Anomalous energy transport in FPU-β chain

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Joint work with Antoine Mellet (University of Maryland)

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EPSRC

Engineering and Physical Sciences Research Council

CCA

Energy transport in a solid

Heat equation

$$\partial_t \rho(t,x) = D\Delta_x \rho(t,x) \begin{cases} \partial_t \rho(t,x) & = & -\nabla_x \cdot \vec{\jmath}(t,x) & \text{(mass cons.)} \\ \vec{\jmath}(t,x) & = & -D\nabla_x \rho(t,x) & \text{(Fourier law)} \end{cases}$$

Energy transport in a solid

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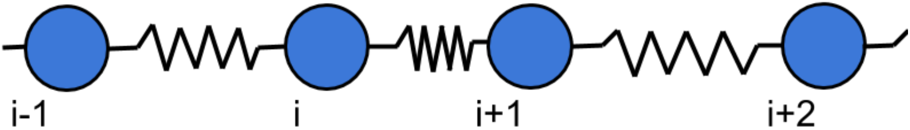
MICROSCOPIC MODEL: FPU lattice/chain (Fermi Pasta Ulam)



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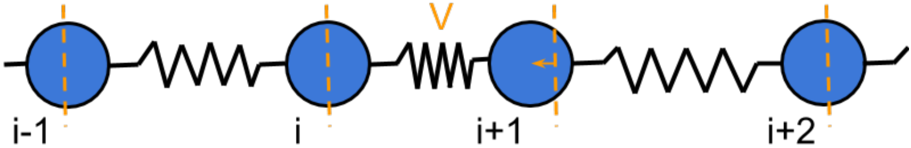


$$H(q,p) = \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + V_h(q) + \sqrt{\lambda}V(q)$$

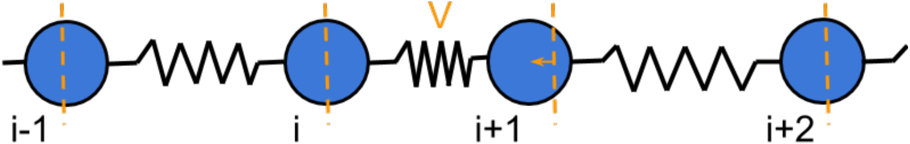
*Bonetto et al. [2000]

Anomalous energy transport in a solid (1D-2D)

MACROSCOPIC MODEL:
Fractional heat equation

$$\partial_t \rho(t,x) = -D(-\Delta_x)^{\gamma/2}\rho(t,x) \quad \begin{cases} \partial_t \rho(t,x) = -\nabla_x \cdot \vec{\jmath}(t,x) & \text{(mass cons.)} \\ \vec{\jmath}(t,x) = \text{Anomalous} & \text{Fourier Law} \end{cases}$$

MICROSCOPIC MODEL: FPU-β lattice/chain (Fermi Pasta Ulam)



$$H(q,p) = \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{2}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 + \beta\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^4$$

$$(-\Delta_x)^{\gamma/2}\rho = \mathcal{F}^{-1}\left(|k|^\gamma \mathcal{F}(\rho)(k)\right)$$

Previous results

MICRO
FPU chain/lattice

$$H(q,p) = \frac{1}{2}p^2 + V(q)$$

Harmonic potential
No relaxation to eq.

Cubic potential
(FPU-alpha chain)

Quartic potential
(FPU-beta chain)

On site potential:
- classic diffusion
(Aoki, Lukkarinen, Spohn)

Next neighbour pot.:
- 1d, 2d anomalous dif.
- 3d or higher, classic dif.
(Lepri, Livi, Politi)

Harmonic potential + noise conserving momentum and energy
(Olla, Basile, et.al)

MACRO
(Anomalous) diffusion

Nonlinear heat equation

$$\partial_t T = \nabla_x \cdot (\kappa(T)\nabla_x T)$$

Fractional heat equation

$$\partial_t T = -\kappa(-\Delta)^\gamma T$$

Heat equation

$$\partial_t T = \kappa \Delta T$$

Previous results

MICRO
FPU chain/lattice

$$H(q,p) = \frac{1}{2}p^2 + V(q)$$

Spohn (formal)

Heat – lattice vibrations – phonons

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No relaxation to eq.

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KINETIC

$$\partial_t W + \omega'(k)\partial_x W = C(W)$$

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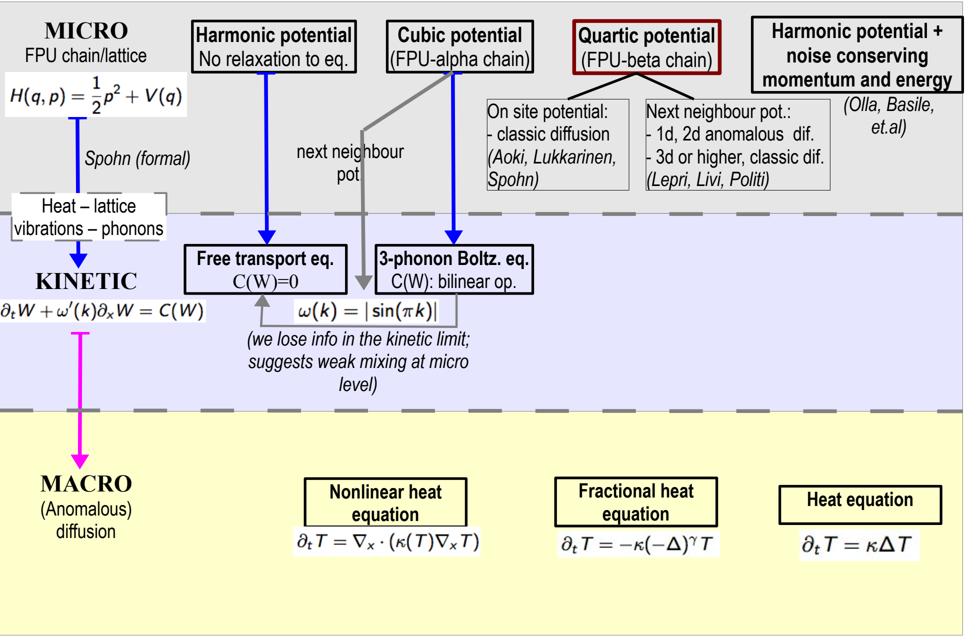

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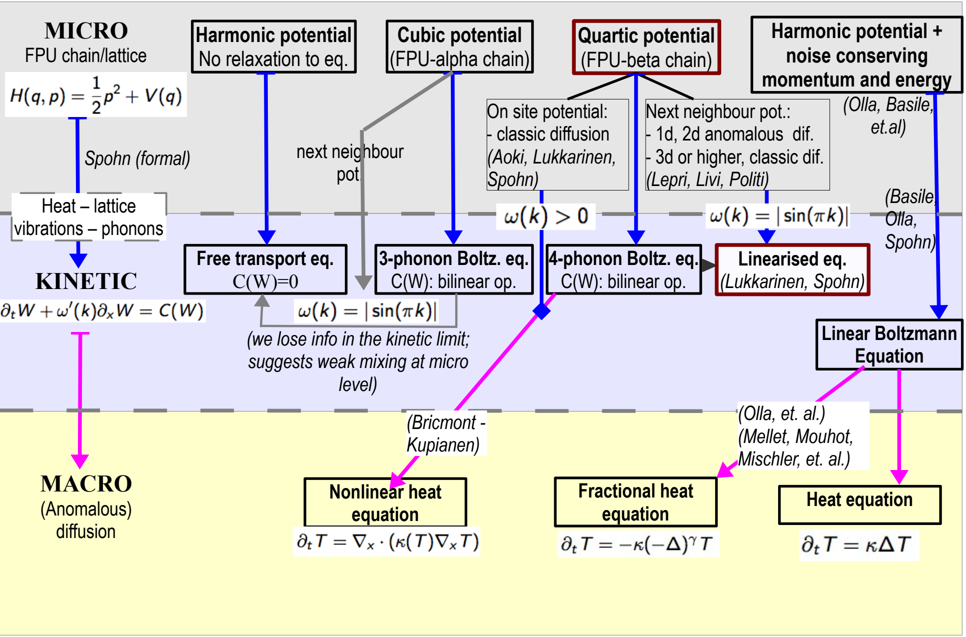

The kinetic model: the Boltzmann-phonon equation

The formal kinetic limit (Spohn, *'The phonon Boltzmann equation, properties and link to weakly anharmonic lattice dynamics'*) - Peierls.

$$\partial_t W(t,x,k) + \omega'(k)\partial_x W(t,x,k) = C(W) \quad (t,x,k) \in (0,\infty) \times \mathbb{R} \times \mathbb{T}$$

k: wavenumber, $\omega(k)$: dispersion relation.

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Four phonon-collision operator

$$C(W) = \qquad \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}}$$

$$dk_1\, dk_2\, dk_3$$

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$$\times (W_1 W_2 W_3 + W(\sigma_1 W_2 W_3 + W_1 \sigma_2 W_3 + W_1 W_2 \sigma_3))\, dk_1\, dk_2\, dk_3$$

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$$C(W) = \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} \int_{\mathbb{T}}\int_{\mathbb{T}}\int_{\mathbb{T}} F(k,k_1,k_2,k_3)^2$$

$$\times \left(W_1 W_2 W_3 + W(\sigma_1 W_2 W_3 + W_1 \sigma_2 W_3 + W_1 W_2 \sigma_3)\right) dk_1\, dk_2\, dk_3$$

with

$$F := F(k,k_1,k_2,k_3)^2 = \prod_{i=0}^{3} \frac{2\sin^2(\pi k_i)}{\omega(k_i)}.$$

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$$\begin{aligned} C(W) &= \int\int\int F^2 \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ &\quad [W_1 W_2 W_3 + W W_2 W_3 - W W_1 W_3 - W W_1 W_2]\, dk_1\, dk_2\, dk_3. \end{aligned}$$

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Equilibrium: $\dfrac{1}{\beta\omega+\alpha}$; Cons.: $\int_{\mathbb{T}} W\, dk = cte$, $\int_{\mathbb{T}} \omega(k) W\, dk = cte$

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Linearised 4-phonon Boltzmann equation

Around $W^\varepsilon = \frac{1}{\beta_0 \omega(k)}(1 + \varepsilon f^\varepsilon)$, $L(f) = W_0^{-1} DC(W_0)(W_0 f)$:

$$\partial_t f(t,x,k) + \omega'(k)\partial_x f(t,x,k) = L(f) \qquad (t,x,k) \in (0,\infty) \times \mathbb{R} \times [0,1]$$

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'Regularise' L (Lukkarinen and Spohn [2008])

$L(f) = K(f) - Vf$ Ker $L = \text{span}\,\{1, \omega^{-1}\}$; $\int_{\mathbb{T}} f\, dk = cte$, $\int_{\mathbb{T}} \omega^{-1} f\, dk = cte$

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Proposition (Lukkarinen and Spohn [2008])

$$c_1|\sin(\pi k)|^{5/3} \le V(k) \le c_2|\sin(\pi k)|^{5/3} \qquad c_1, c_2 > 0$$

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where

$$\kappa = \int L^{-1}(\omega'(k)) \omega'(k) dk \approx -\int_0^1 \frac{(\omega'(k))^2}{V} dk = \infty, \qquad V \sim \sin(\pi k)^{5/3}$$

Note: formal restriction to the linearised case

$$\varepsilon^{\alpha}\partial_t W^{\varepsilon} + \varepsilon\omega'(k)\partial_x W^{\varepsilon} = C(W^{\varepsilon}),$$

Linearise

$$W^{\varepsilon} = W_0(1+\varepsilon f^{\varepsilon}) \text{ where } W_0 = \frac{1}{\beta_0\omega(k)},\ \beta_0 > 0.$$

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Proceeding as before

$$\partial_t T^\varepsilon + \partial_x \frac{1}{\varepsilon}\int \omega'(k) f^\varepsilon \, dk = 0$$

$$\varepsilon^{-1}\int \omega' f^\varepsilon dk = \int L^{-1}(\omega')\omega'\partial_x f^\varepsilon dk - \int L^{-1}(\omega')Q(f^\varepsilon, f^\varepsilon)dk + \mathcal{O}(\varepsilon).$$

But $Q(T,T) = 0$.

Difficulties

- It is not a linear Boltzmann equation (or radiative transfer equation);

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  - Intuition: $f \in L^2$ but $\omega^{-1} \notin L^2$.

Theorem (Fractional diffusion limit for the linearised equation)

Let f^ε be a solution of $\varepsilon^{8/5}\partial_t f^\varepsilon + \varepsilon\omega'(k)\partial_x f^\varepsilon = L(f^\varepsilon)$ (BPE) with initial data $f_0 \in L^2(\mathbb{R}\times\mathbb{T})$. Then, up to a subsequence,

$$f^\varepsilon(t,x,k) \rightharpoonup T(t,x) \qquad L^\infty((0,\infty);L^2(x,k))\text{-weak}$$

where T is the weak limit of

$$T^\varepsilon = \frac{1}{\langle V\rangle}\int_0^1 V(k) f^\varepsilon(k)dk$$

and solves the fractional diffusion equation

$$\partial_t T + \kappa(-\Delta_x)^{4/5}T = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}$$

$\kappa\in(0,\infty)$ with initial condition

$$T(0,x) = \frac{1}{\langle V\rangle}\int_0^1 V f_0(t,x,k)\,dk.$$

Comments on the proof and results (I)

$$f^{\varepsilon}(t,x,k) = \underbrace{T^{\varepsilon}(t,x)}_{\text{energy}} + \underbrace{S^{\varepsilon}(t,x)\omega(k)^{-1}}_{\text{spurious term}} + \underbrace{\varepsilon^{\frac{4}{5}} h^{\varepsilon}(t,x,k)}_{\text{remainder}}$$

$T^{\varepsilon} \in L^{\infty}(0,\infty; L^2(\mathbb{R}))$,

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Proposition

The function $\tilde{S}^\varepsilon$ converges in distribution sense to

$$\tilde{S}(t,x) = -\frac{\kappa_2}{\kappa_3}(-\Delta)^{3/10}T(t,x).$$

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and

$$\partial_t T + \left(\kappa_1 - \frac{\kappa_2^2}{\kappa_3}\right)(-\Delta)^{4/5} T = 0$$

Comments on the proof and results (III)

Method: Laplace-Fourier Transform Method (like in Mellet, Mischler, Mouhot, 'Fractional diffusion limit for collisional kinetic equations').

$$\widehat{f^{\varepsilon}}(p,\xi,k) = \int_{\mathbb{R}} \int_0^{\infty} e^{-pt} e^{-i\xi x} f^{\varepsilon}(t,x,k)\, dt\, dx.$$

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$$\varepsilon^\alpha p \widehat{f^\varepsilon} - \varepsilon^\alpha \widehat{f_0} + i\varepsilon\omega'(k)\xi\widehat{f^\varepsilon} = K(\widehat{f^\varepsilon}) - V\widehat{f^\varepsilon}$$

We recall that $L(f) = K(f) - Vf$ with $K(f) = \int K(k,k')f(k')\,dk'$. The fact that $\int L(f)\,dk = 0$ and $\int \frac{1}{\omega(k)} L(f)\,dk = 0$ for all f implies

$$V(k) = \int K(k',k)dk', \qquad \frac{V(k)}{\omega(k)} = \int K(k',k)\frac{1}{\omega(k')}\,dk'$$

Multiplying the equation by $K(k',k)$ and integrating with respect to k and k', we get...

- On the constant mode:

$$\mathcal{F}_1^\varepsilon(\widehat{f}^0) + a_1^\varepsilon(p,\xi)\widehat{T}^\varepsilon(p,\xi) + a_2^\varepsilon(p,\xi)\widehat{\tilde{S}^\varepsilon}(p,\xi) + R_1^\varepsilon(p,\xi) = 0, \qquad (3)$$

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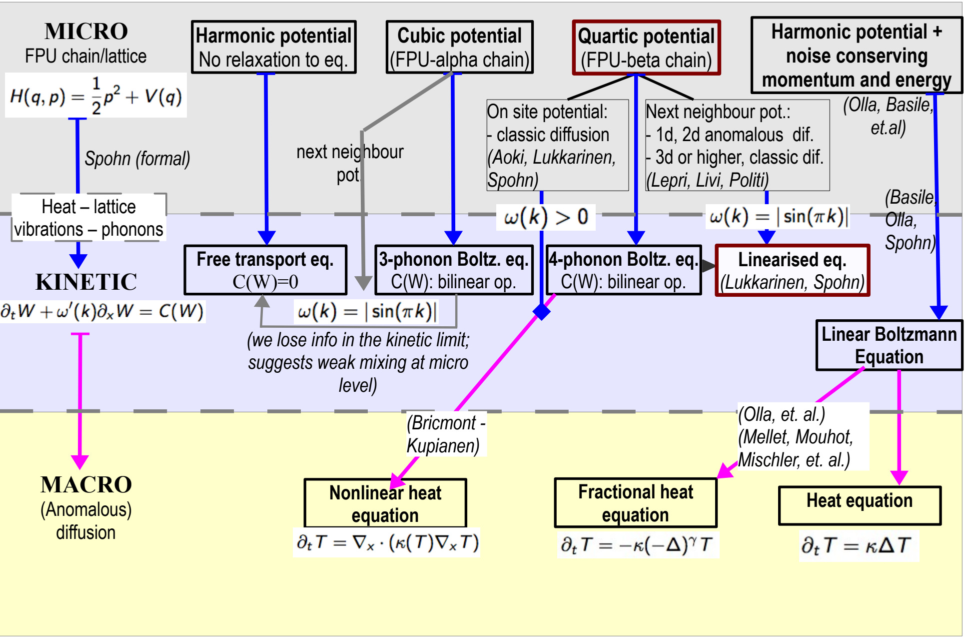

MICRO
FPU chain/lattice
$H(q,p) = \frac{1}{2}p^2 + V(q)$
Spohn (formal)
Heat – lattice vibrations – phonons
KINETIC
$\partial_t W + \omega'(k)\partial_x W = C(W)$
MACRO
(Anomalous) diffusion
Harmonic potential
No relaxation to eq.
Cubic potential
(FPU-alpha chain)
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(FPU-beta chain)
Harmonic potential + noise conserving momentum and energy
(Olla, Basile, et.al)
next neighbour pot
On site potential:
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(Aoki, Lukkarinen, Spohn)
Next neighbour pot.:
- 1d, 2d anomalous dif.
- 3d or higher, classic dif.
(Lepri, Livi, Politi)
$\omega(k) > 0$
$\omega(k) = |\sin(\pi k)|$
(Basile, Olla, Spohn)
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3-phonon Boltz. eq.
C(W): bilinear op.
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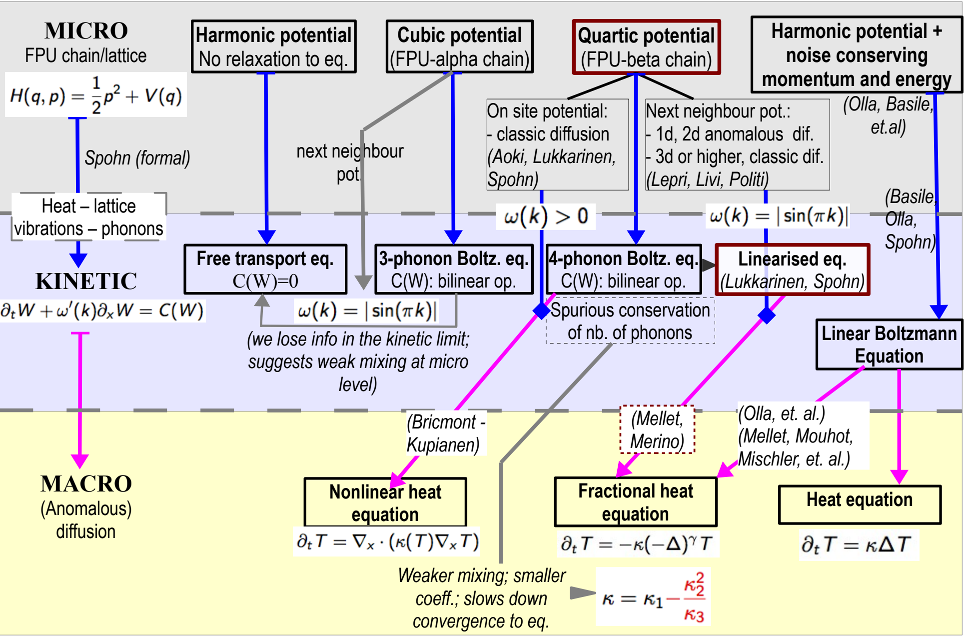
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$\kappa = \kappa_1 - \frac{\kappa_2^2}{\kappa_3}$

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- 'Similar' equations exist in other context:
  - Weak wave turbulence (4-wave kinetic equation).
  - Nordheim equation.

References I

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