A Sufficient Condition for the Kolmogorov 4/5 Law for Stationary Martingale Solutions to the 3D Navier-Stokes Equations

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Stochastic Navier-Stokes equations

$$\partial_t u + (u \cdot \nabla) u + \nabla p - \nu \Delta u = \partial_t W, \qquad \text{(NSE)}$$
$$\operatorname{div} u = 0, \qquad \text{on } \mathbb{T}^3$$

- ν inverse of Reynolds number, $\nu = \frac{1}{Re}$
- W white-in-time, colored-in-space Gaussian process

$$\partial_t W(t,x) = \sum_{k=1}^{\infty} \sigma_k e_k(x) d\beta_k(t),$$

  - $\{\beta_k(t)\}_k$ independent one-dimensional Brownian motions
  - $\{e_k\}_k$ orthonormal eigenfunctions of the Stokes operator on $\mathbb{T}^3$
  - $\{\sigma_k\}_k$ fixed constants satisfying coloring condition

$$\varepsilon = \frac{1}{2} \sum_{k=1}^{\infty} |\sigma_k|^2 < \infty$$

Symmetries of Navier-Stokes equations (on $\mathbb{R}^3$)

If $u(t,x)$ is a solution of (NSE), then

- Space translations: $u(t, x+z)$ for $z \in \mathbb{R}^3$ is a solution as well
- Time translations: $u(t+\tau, x)$, $\tau \in \mathbb{R}$ solution
- Galilean transformations: $U_0 + u(t, x - U_0 t)$, $U_0 \in \mathbb{R}^3$ solution
- Parity: $-u(t, -x)$ solution
- Rotations: $Ru(t, R^\top x)$, $R \in SO(\mathbb{R}^3)$ solution
- Scaling: $\lambda^s u(\lambda^{1-s} t, \lambda x)$ solution for $\lambda \in \mathbb{R}^+$, $s = -1$.

But ...

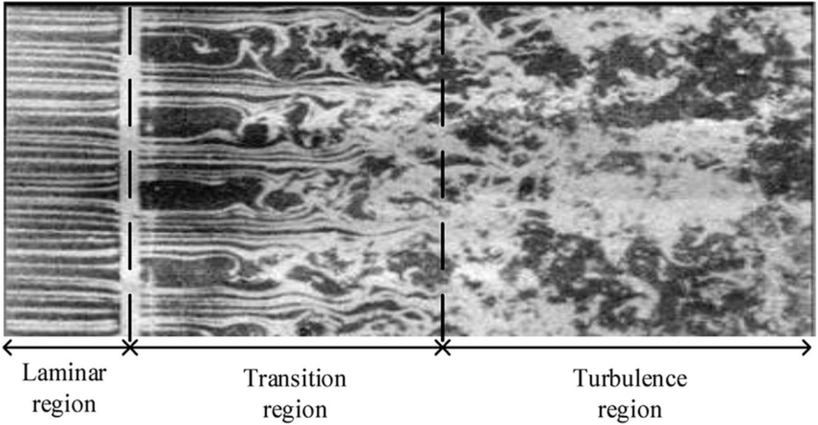

Figure: Homogeneous turbulence behind a grid (picture from Frisch (1995))

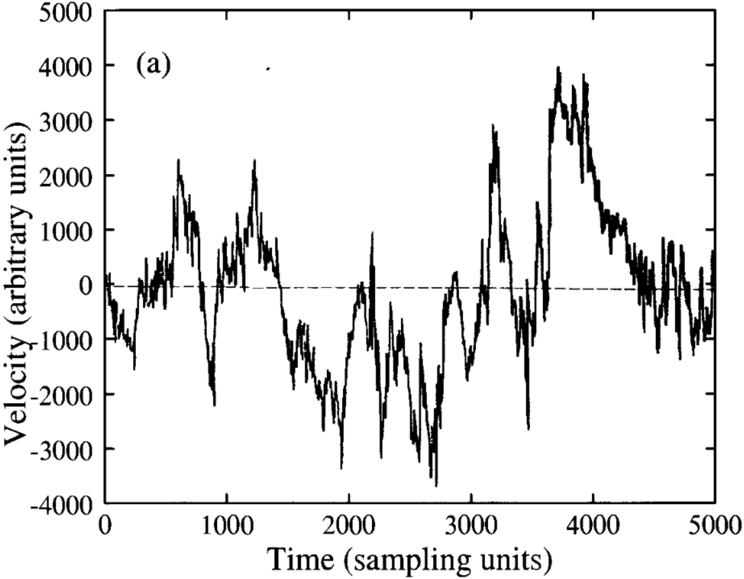

Figure: One second of a signal recorded by a hot-wire (sampled at 5kHz) in wind tunnel (picture from Frisch (1995))

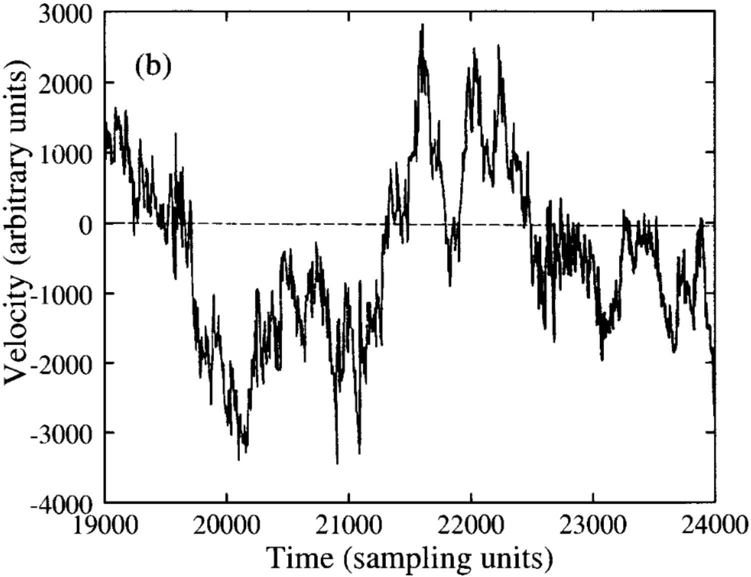

Figure: One second of a signal recorded by a hot-wire (sampled at 5kHz) in wind tunnel four seconds later (picture from Frisch (1995))

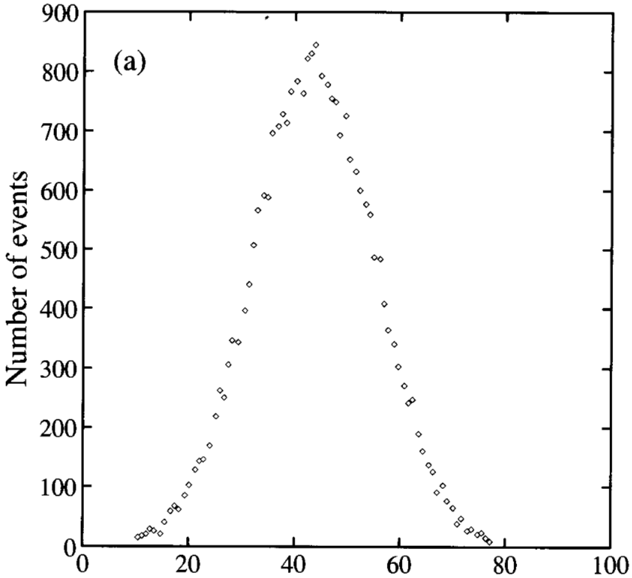

Figure: Histogram for previous figure sampled 5000 times over time-span of 150s (picture from Frisch (1995))

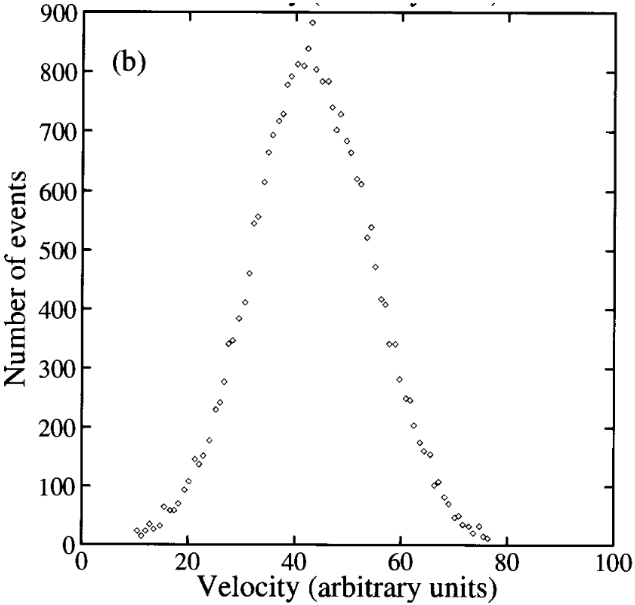

Figure: Same histogram a few minutes later (picture from Frisch (1995))

Kolmogorov 1941 (K41)

- Probabilistic description of turbulence
- At high Reynolds numbers the symmetries of (NSE) are restored in a statistical sense, flow is statistically stationary and locally homogeneous and isotropic:
  - Statistically stationary: $u(t+\tau,\cdot) \stackrel{\text{law}}{=} u(t,\cdot)$, $\tau > 0$
  - Velocity increment $\delta_h u(x) = u(x+h) - u(x)$, $h \in \mathbb{R}^3$
  - Local homogeneity: $\delta_h u(x+y) \stackrel{\text{law}}{=} \delta_h u(x)$
  - Local isotropy: $R\delta_{R^\top h} u(x) \stackrel{\text{law}}{=} \delta_h u(x)$, $R \in SO(3)$
- Kolmogorov formulates hypotheses that should hold for such flows based on experimental observations and derives additional predicitions about these

K41

- **First hypothesis of similarity**: For locally homogeneous and isotropic turbulence, the laws of $\delta_h u$ are uniquely determined by the kinematic viscosity ν and the mean energy dissipation rate $\varepsilon \sim \nu \|\nabla u\|_{L^2_x}^2$.

- **Second hypothesis of similarity**: Let $\lambda_K = \nu^{3/4}\varepsilon^{-1/4}$ the *Kolmogorov scale*. If $|h|$ is large in comparison to λ_K, then the laws of the velocity increments $\delta_h u$ are uniquely defined by the mean energy dissipation rate ε and do not depend on ν.

K41

Intermittency

- 2nd hypothesis is debated, it would lead to scale invariance: There exists $s \in \mathbb{R}$ such that $\delta_{\lambda h} u(x) \overset{\text{law}}{=} \lambda^s \delta_h u(x)$ for all $\lambda \in \mathbb{R}_+$, $|h| \gg \lambda_K$

- For example this would imply,

$$\mathbf{E}\,(\delta_h u)^2 = C\varepsilon^{2/3}|h|^{2/3},$$

  where C is a universal constant
  (Since units $(\delta_h u)^2 \sim [L]^2/[T]^2$, $\varepsilon \sim [L]^2/[T]^3$ and by scale invariance $(\delta_h u)^2 \sim \ell^{2s} \Rightarrow$ only possible exponent is $s = 1/3$)

- Physical experiments indicate scale invariance might not be true/C is not universal $\Rightarrow$ Intermittency corrections

- Theory of intermittency is largely based on empirical considerations, no direct derivation from fluid dynamics equations/mathematical theory, relates to higher regularity/smoothness of solutions

Intermittency: Experimental observations

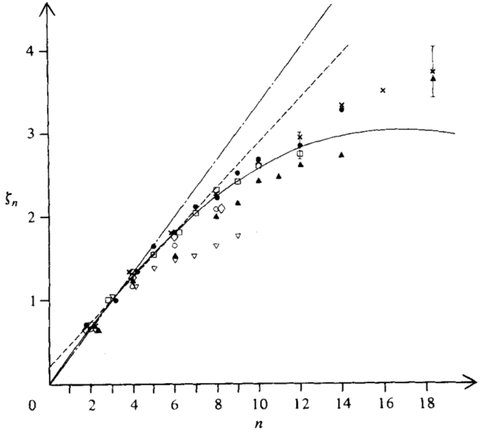

FIGURE 14. Variation of exponent ζ_n as a function of the order n. ●, $R_\lambda = 515$ (duct); □, 536; ×, 852. Symbols ○, ▲, ▽, ◇ are respectively the exponents given by Mestayer (1980); Vasilenko *et al.* (1975); Van Atta & Park (1972); and Antonia *et al.* (1982*a*). The solid curve is LN with $\mu = 0.2$, the dotted curve the β-model and the chain-dotted line Kolmogorov's (1941) model.

Figure: From Anselmet, Gagne, Hopfinger, Antonio (1983)

4/5 law

- For $|h| \in [\ell_D, \ell_I]$, (dissipation scale $\ell_D \approx \varepsilon^{-1/4}\nu^{3/4}$, integral scale ℓ_I),
$$\mathbf{E}\left(\delta_h u \cdot \frac{h}{|h|}\right)^3 \sim -\frac{4}{5}\varepsilon|h|$$
where $\delta_h u(x) = u(x+h) - u(x)$.
- Balance of 'dissipation' due to nonlinear effects with energy input
- 4/5 law should be independent of intermittency corrections/higher order regularity of solutions!
- Second hypothesis of similarity should not be needed for derivation
- 4/5 law should be deducible directly from fluid mechanics equations without further assumptions!

Statistically stationary solutions

- Kolmogorov derived his laws of turbulence for this case
- Statistically stationary means: $u(\cdot + \tau)$ has the same distribution as $u(\cdot)$, i.e. $u(\cdot + \tau) \overset{\text{law}}{=} u(\cdot)$ for every $\tau > 0$

$$\mathbf{E}F(u(\cdot + \tau)) = \mathbf{E}F(u(\cdot)), \quad F \text{ continuous function}$$

- Energy balance:

$$\nu \mathbf{E} \|\nabla u\|_{L^2_x}^2 = \varepsilon,$$

where $\varepsilon = \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k^2 < \infty$

Energy balance for statistically stationary solutions

Itô's formula: Stochastic process $u = (u^{(1)}, u^{(2)}, \dots)$,

$$du^{(i)} = \mu^{(i)} dt + \sum_k v_k^{(i)} d\beta_k$$

$f = f(t, u)$ satisfies

$$df = \frac{\partial f}{\partial t} dt + \sum_i \frac{\partial f}{\partial u^{(i)}} du^{(i)} + \frac{1}{2} \sum_{i,j} \frac{\partial^2 f}{\partial u^{(i)} \partial u^{(j)}} d[u^{(i)}, u^{(j)}](t)$$

$$= \left(\frac{\partial f}{\partial t} + \frac{1}{2} \sum_{i,j,k} \frac{\partial^2 f}{\partial u^{(i)} \partial u^{(j)}} v_k^{(i)} v_k^{(j)} + \frac{\partial f}{\partial u^{(i)}} \mu^{(i)} \right) dt + \sum_{i,k} \frac{\partial f}{\partial u^{(i)}} v_k^{(i)} d\beta_k,$$

since

$$d[\beta_\ell, \beta_m](t) = \delta_{\ell m} dt,$$

$$d[u^{(i)}, u^{(j)}](t) = \frac{1}{2} \sum_k v_k^{(i)} v_k^{(j)} dt,$$

Energy balance for statistically stationary solutions

For $f(t,u) = \frac{1}{2}|u|^2$,

$$du = (-(u\cdot\nabla)u + \nu\Delta u - \nabla p)dt + \sum_k \sigma_k e_k d\beta_k :$$

$$\frac{1}{2}d(|u|^2) = \left(\frac{1}{2}\sum_k \sigma_k^2|e_k|^2 - \frac{1}{2}u\cdot\nabla|u|^2 + \nu\Delta u\cdot u - \nabla p\cdot u\right)dt + \sum_k \sigma_k e_k\cdot u\, d\beta_k$$

Integrate over torus $\mathbb{T}^3$:

$$\frac{1}{2}d\left(\int|u|^2dx\right) = \left(\frac{1}{2}\sum_k \sigma_k^2 - \nu\int|\nabla u|^2dx\right)dt + \sum_k \sigma_k\int e_k\cdot u\,dx\,d\beta_k$$

Integrate in time, take expectation:

$$\frac{1}{2}\mathbf{E}\int|u|^2(t)dx - \frac{1}{2}\mathbf{E}\int|u_0|^2dx = \varepsilon t - \nu\mathbf{E}\int_0^t\int|\nabla u|^2dxds$$

Assume u is stationary

$$\varepsilon = \nu\mathbf{E}\int|\nabla u|^2dx = \nu\mathbf{E}\,\|\nabla u\|_{L_x^2}^2$$

Martingale solutions

- ‘Stochastic equivalent’ of Leray-Hopf weak solutions
- Stochastic basis $(\Omega, (\mathcal{F}_t)_{t\in[0,T]}, \mathcal{P}, \{\beta_k\}_k)$, progressively measurable stochastic process $u : [0, T] \times \Omega \to L^2_{\text{div}}$
- sample paths of u in $C_t(H^\alpha_x) \cap L^\infty_t L^2_{\text{div}} \cap L^2_t H^1_{\text{div}}$, $\alpha < 0$
- $\mathbf{P}$-a.s. $\omega \in \Omega$ $u(\omega)$ is a weak solution of the stochastic Navier-Stokes equations

$$du + (u \cdot \nabla)udt + \nabla pdt = \nu \Delta udt + dW_t,$$

- Energy inequality: a.e. $t_2 > t_1 \geq 0$

$$\frac{1}{2}\mathbf{E}\,\|u(t_2)\|^2_{L^2_x} - \frac{1}{2}\mathbf{E}\,\|u(t_1)\|^2_{L^2_x} + \nu\mathbf{E}\int_{t_1}^{t_2} \|\nabla u(s)\|^2_{L^2_x}\,ds \leq \varepsilon(t_2 - t_1)$$

  energy input $\varepsilon = \frac{1}{2}\sum_k \sigma_k^2$
- Such solutions exist (Bensoussan, Temam (1973))

Martingale solutions

- Stationary martingale solutions: Same as before but paths $u(\cdot + \tau)$ have same law as $u(\cdot)$ for each $\tau \geq 0$.
- Such solutions exist (Flandoli, Gatarek (1995))
- Stationary energy inequality

$$\nu \mathbf{E} \|\nabla u\|_{L^2_x}^2 \leq \varepsilon$$

- Homogeneous forcing $\Rightarrow$ homogeneous stationary solution exists

Weak anomalous dissipation

- Stochastic heat equation: $\partial_t v - \nu \Delta v = \partial_t W$
  Same energy balance for stationary solutions:
  $$\varepsilon = \nu \mathbf{E} \|\nabla v\|_{L_x^2}^2$$
  But no energy cascade!
- Energy balance doens't say anything about nonlinear effects nor contains enough information to derive 4/5 law
- Poincaré inequality (domain bounded)/Fourier transform:
  $$\nu \|u^\nu\|_{L_x^2}^2 \leq C\nu \|\nabla u^\nu\|_{L_x^2}^2 \lesssim \varepsilon$$
- *Weak anomalous dissipation* if
  $$\lim_{\nu \to 0} \nu \mathbf{E} \|u^\nu\|_{L_x^2}^2 = 0. \qquad \text{(WAD)}$$
- Energy in low frequencies gets transferred to high frequencies
- (WAD) holds for passive scalar $\partial_t f + v \cdot \nabla f - \nu \Delta f = \partial_t W$, where v is weakly mixing (Bedrossian, Coti Zelati, Glatt-Holtz (2016)) or solution of stochastic 2D (NSE) or hyperviscous 3D (NSE) (Bedrossian, Blumenthal, Punshon-Smith (2018))

What we prove

- Velocity increment: $\delta_h u(x) := u(x+h) - u(x)$, $h \in \mathbb{R}^3$
- Averaged 3rd order longitudinal structure function:

$$S_{||}(\ell) = \frac{1}{4\pi}\mathbf{E}\int_{\mathbb{S}^2}\int_{\mathbb{T}^3}(\delta_{\ell\hat{n}}u\cdot\hat{n})^3 dx dS(\hat{n})$$

Theorem (4/5 law)

Let $\{u^\nu\}_{\nu>0}$ *be a sequence of stationary martingale solutions to* (NSE), *which satisfy* (WAD), *and let* $S_{||}$ *be the third order structure function. Then, there exists* $\ell_D = \ell_D(\nu)$ *with* $\lim_{\nu\to 0}\ell_D = 0$ *such that*

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_D,\ell_I]}\left|\frac{S_{||}(\ell)}{\ell} + \frac{4}{5}\varepsilon\right| = 0. \quad (1)$$

Remarks

- Only assumption is *weak anomalous dissipation*: $\nu \mathbf{E} \|u^\nu\|_{L^2_x} \xrightarrow{\nu \to 0} 0$
- No smoothness/regularity of solutions required
  - Rigorous derivation under assumption of regularity/integrability that weak martingale solutions satisfy ($\approx$ stochastic equivalent of Leray-Hopf solutions)
- No energy balance needed, inequality $\nu \mathbf{E} \|\nabla u\|_2^2 \leq \varepsilon$ is enough
- Assuming symmetries (homogeneity, isotropy), same results holds without averaging.
- Proof ideas from Frisch (1995), Monin, Yaglom (1965), Eyink (2003), Nie, Tanveer (1995), but mathematically rigorous and with weaker assumptions (only (WAD))
- Lagrangian version of 4/5 law: Drivas (2018)

4/3 law

- Velocity increment: $\delta_h u(x) := u(x+h) - u(x),\ h \in \mathbb{R}^3$
- Averaged 3rd order structure function:

$$S_0(\ell) = \frac{1}{4\pi}\mathbf{E}\int_{\mathbb{S}^2}\int_{\mathbb{T}^3} |\delta_{\ell\hat{n}} u|^2 \delta_{\ell\hat{n}} u \cdot \hat{n}\, dx dS(\hat{n})$$

Theorem (4/3 law)

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$$\lim_{\ell_I \to 0} \limsup_{\nu \to 0} \sup_{\ell \in [\ell_D, \ell_I]} \left| \frac{S_0(\ell)}{\ell} + \frac{4}{3}\varepsilon \right| = 0. \tag{2}$$

What we prove

- Velocity increment: $\delta_h u(x) := u(x+h) - u(x)$, $h \in \mathbb{R}^3$
- Averaged 3rd order longitudinal structure function:

$$S_{||}(\ell) = \frac{1}{4\pi} \mathbf{E} \int_{\mathbb{S}^2} \int_{\mathbb{T}^3} (\delta_{\ell \hat{n}} u \cdot \hat{n})^3 dx dS(\hat{n})$$

Theorem (4/5 law)

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Kármán-Howarth-Monin relation

Define

$$\Gamma(t,h) = \mathbf{E}\int_{\mathbb{T}^3} u(t,x)\otimes u(t,x+h)dx,$$

$$D^k(t,h) = \mathbf{E}\int_{\mathbb{T}^3} (\delta_h u\otimes \delta_h u)\delta_h u^{(k)}\, dx,$$

$$a(h) = \frac{1}{2}\sum_k \sigma_k^2 \int_{\mathbb{T}^3} e_k(x)\otimes e_k(x+h)dx,$$

Let $\eta(h)$ smooth, isotropic, compactly supported test function of the form

$$\eta(h) = \phi(|h|)I + \varphi(|h|)\hat{h}\otimes\hat{h}, \qquad \hat{h} = \frac{h}{|h|},$$

$\phi(\ell)$ and $\varphi(\ell)$ smooth and compactly supported on $(0,\infty)$. Then,

$$\int_{\mathbb{R}^3}\eta(h) : \Gamma(T,h)\, dh - \int_{\mathbb{R}^3}\eta(h) : \Gamma(0,h)\, dh = 2T\int_{\mathbb{R}^3}\eta(h) : a(h)dh$$
$$-\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\partial_k\eta(h) : D^k(t,h)\, dhdt + 2\nu\int_0^T\int_{\mathbb{R}^3}\Delta\eta(h) : \Gamma(t,h)\, dhdt.$$

Kármán-Howarth-Monin relation

If u is stationary,

$$\frac{1}{2}\int_0^T\int_{\mathbb{R}^3}\partial_k\eta(h):D^k(t,h)\,dhdt$$
$$=2T\int_{\mathbb{R}^3}\eta(h):a(h)dh+2\nu\int_0^T\int_{\mathbb{R}^3}\Delta\eta(h):\Gamma(t,h)\,dhdt,$$

where

$$\Gamma(t,h)=\mathbf{E}\int_{\mathbb{T}^3}u(t,x)\otimes u(t,x+h)dx,$$
$$D^k(h)=\mathbf{E}\int_{\mathbb{T}^3}(\delta_h u\otimes\delta_h u)\delta_h u^{(k)}\,dx,$$
$$a(h)=\frac{1}{2}\sum_k\sigma_k^2\int_{\mathbb{T}^3}e_k(x)\otimes e_k(x+h)dx,$$
$$\eta(h)=\phi(|h|)I+\varphi(|h|)\hat{h}\otimes\hat{h},\qquad \hat{h}=\frac{h}{|h|}.$$

Proof idea

- Mollify equations for $u(t,x)$ and $u(t,x+h)$

$$f_\kappa = (f)_\kappa := \gamma_\kappa \star f.$$

  For each $x \in \mathbb{T}^3$

$$du_\kappa(t,x) + \mathrm{div}_x (u \otimes u)_\kappa(t,x)\mathrm{d}t + \nabla p_\kappa(t,x)\mathrm{d}t = \nu \Delta u_\kappa(t,x)\mathrm{d}t + dW_\kappa(t,x),$$

- Compute evolution of $u_\kappa(t,x) \otimes u_\kappa(t,x+h)$ using stochastic product rule
- Integrate against isotropic test function $\eta(h)$, estimate all terms and pass $\kappa \to 0$.
- Pressure term vanishes because of isotropic test function

4/3 law

- Choose $\eta(h) = \phi(|h|)I$ in KHM-relation and average Γ and a over sphere

$$\bar{\Gamma}(\ell) = \frac{1}{4\pi}\int_{\mathbb{S}^2} I : \Gamma(\ell\hat{n})dS(\hat{n}), \quad \bar{a}(\ell) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \mathrm{tr}(a(\ell\hat{n}))dS(\hat{n})$$

- Flux term

$$\sum_k \int_{\mathbb{T}^3} (D^k(h) : I)\partial_k\phi(h)dh = \int_{\mathbb{T}^3} \mathbf{E}\int_{\mathbb{T}^3} |\delta_h u|^2 \delta_h u \cdot \nabla\phi(h)\, dx\, dh,$$

- KHM-relation becomes

$$\frac{1}{4}\int_{\mathbb{R}^+} S_0(\ell)\ell^2\phi'(\ell)d\ell = \int_{\mathbb{R}^+} \ell^2\phi(\ell)\left(\nu\bar{\Gamma}'' + \nu\frac{2}{\ell}\bar{\Gamma}' + \bar{a}(\ell)\right)d\ell$$

- That's an ODE (in the sense of distributions):

$$\partial_\ell\left(\ell^3\frac{S_0}{\ell}\right) = \ell^2\left(S_0' + \frac{2}{\ell}S_0\right) = -4\ell^2\left(\nu\bar{\Gamma}'' + \nu\frac{2}{\ell}\bar{\Gamma}' + \bar{a}\right).$$

- Integrate ODE

$$\frac{S_0(\ell)}{\ell} = -\frac{1}{\ell^3}\int_0^\ell \tau^2 \left(4\nu\bar{\Gamma}''(\tau) + 4\nu\frac{2}{\tau}\bar{\Gamma}'(\tau) + 4\bar{a}(\tau)\right) d\tau$$

- Estimate terms

$$\frac{4}{\ell^3}\int_0^\ell \tau^2 \bar{a}(\ell) d\tau = \frac{4}{3}\bar{a}(0) + \frac{4}{\ell^3}\int_0^\ell \tau^2 \left(\bar{a}(\tau) - \bar{a}(0)\right) d\tau$$

Continuity of a $\Rightarrow$ 2nd term is $o_{\ell\to 0}(1)$

- Integrate by parts once more

$$\frac{1}{\ell^3}\int_0^\ell \left[\tau^2\bar{\Gamma}''(\tau) + 2\tau\bar{\Gamma}'(\tau)\right] d\tau = \frac{\bar{\Gamma}'(\ell)}{\ell}$$

- We obtain

$$\frac{S_0(\ell)}{\ell} = -\frac{4\nu\bar{\Gamma}'(\ell)}{\ell} - \frac{4}{3}\varepsilon + o_{\ell\to 0}(1)$$

Remains to estimate

$$\left|\frac{\nu}{\ell}\bar{\Gamma}'(\ell)\right| \lesssim \frac{\nu}{\ell}\left(\mathbf{E}\,\|\nabla u\|_{L^2_x}^2\right)^{1/2}\left(\mathbf{E}\,\|u\|_{L^2_x}^2\right)^{1/2} \lesssim \frac{(\varepsilon\nu)^{1/2}}{\ell}\left(\mathbf{E}\,\|u\|_{L^2_x}^2\right)^{1/2}$$

Weak anomalous dissipation condition $\Rightarrow$ can choose $\ell_D(\nu) \to 0$ such that $\left(\nu\mathbf{E}\,\|u\|_{L^2_x}^2\right)^{1/2} = o(\ell_D)$:

$$\lim_{\nu\to 0}\sup_{\ell\in(\ell_D,\ell_I)}\left|\frac{\nu}{\ell}\bar{\Gamma}'(\ell)\right| = 0$$

Hence

$$\lim_{\ell_I\to 0}\limsup_{\nu\to 0}\sup_{\ell\in[\ell_D,\ell_I]}\left|\frac{S_0(\ell)}{\ell}+\frac{4}{3}\varepsilon\right| = 0.$$

Remarks

- ℓ_I depends only of continuity of a.
- ℓ_D needs to satisfy $\ell_D \gg \varepsilon^{1/2}\nu^{1/2}(\mathbf{E}\,\|u\|_2^2)^{1/2}$.
- Taylor micro scale: $\lambda_T(\nu) \approx \varepsilon^{-1/2}\nu^{1/2}(\mathbf{E}\,\|u\|^2)^{1/2}$
- Kolmogorov dissipation scale: $\lambda_K(\nu) \approx \varepsilon^{-1/4}\nu^{3/4}$.
- Different test function η and a few more terms: Obtain $4/5$ law under same assumptions

Necessary conditions I

Theorem

$\{u\}_{\nu>0}$ *sequence of stationary martingale solutions to the Navier-Stokes equations satisfying*

- *Energy balance: For ν sufficiently small*

$$\nu \mathbf{E} \left\| \nabla u \right\|_{L^2_x}^2 = \varepsilon,$$

- *Regularity: There exists $C > 0$ and $s > 1$ such that*

$$\sup_{\nu \in (0,1)} \nu \mathbf{E} \left\| |\nabla|^s u \right\|_{L^2_x}^2 \leq C,$$

then

$$\lim_{\ell \to 0} \sup_{\nu \in (0,1)} \left(\left| \frac{S_{||}(\ell)}{\ell} \right| + \left| \frac{S_0(\ell)}{\ell} \right| \right) = 0$$

Necessary conditions II

Corollary

Let $\{u\}_{\nu>0}$ *be a sequence of stationary martingale solutions to* (NSE) *such that for some* $c \neq 0$ *and some* $\ell_D(\nu)$ *with* $\lim_{\nu\to 0} \ell_D = 0$ *there holds*

$$\lim_{\ell_I\to 0} \lim_{\nu\to 0} \sup_{\ell\in[\ell_D,\ell_I]} \left| \frac{S_0(\ell)}{\ell} + c \right| = 0.$$

Then, for all $s > 5/4$,

$$\liminf_{\nu\to 0} \mathbf{E}\nu \left\| |\nabla|^s u \right\|_{L^2_x}^2 = \infty.$$

If $\inf_\nu \mathbf{E}\nu \|\nabla u\|_{L^2_x}^2 > 0$, then (WAD) implies that

$$\liminf_{\nu\to 0} \mathbf{E}\nu \left\| |\nabla|^s u \right\|_{L^2_x}^2 = \infty, \qquad \forall s > 1.$$