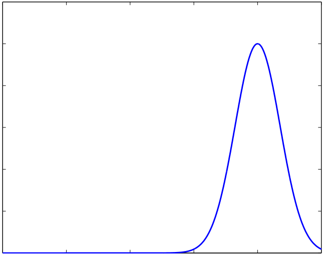
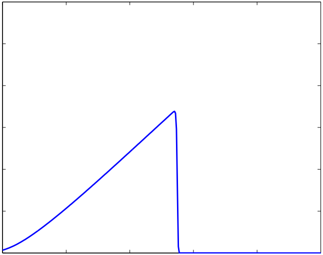


(a) Linear ($\partial_t u + \partial_x u = 0$)



(b) Nonlinear ($\partial_t u + \left(u^2/2\right)_x = 0$)

Discontinuities (**shocks**) appear, and (1) cannot be interpreted in the classical sense.

Weak solutions

$$\partial_t u + \nabla \cdot f(u) = 0 \tag{1}$$

The solutions of (1) are in general discontinuous, so we must interpret (1) in the sense of distributions:

Definition

A function $u \in L^\infty(\mathbb{R}^d \times \mathbb{R}_+)$ is a **weak solution** of (1) if

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}_+} u\varphi_t + f(u) \cdot \nabla\varphi \; dxdt + \int_{\mathbb{R}^d} u(x,0)\varphi(x,0) \; dx = 0$$

for all $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+)$.

Entropy conditions

$$\partial_t u + \nabla \cdot f(u) = 0 \tag{1}$$

- After the formation of shocks there are *infinitely* many weak solutions.
- Motivated from physics, *entropy conditions* are imposed to single out a unique "physical" solution.

Definition

An **entropy pair** is a convex function $\eta(u)$, together with a function $q(u)$ such that $q'(u) = \eta'(u) \cdot f'(u)$.

- Entropy should be **dissipated** at shocks:

Definition

A weak solution u is an **entropy solution** of (1) if

$$\eta(u)_t + \nabla \cdot q(u) \leqslant 0$$

for all entropy pairs (η, q) (in the sense of distributions).

Section 1

Stability of the initial-value problem

Eq. (1) as a dynamical system – a cartoon

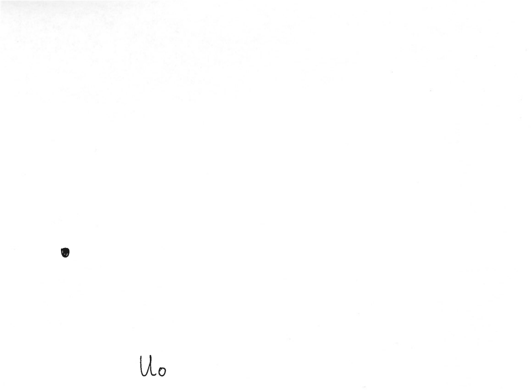
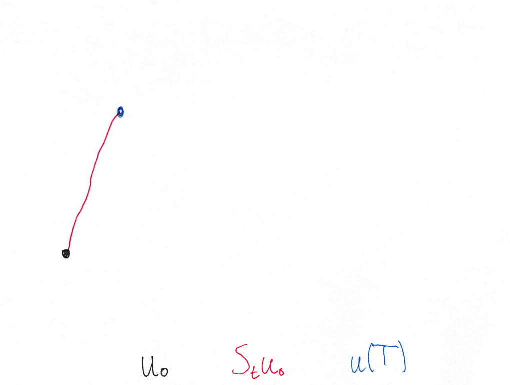


Figure : Cartoon of solution space

- Each point • represents an initial data function u_0.

Eq. (1) as a dynamical system – a cartoon



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- Each u_0 is evolved in time to $u(T) = S_T u_0$.

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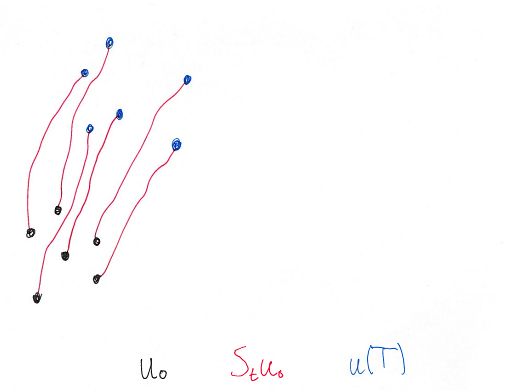

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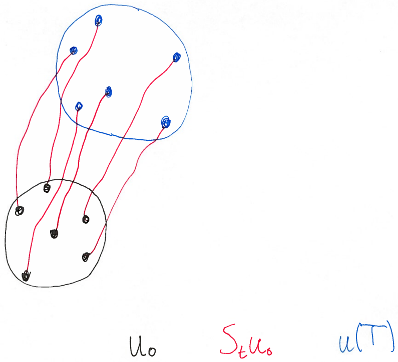

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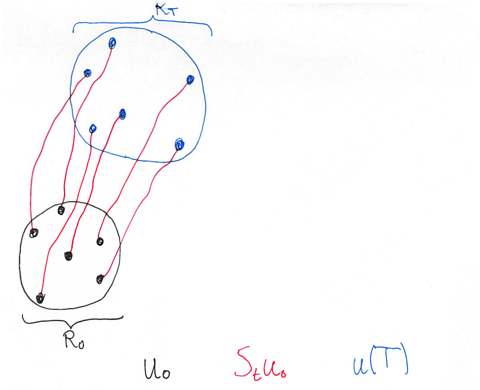

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- Each point • represents an initial data function u_0.
- Each u_0 is evolved in time to $u(T) = S_T u_0$.
- The spread R_T of $u(T)$ depends on the spread R_0 of u_0.

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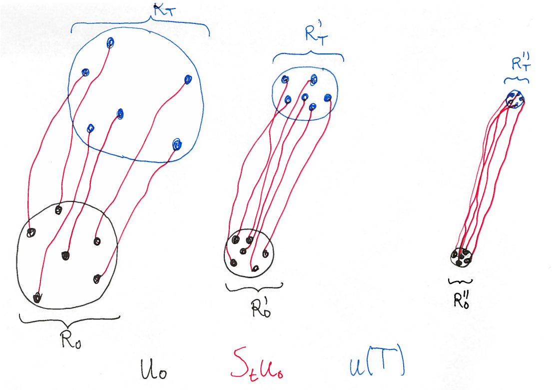

Figure : Cartoon of solution space

- Each point • represents an initial data function u_0.
- Each u_0 is evolved in time to $u(T) = S_T u_0$.
- The spread R_T of $u(T)$ depends on the spread R_0 of u_0.
- The system is **stable** (with respect to initial data) if $R_T \to 0$ when $R_0 \to 0$.

Well-posedness: scalar equations

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

Theorem (Kruzkov 1970)

For **scalar** *conservation laws in* **any number of dimensions** $d \geqslant 1$, *there* **exists a unique entropy solution** *of* (1). *The solutions are stable with respect to initial data:*

$$\int_{\mathbb{R}^d} |u(x,t) - \tilde{u}(x,t)|\, dx \leqslant \int_{\mathbb{R}^d} |u_0(x) - \tilde{u}_0(x)|\, dx \qquad \text{for all } t > 0$$

for entropy solutions u *and* $\tilde{u}$ *with initial data* u_0 *and* $\tilde{u}_0$.

There is a wealth of stable, convergent numerical methods for scalar conservation laws (**Lax & Friedrichs, Crandall & Majda, Tadmor, Osher, Roe, Johnson & Szepessy**, $+++$)

Well-posedness: systems of equations

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For **systems of equations** ($N > 1$), much less is known.

Theorem (Lax 1957, Glimm 1965, Bressan et al. 2000)

For systems of equations in **one dimension** $d = 1$*, there exists a unique entropy solution of* (1) *whenever the initial data is* **sufficiently small** *(i.e., sufficiently close to a constant solution).*

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De Lellis et al.

Two-dimensional isentropic Euler is ill-posed in the sense of entropy solutions.

There is **no** general convergence theory of numerical methods for multidimensional systems of conservation laws.

Summary

To summarize:

- Scalar, multidimensional conservation laws are well-posed: there is existence, uniqueness and stability of solutions.
- For scalar conservation laws, there are efficient, high-order accurate numerical schemes which are stable and convergent.
- There is no general existence, uniqueness and stability theory for multidimensional systems of conservation laws.
- No numerical scheme is known to converge for “large” initial data.

The Euler equations

- As an example we consider the two-dimensional Euler equations for compressible, polytropic ideal gases,

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho v_x\\ \rho v_y\\ E\end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix}\rho v_x\\ \rho v_x^2 + p\\ \rho v_x v_y\\ (E+p)v_x\end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix}\rho v_y\\ \rho v_x v_y\\ \rho v_y^2 + p\\ (E+p)v_y\end{pmatrix} = 0.$$

  The density ρ, velocity field (v_x, v_y), pressure p and total energy E are related by the equation of state

$$E = \frac{p}{\gamma - 1} + \frac{\rho(v_x^2 + v_y^2)}{2}.$$

- To approximate this system we use a standard finite volume method.

Instability of hyperbolic systems

- Consider initial data

$$\begin{pmatrix}\rho & v_x & v_y & p\end{pmatrix} = \begin{cases} \begin{pmatrix}1 & 0.5 & 0 & 2.5\end{pmatrix} & \text{if } y \leqslant 0.25 \text{ or } y \geqslant 0.75 \\ \begin{pmatrix}2 & -0.5 & 0 & 2.5\end{pmatrix} & \text{if } 0.25 < y < 0.75. \end{cases}$$

(periodically in $x, y \in [0,1]$). This is a steady state.

- We add a small perturbation of order 10^{-2} to the initial data.

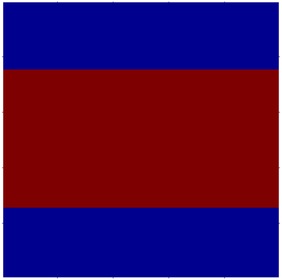
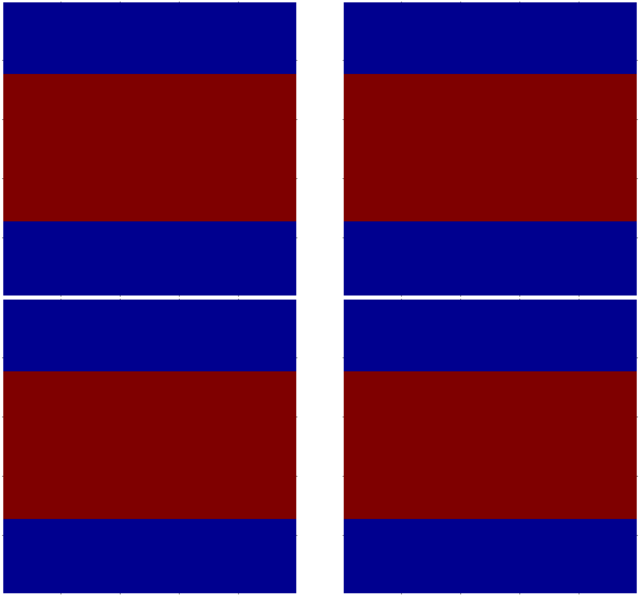
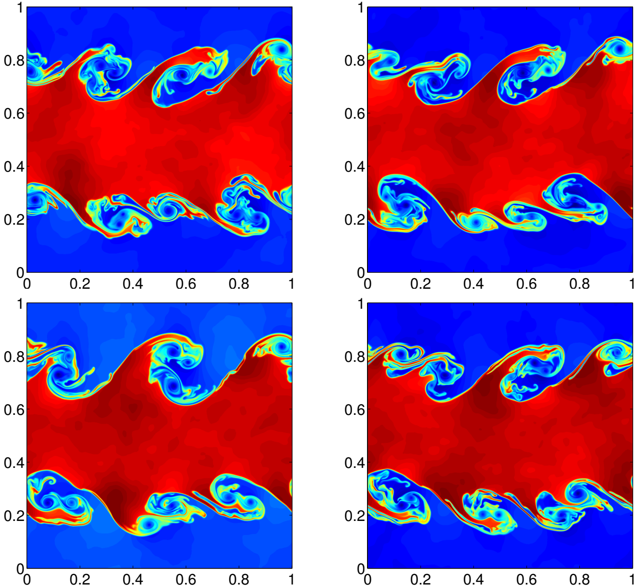


Figure : Movie of density

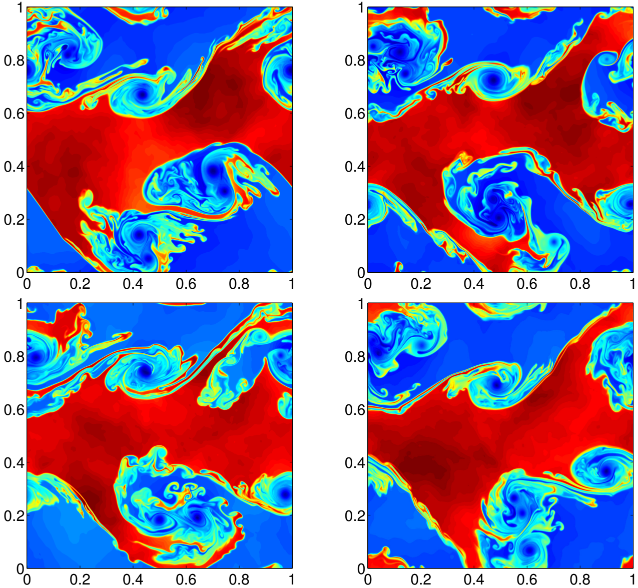
Four different perturbations. Density at $t = 0$



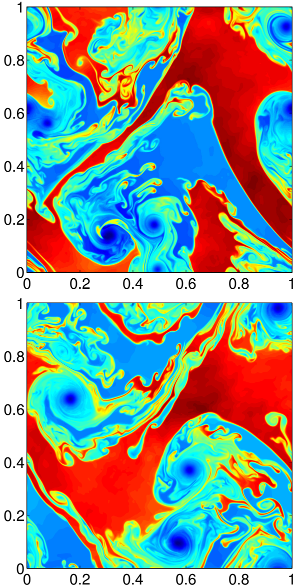
Four different perturbations. Density at $t = 0.5$



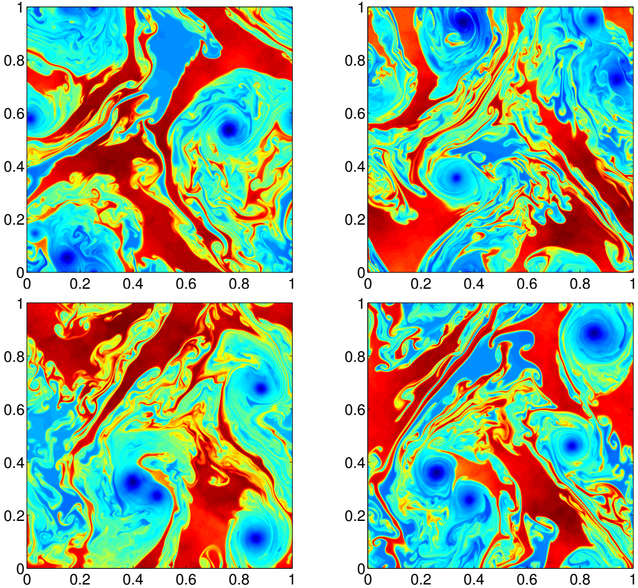
Four different perturbations. Density at $t = 1$



Four different perturbations. Density at $t = 1.5$



Four different perturbations. Density at $t = 2$



Nonconvergence as $\Delta x \to 0$

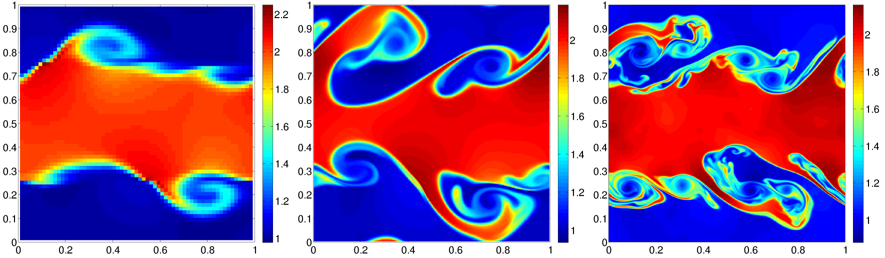


Figure : Mass density at $t = 2$, computed on meshes of 64^2, 256^2 and 1024^2 gridpoints.

- Computed solutions displays details at the scale of numerical viscosity $\sim O(\Delta x)$.
- Mesh refinement $\Delta x \to 0$ induces finer and finer scales in the flow.
- There is no convergence as $\Delta x \to 0$.

Nonconvergence as $\Delta x \to 0$

- The non-convergence can be quantified:

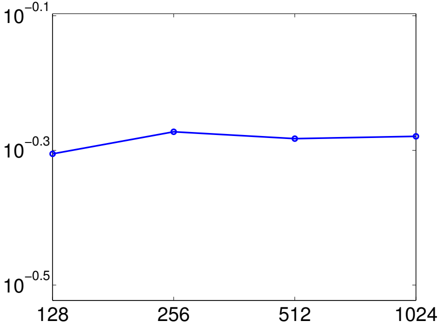


Figure : "Cauchy rates" $\left\| u^{\Delta x}(t) - u^{\Delta x/2}(t) \right\|_{L^1\left([0,1]^2\right)}$.

- Using different perturbation, or a different numerical scheme, or a different mesh, gives very different solutions.

Instability of the Cauchy problem

The Cauchy problem is **unstable**: decreasing initial uncertainty does not decrease uncertainty at $t = T$.

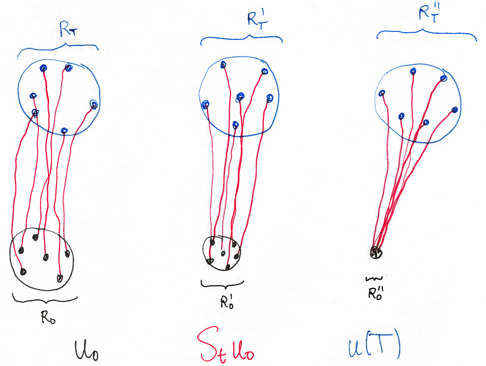


Figure : Cartoon of solution space

Instability of the Cauchy problem

Conclusions

Certain initial-value problems are **unstable** with respect to initial data:

$$\|u(t) - \tilde{u}(t)\|$$

can be large even if

$$\|u_0 - \tilde{u}_0\|$$

is small.

- We wish to quantify the spread/distribution of solutions $u(T)$ for a **fixed initial data** u_0.
- Some available options:
  - Statistical solutions (Foias, Temam, ...)
  - Measure-valued solutions (DiPerna, Murat, Tartar)
  - Multivalued semigroups (Ball, Melnik, Valero, ...)

Section 2

Measure-valued solutions

Statistical measures of $M = 3$ samples

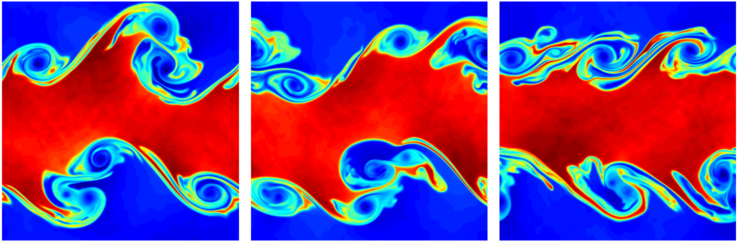
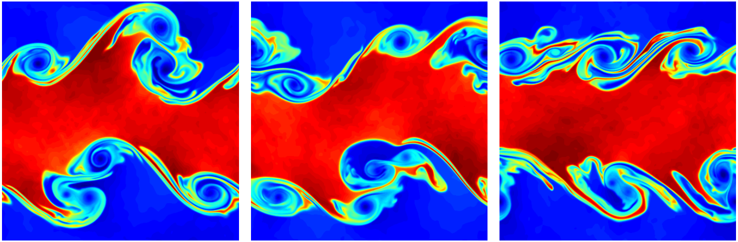


Figure : Density $\rho(x, t)$ at $t = 2$ for $M = 3$ different samples

Statistical measures of $M = 3$ samples



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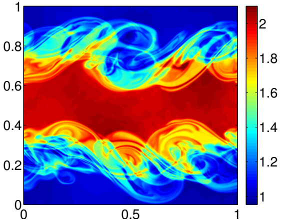
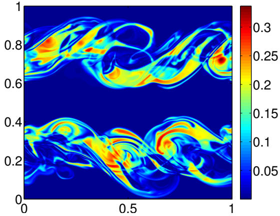


(a) Mean $E(\rho(x,t))$



(b) Variance $E(\rho(x,t)^2) - E(\rho(x,t))^2$

Figure : Mean and variance at $t = 2$

Statistical measures of $M = 5$ samples

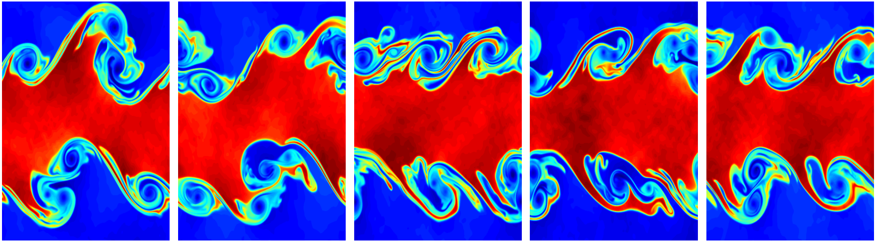
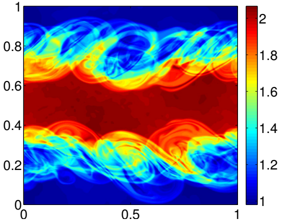
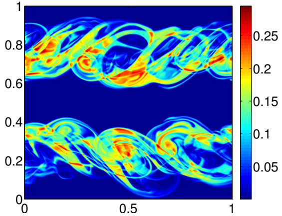


Figure : Density $\rho(x, t)$ at $t = 2$ for $M = 5$ different samples



(a) Mean $\mathrm{E}(\rho(x, t))$



(b) Variance $\mathrm{E}(\rho(x, t)^2) - \mathrm{E}(\rho(x, t))^2$

Figure : Mean and variance at $t = 2$

Statistical measures of $M = 10$ samples

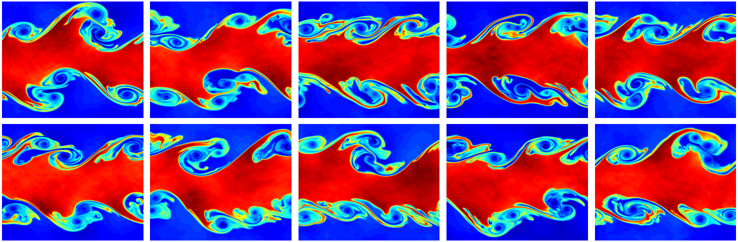
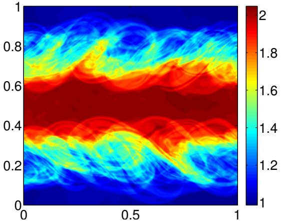


Figure : Density $\rho(x, t)$ at $t = 2$ for $M = 10$ different samples



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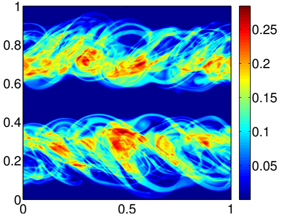


(b) Variance $\mathrm{E}(\rho(x, t)^2) - \mathrm{E}(\rho(x, t))^2$

Figure : Mean and variance at $t = 2$

Statistical measures of $M = 20$ samples

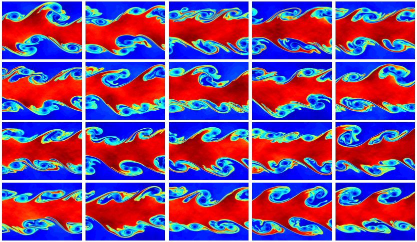
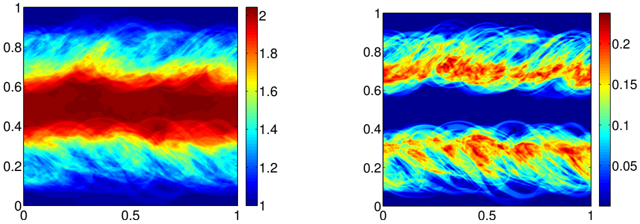


Figure : Density $\rho(x, t)$ at $t = 2$ for $M = 20$ different samples



(a) Mean $E(\rho(x, t))$ (b) Variance $E(\rho(x, t)^2) - E(\rho(x, t))^2$

Figure : Mean and variance at $t = 2$

Statistical measures of $M = 40$ samples

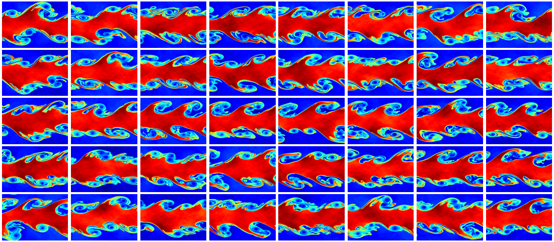
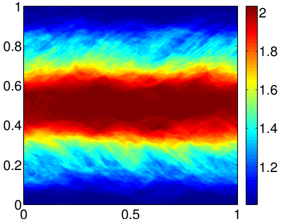
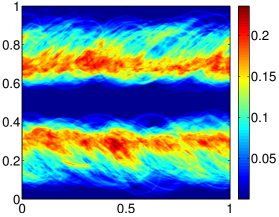


Figure : Density $\rho(x, t)$ at $t = 2$ for $M = 40$ different samples



(a) Mean $E(\rho(x, t))$



(b) Variance $E(\rho(x, t)^2) - E(\rho(x, t))^2$

Figure : Mean and variance at $t = 2$

Statistical interpretation of solutions

- Individual samples show chaotic, turbulent behavior.
- However, statistical properties like the mean and variance seem to converge as the number of samples M increases.
- We reformulate the PDE in a probabilistic setting: **measure-valued solutions**.

Young measures

- Instead of assigning only one value $u(x,t) \in \mathbb{R}^N$ to each point (x,t), *we view the solution at* (x,t) *as a probability measure on* $\mathbb{R}^N$*.*

Definition

A **Young measure** is a function ν mapping

$$(x,t) \mapsto \nu_x^t \in Prob(\mathbb{R}^N).$$

- Here, $Prob(\mathbb{R}^N) := \{\text{probability measures on } \mathbb{R}^N\}$.

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- Here, $Prob(\mathbb{R}^N) := \{\text{probability measures on } \mathbb{R}^N\}$.
- Any function $u : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^N$ can be identified with the Young measure

$$\nu_x^t := \delta_{u(x,t)} \qquad (\nu \text{ is } \textbf{atomic}).$$

- For $g \in C_0(\mathbb{R}^N)$ we evaluate "$g(\nu_x^t)$" as

$$\langle \nu_x^t, g \rangle := \int_{\mathbb{R}^N} g(\xi)\, d\nu_x^t(\xi) \qquad (\text{expectation of } g \text{ w.r.t. } \nu_x^t).$$

Measure-valued solutions

We replace the initial value problem

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

with the *measure-valued initial value problem*

$$\begin{aligned} \partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\ \nu_{(x,0)} &= \delta_{u_0(x)} \end{aligned} \tag{2}$$

(where $\mathrm{id}(\xi) := \xi$).

Measure-valued solutions

$$
\begin{aligned}
\partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\
\nu_{(x,0)} &= \delta_{u_0(x)}.
\end{aligned} \tag{2}
$$

Definition (DiPerna, 1985)

(i) A Young measure ν is a **measure-valued (MV) solution** of (2) if

$$
\int_{\mathbb{R}_+} \int_{\mathbb{R}} \langle \nu, \mathrm{id} \rangle \varphi_t + \langle \nu, f \rangle \cdot \nabla \varphi \; dxdt + \int_{\mathbb{R}} u_0(x) \varphi(x,0) \; dx = 0.
$$

for all $\varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$.

(ii) A Young measure ν is an **entropy measure-valued solution** of (2) if

$$
\int_{\mathbb{R}_+} \int_{\mathbb{R}} \langle \nu, \eta \rangle \varphi_t + \langle \nu, q \rangle \cdot \nabla \varphi \; dxdt + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x,0) \; dx \geqslant 0
$$

for all $0 \leqslant \varphi \in C_c^1(\mathbb{R} \times \mathbb{R}_+)$ for an entropy pair (η, q).

Note: If ν is an atomic (entropy) measure-valued solution, $\nu = \delta_u$, then u is an (entropy) weak solution, and vice versa.

MV and kinetic solutions

- Consider a **scalar conservation law**

$$\partial_t u + \nabla \cdot f(u) = 0. \tag{1}$$

- A **kinetic solution** $\chi = \chi(x,t;v) : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of (1) satisfies

$$\partial_t \chi + a(v) \cdot \nabla \chi = \partial_v m$$

for $a(v) := f'(v)$ and a measure $m = m(x,t) \geqslant 0$ on $\mathbb{R}$.

Theorem (Lions, Perthame, Tadmor (1994))

$u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is an entropy solution of (1) if and only if

$$\chi(x,t;v) := H(v) - H(v - u(x,t))$$

is a kinetic solution. Here, $H(v) := \begin{cases} 0 & v < 0 \\ 1 & v \geqslant 0 \end{cases}$.

MV and kinetic solutions

- Consider a **scalar conservation law**

$$\partial_t u + \nabla \cdot f(u) = 0. \tag{1}$$

- A **kinetic solution** $\chi = \chi(x,t;v) : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ of (1) satisfies

$$\partial_t \chi + a(v) \cdot \nabla \chi = \partial_v m$$

for $a(v) := f'(v)$ and a measure $m = m(x,t) \geqslant 0$ on $\mathbb{R}$.

Theorem (Lions, Perthame, Tadmor (1994))

$u \in L^\infty(\mathbb{R} \times \mathbb{R}_+)$ is an entropy solution of (1) if and only if

$$\chi(x,t;v) := H(v) - H(v - u(x,t))$$

is a kinetic solution. Here, $H(v) := \begin{cases} 0 & v < 0 \\ 1 & v \geqslant 0 \end{cases}$.

- If χ is a kinetic solution then

$$\nu_{x,t} := \partial_v H - \partial_v \chi(x,t)$$

is an MV solution.
- This framework only holds for scalar equations!

Section 3

Stability of measure-valued solutions

Weak-strong stability

We would like to show that the measure-valued Cauchy problem

$$\begin{aligned} \partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\ \nu_{(x,0)} &= \delta_{u_0(x)}. \end{aligned} \tag{2}$$

is stable in the following sense:

Definition

We say that (2) is **MV stable** if the following property holds:

For every u_0 there exists an entropy MV solution ν of (2) such that if

$$\|u_0 - \tilde{u}_0\| \ll 1$$

then

$$d(\nu, \tilde{\nu}) \ll 1$$

for all EMV solutions $\tilde{\nu}$.

The L^p Wasserstein distance

- To measure the distance between Young measures ν and $\tilde{\nu}$, we use the metric

$$d_p\left(\nu^t, \tilde{\nu}^t\right) := \left(\int_{\mathbb{R}^d} W_p(\nu_x^t, \tilde{\nu}_x^t)^p \, dx\right)^{1/p}.$$

- W_p is the **Wasserstein distance** (a metric between probability measures).
- If ν and $\tilde{\nu}$ are **atomic** ($\nu = \delta_u$ and $\tilde{\nu} = \delta_{\tilde{u}}$), then

$$d_p\left(\nu^t, \tilde{\nu}^t\right) = \left\|u(\cdot, t) - \tilde{u}(\cdot, t)\right\|_{L^p(\mathbb{R}^d)}.$$

MV stability for scalar conservation laws

Consider a **scalar conservation law**.

Theorem (USF, Käppeli, Mishra, Tadmor 2014)

Let u be the entropy solution of the conservation law, and let $\tilde{\nu}$ be an entropy MV solution with initial data $\tilde{\nu}^0$. Then for all $t > 0$,

$$d_1\left(\tilde{\nu}^t, \delta_{u(\cdot,t)}\right) \leqslant d_1\left(\tilde{\nu}^0, \delta_{u_0}\right),$$

i.e.,

$$\int_{\mathbb{R}^d} W_1\left(\tilde{\nu}_x^t, \delta_{u(x,t)}\right)\, dx \leqslant \int_{\mathbb{R}^d} W_1\left(\tilde{\nu}_x^0, \delta_{u_0(x)}\right)\, dx.$$

Proof.

Follows from [Kruzkov 1970] and [DiPerna 1985, Theorem 4.1]. □

This theorem says that the MV Cauchy problem for **scalar conservation laws** *is MV stable (if σ is close to u_0, then ν is close to u).*

MV stability of systems

Consider a **general hyperbolic system of conservation laws**.

Theorem (USF, Käppeli, Mishra, Tadmor 2014)

Let u be a **classical solution** *of* (1)*, and let $\tilde{\nu}$ be an entropy MV solution with initial data $\tilde{u}_0$. Then for all $t > 0$,*

$$d_2\left(\tilde{\nu}^t, \delta_{u(\cdot,t)}\right) \leqslant C\|\tilde{u}_0 - u_0\|_{L^2},$$

i.e.,

$$\int_{\mathbb{R}^d} W_2\left(\tilde{\nu}_x^t, \delta_{u(x,t)}\right)^2 \, dx \leqslant C \int_{\mathbb{R}^d} |\tilde{u}_0(x) - u_0(x)|^2 \, dx$$

for some $C = C(t)$.

Proof.

Generalization of [Demoulini, Stuart, Tzavaras 2012, Theorem 2.2] (using the method of relative entropies). □

This theorem says that the MV Cauchy problem is MV stable **whenever there is a smooth solution** *(if $\tilde{u}_0$ is close to u_0, then $\tilde{\nu}$ is close to u).*

Section 4

Computing approximate measure-valued solutions

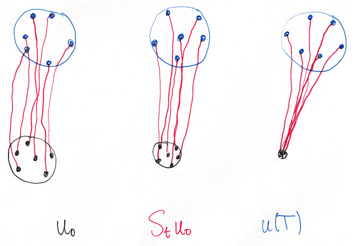
Computing MV solutions

We would like to compute numerical approximations to the MV Cauchy problem

$$
\begin{aligned}
\partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\
\nu_{(x,0)} &= \delta_{u_0(x)}.
\end{aligned}
\tag{2}
$$

Idea of algorithm (version 0)

- The solution ν should contain not only information about u_0, *but of all infinitesimally small perturbation of* u_0.



Computing MV solutions

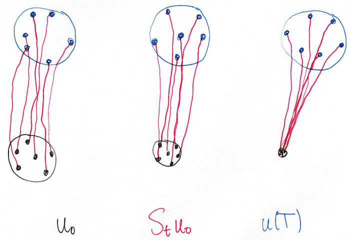
We would like to compute numerical approximations to the MV Cauchy problem

$$\begin{aligned} \partial_t \langle \nu, \mathrm{id} \rangle + \nabla \cdot \langle \nu, f \rangle &= 0 \\ \nu_{(x,0)} &= \delta_{u_0(x)}. \end{aligned} \tag{2}$$

Idea of algorithm (version 1)

For a small $\varepsilon > 0$, average over all solutions of (1) with perturbed initial data $\tilde{u}_0$:

$$\|u_0 - \tilde{u}_0\| < \varepsilon.$$



Computing MV solutions

Idea of algorithm (version 2)

1. Pick perturbations $\tilde{u}_0^1, \dots, \tilde{u}_0^M$ with $\|u_0 - \tilde{u}_0^k\| < \varepsilon$.
2. (Numerically) approximate

$$\begin{aligned} \partial_t u^k + \nabla \cdot f(u^k) &= 0 \\ u^k(x,0) &= \tilde{u}_0^k(x). \end{aligned}$$

3. Let

$$\nu_x^{M,t} := \frac{1}{M} \sum_{k=1}^{M} \delta_{u^k(x,t)}.$$

Computing approximate MV solutions: Monte Carlo approximation

Algorithm

Let $X^1(\omega; x), \dots, X^M(\omega; x)$ be i.i.d. random fields.

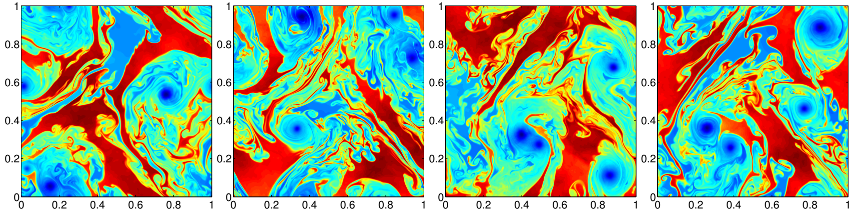
1. Add a small random perturbation to the initial data,
$$u_0^k(\omega; x) = u_0(x) + \varepsilon X^k(\omega; x), \qquad k = 1, \dots, M$$
2. For some fixed $\omega \in \Omega$, compute an approximate solution $u^{\Delta x, k}(x, t)$ of the Cauchy problem with initial data $u_0^k(\omega; x)$.
3. Define $\nu^{\Delta x, M}$ as
$$\nu_x^{\Delta x, M, t} := \frac{1}{M} \sum_{k=1}^{M} \delta_{u^{\Delta x, k}(x,t)},$$
the "law" of $u^{\Delta x, 1}, \dots, u^{\Delta x, M}$.

Theorem (USF, Käppeli, Mishra, Tadmor 2014)

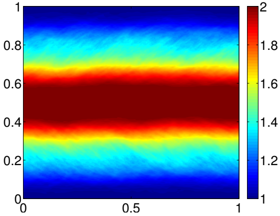
The above Monte Carlo method converges as $M \to \infty$.

Numerical example: single samples

- We consider the unstable Kelvin-Helmholtz example.
- We compute $M = 400$ different samples and assemble the **approximate EMV solution** ν^M:



$\Downarrow$

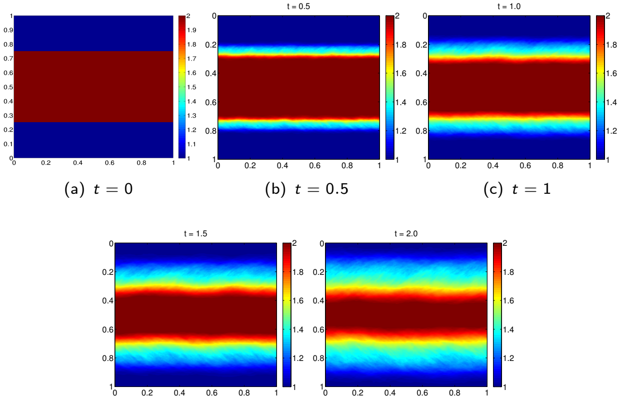


Numerical example: Time evolution

We compute the **mean**

$$\overline{u}^{\Delta x} := \langle \nu^{\Delta x}, \mathrm{id} \rangle = \int_{\mathbb{R}^N} \xi \, d\nu(\xi) = \int_{\Omega} u^{\Delta x}(\omega) \, dP(\omega).$$

over $M = 400$ samples on a grid of 1024^2 mesh points.

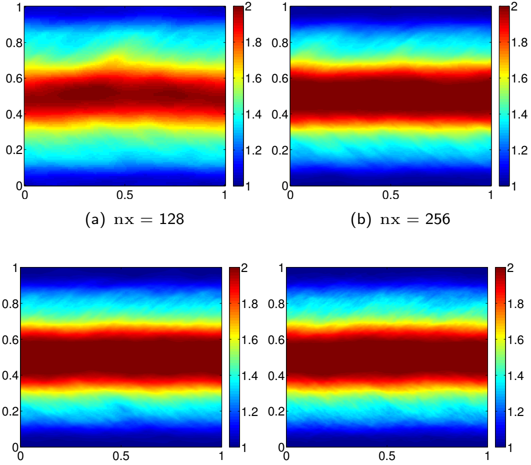


(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$

(d) $t = 1.5$ (e) $t = 2$

Numerical example: Statistical quantities converge

The mean $\overline{u}^{\Delta x}$ converges:



(a) $\mathrm{nx} = 128$ (b) $\mathrm{nx} = 256$

(c) $\mathrm{nx} = 512$ (d) $\mathrm{nx} = 1024$

Figure : Density mean at $t = 2$ under mesh refinement.

Numerical example: Statistical quantities converge

We compute the **variance** $\mathrm{Var}^{\Delta x} := \langle \nu^{\Delta x}, \mathrm{id}^2 \rangle - \langle \nu^{\Delta x}, \mathrm{id} \rangle^2$.

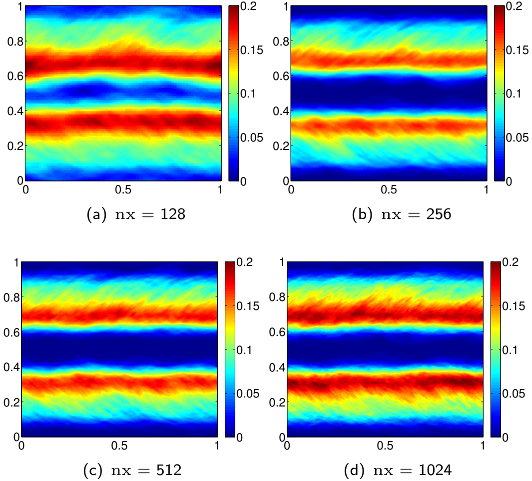


Figure : Density variance at $t = 2$ under mesh refinement.

Numerical example: Statistical quantities converge

We plot the "L^1 Cauchy rates"

$$\left\|\overline{u}^{\Delta x}(t) - \overline{u}^{\Delta x/2}(t)\right\|_{L^1([0,1]^2)}.$$

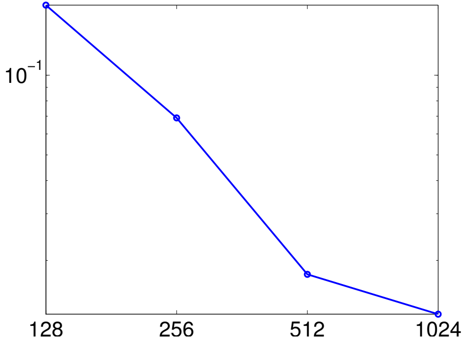


Figure : Cauchy rates for mean of density at $t = 2$ for $\mathrm{nx} = 128, 256, 512, 1024$.

Numerical example: Statistical quantities converge

We plot the "L^1 Cauchy rates"

$$\left\| \mathrm{Var}^{\Delta x}(t) - \mathrm{Var}^{\Delta x/2}(t) \right\|_{L^1([0,1]^2)}$$

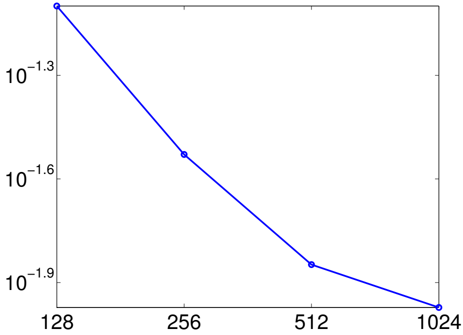
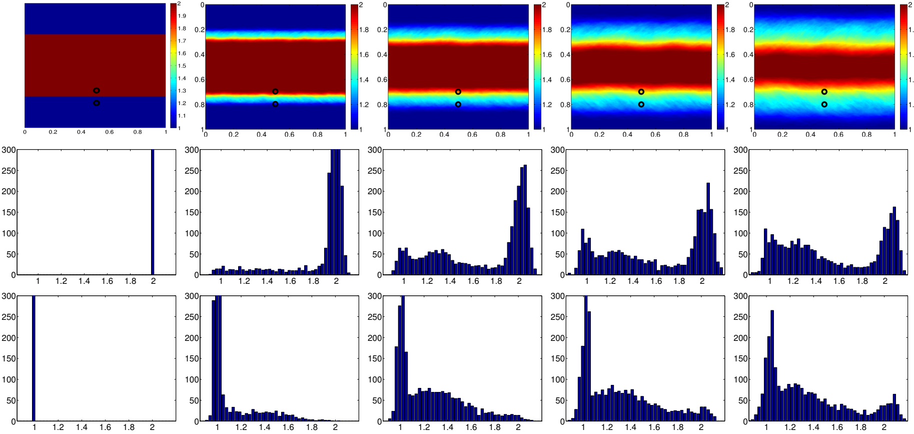


Figure : Cauchy rates for variance of density at $t = 2$ for $\mathrm{nx} = 128, 256, 512, 1024$.

Numerical example: Time evolution

The approximate PDF for mass density ρ at two points x on a grid of 1024^2 mesh points.

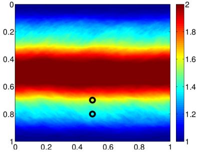


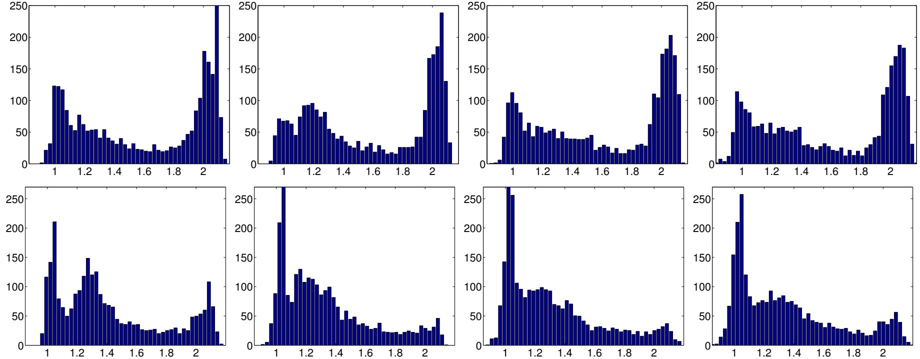
(a) $t = 0$ (b) $t = 0.5$ (c) $t = 1$ (d) $t = 1.5$ (e) $t = 2$

Figure : Top row: mean ρ. Middle and bottom rows: PDF at $x = (0.5, 0.7)$ and $x = (0.5, 0.8)$.

Numerical example: Convergence at isolated points

The approximate PDF for density ρ at two points x on a series of meshes.





(a) nx = 128 (b) nx = 256 (c) nx = 512 (d) nx = 1024

Figure : Top row: $x = (0.5, 0.7)$. Bottom row: $x = (0.5, 0.8)$

Numerical example: Convergence in $L^1(W_1)$

We compute the "$L^1(W_1)$" distance

$$d_1\left(\nu^{\Delta x,t}, \nu^{\Delta x/2,t}\right) := \int_{\mathbb{R}^2} W_1\left(\nu_x^{\Delta x,t}, \nu_x^{\Delta x/2,t}\right)\, dx.$$

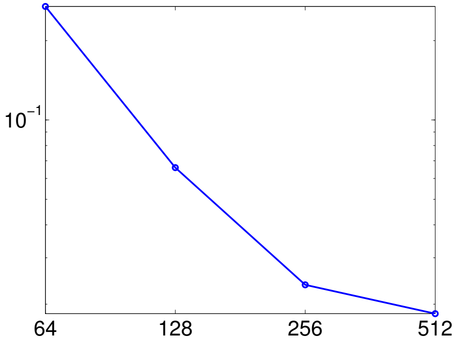


Figure : $L^1(W_1)$ error for $\Delta x = 1/64, \dots, 1/512$.

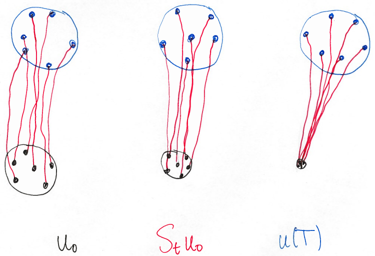
(As $\Delta x \to 0$, the Monte Carlo error dominates, and the convergence flattens out.)

Section 5

Statistical and measure-valued solutions

Statistical vs. MV solutions

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0, \qquad x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$



- A **measure-valued solution** of (1) allows uncertainty in the solution *at every* (x,t).
- A **statistical solution** (Foias, 1972) μ^t is a probability measure *over solution space*.

MV and statistical solutions

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) &= 0, \qquad x \in \mathbb{R}^d,\ t > 0 \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

- A **statistical solution** μ^t of (1) is a probability measure *over solution space*:

$$\mu^t \in Prob(\mathcal{F}), \qquad \mathcal{F} := L^p(\mathbb{R}^d, \mathbb{R}^N).$$

satisfying (1) in a weak sense (Foias, 1972).

MV solutions and correlations

- An MV solution gives statistics at fixed (x, t), *but says nothing about the relation between the solution at* (x_1, t) *and* (x_2, t) (i.e., about correlations between different points).
- Additional information is given by **correlation Young measures**

$$\nu^t_{x_1,\dots,x_k} \in Prob(\mathbb{R}^{kN}),$$

describing the correlation of the solution at points $x_1, \dots, x_k \in \mathbb{R}^d$.

- An **infinite Young measure** is a family of correlation Young measures

$$\nu = (\nu_{x_1}, \nu_{x_1,x_2}, \dots).$$

Theorem (USF, S. Lanthaler 2014)

There is a one-to-one correspondence between infinite Young measures ν and probability measures $\mu \in Prob(\mathcal{F})$.

Section 6

Summary and outlook

Open questions

Ongoing work:

- When can we say that a Cauchy problem is MV stable?
- What "entropy-type conditions" (e.g., maximal entropy decay) ensures stability uniqueness?
- Computationally efficient alternatives to the Monte Carlo method (multi-level Monte Carlo, stochastic collocation, generalized polynomial chaos).

Regarding statistical solutions,

- When is there a unique *statistical* solution (or equivalently, infinite Young measure solution)?

Thank you for your attention!

References

U. S. Fjordholm and S. Mishra.
Computing measure-valued solutions of conservation laws with a strictly convex entropy.
In preparation, 2014.

U. S. Fjordholm, R. Käppeli, S. Mishra, and E. Tadmor.
Numerical approximation of measure-valued solutions of hyperbolic systems of conservation laws.
Submitted for publication, 2014.