From Boltzmann to Landau: convergence of solutions and propagation of integrability in the Coulomb case

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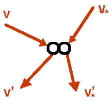
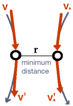
We are interested in the behavior of rarefied gases with long-range intermolecular interactions:

- Forces between gas particles are inversely proportional to the distance between them: r^{-s}, $s \geq 2$.
- Interacting particles barely change each others' trajectories: $|v' - v|, |v'_* - v_*| \sim 0$, or
- Trajectories of particles very close to each other are almost parallel: $|v - v_*| = |v' - v'_*| \sim 0$
- Particles interact at very small angles: $(v - v_*) \cdot (v' - v'_*) \sim |v - v_*|^2$.





The model:

For $t > 0$, let $0 \leq f = f(v, x, t) \in L^1_x \times L^1_v(\mathbb{R}^3)$ be the particles' probability distribution function.
If $s > 2$, the flow of a rarefied gas is modeled by the Boltzmann equation:

$$\partial_t f(v, x, t) + v \cdot \nabla_x f(v, x, t) = Q_B(f, f)(v, x, t)$$

$s = 2$ corresponds to *Coulomb forces* - use the *Landau equation* instead:

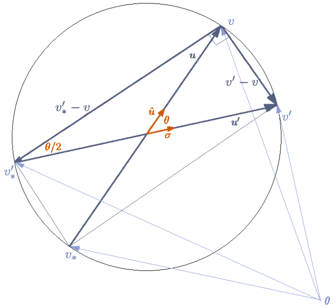
$$\partial_t f(v, x, t) + v \cdot \nabla_x f(v, x, t) = Q_L(f, f)(v, x, t)$$

Transition from Boltzmann to Landau equations is called the *grazing collisions limit.*

Structure of this talk

1. Introduction
   - description of the Boltzmann and Landau equations
   - the grazing collisions limit
   - well-posedness
2. An angle-potential concentrated collision kernel
   - motivation
   - connection with Q_L
3. Existence of L^p Boltzmann solutions:
   - L^p estimates on the collision operator
   - Bressan's theorem for existence and uniqueness
4. Future study
   - compactness
   - convergence to a Landau solution?

The Boltzmann equation: geometry of collisions



Notation

$$\boldsymbol{u} := v - v_*$$

$$\boldsymbol{u}' := v' - v'_*$$

$$\boldsymbol{\sigma} := \frac{\boldsymbol{u}'}{|\boldsymbol{u}'|} = \frac{\boldsymbol{u}'}{|\boldsymbol{u}|}$$

$$\cos\theta = \hat{\boldsymbol{u}} \cdot \boldsymbol{\sigma}$$

Suppose that

v, v_* are velocities of two colliding particles,

v', v'_* are their pre- (or post-) collisional velocities (elastic collisions are reversible).

Conservation laws

$$v + v_* = v' + v'_* \quad \text{(momentum)}$$

$$|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2 \quad \text{(kinetic energy)}$$

Representation by σ

For $v, v_* \in \mathbb{R}^3$, any v', v'_* can be uniquely represented by

$$v' = v'(v, v_*, \boldsymbol{\sigma}) = v + \frac{1}{2}|u|(\boldsymbol{\sigma} - \hat{\boldsymbol{u}})$$

$$v'_* = v'_*(v, v_*, \boldsymbol{\sigma}) = v_* - \frac{1}{2}|u|(\boldsymbol{\sigma} - \hat{\boldsymbol{u}})$$

Facts

$$|u| = |u'|$$

$$u' - u = 2(v' - v) = -2(v'_* - v_*)$$

$$|v' - v| = |v'_* - v_*| = |u|\sin(\theta/2)$$

The Boltzmann equation as a birth-death process

The Boltzmann equation can be derived from the Chapman-Kolmogorov equations for birth-death processes: for all $t > 0$,

$$\begin{aligned} \frac{d}{dt} f(v) &= \int_{\mathbb{R}^3} \Big(\big[\text{rate of entering state } (v, v_*)\ \big] - \big[\text{rate of leaving state } (v, v_*)\ \big] \Big) \mathrm{d}v_* \\ &= \int_{\mathbb{R}^3} \int_{S^2} \big[\text{rate of leaving some state } (v', v'_*)\big] - \big[\text{rate of leaving state } (v, v_*)\big] \mathrm{d}\sigma \mathrm{d}v_* \\ &= \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|^\gamma, \hat{u} \cdot \sigma) \left(f(v') f(v'_*) - f(v) f(v_*) \right) \mathrm{d}\sigma \mathrm{d}v_* = Q_B(f, f)(v). \end{aligned}$$

- $B(|u|^\gamma, \hat{u} \cdot \sigma)$ is the *collision kernel* - models transition rates.
- $\gamma = \gamma(s) = \frac{s-5}{s-1}$ is the *potential* and depends on the strength of intermolecular forces.

$0 < \gamma \leq 1 \longrightarrow$ *hard potentials*

$\gamma = 0 \longrightarrow$ *Maxwell molecules*

$-2 < \gamma < 0 \longrightarrow$ *soft potentials*

$-3 < \gamma \leq -2 \longrightarrow$ *very soft potentials*

$\gamma = -3 \longrightarrow$ *Coulomb potential - no good for Boltzmann*

The space homogenous Boltzmann equation is

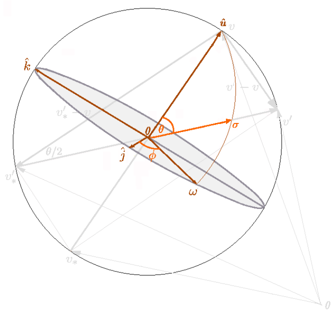
$$\begin{cases} \dfrac{\partial}{\partial t} f(v, t) = \displaystyle\int_{\mathbb{R}^3} \int_{S^2} B(|u|^\gamma, \hat{u} \cdot \sigma) \left(f' f'_* - f f_* \right) \mathrm{d}\sigma \mathrm{d}v_* \\ f(v, 0) = f_0(v). \end{cases}$$

The collision kernel: possibilities for $B(|u|^\gamma, \hat{u}\cdot\sigma)$, $\gamma < 0$

Typically,

$$\begin{aligned} B(|u|^\gamma, \hat{u}\cdot\sigma)\mathrm{d}\sigma &:= |u|^\gamma b(\hat{u}\cdot\sigma)\mathrm{d}\sigma \\ &= |u|^\gamma b(\cos\theta)\sin\theta \mathrm{d}\theta\mathrm{d}\omega \\ &= |u|^\gamma \sin^{-m}(\theta/2)\sin\theta\mathrm{d}\theta\mathrm{d}\omega. \end{aligned}$$

For *very soft potential*, $-3 < \gamma \leq -2,\ 0 < m < 4$.
For *Coulomb potential*, $\gamma = -3,\ m = 4$.
$\Rightarrow$ *Rutherford kernel*



There are other models:
In general, need $b \geq 0$ an even function, symmetric around $\pi/2$, such that

$$\int_0^{\pi/2} b(\cos\theta)\sin^2\theta\cos(\theta/2)\mathrm{d}\theta < \infty$$

NOT TRUE IN THE COULOMB CASE
$\Rightarrow$ TRUNCATE AT $\theta = 0$.

Notation

$\sigma = \hat{u}\cos\theta + \omega\sin\theta,\ 0 \leq \theta \leq \pi$

$\omega = \hat{j}\cos\phi + \hat{k}\sin\phi \in S^2(u^\perp),\ -\pi \leq \phi \leq \pi$

$j = (1,0,0) - \hat{u}\hat{u}_1, \quad k = j \times \hat{u}$

and

$$Q_B(f,h)(v,t) = \iint_{\mathbb{R}^3\times S^2} B(|v-v_*|^\gamma, \sigma)\left(f(v')h(v'_*) - f(v)h(v_*)\right)\mathrm{d}\sigma\mathrm{d}v_*.$$

The Landau equation

In 1936, L. Landau used the *Rutherford collisional cross section* heuristically to come up with

$$Q_L(f,h)(v,t) := \nabla_v \cdot \int_{\mathbb{R}^3} \frac{1}{|u|^{\gamma+2}} \Pi(u) \left(\nabla_v f(v,t) h(v_*,t) - f(v,t) \nabla_{v_*} h(v_*,t) \right) \mathrm{d}v_*$$

$$\Pi(u) := I_{3\times 3} - \frac{u \otimes u}{|u|^2} \in \mathbb{R}^{3\times 3}$$

$$\boldsymbol{\gamma = -d = -3} \longrightarrow \text{"true Landau"}.$$

The Landau equation is

$$\begin{cases} \partial_t f = Q_L(f,f)(v,t) \\ f(v,0) = f_0(v), \end{cases}$$

Remarks:

- No post collisional velocities v', v'_* or integration over the sphere: particles stay away from each other - barely change each other's trajectories. No actual collisions.
- BTE cannot handle the case $\gamma = -3$!

For $Q = Q_B$ or Q_L, the following holds:

- Conservation Laws

$$\frac{\partial}{\partial t}\int f(v,t)(1,v,|v|^2)\mathrm{d}v = \int Q(f,f)(1,v,|v|^2)\mathrm{d}v = 0$$

by symmetry of Q. Therefore mass (1), momentum (v) and energy $(|v|^2)$ are conserved:

$$\int f(v,t)(1,v,|v|^2)\mathrm{d}v = \int f_0(v)(1,v,|v|^2)\mathrm{d}v$$

- The H-Theorem

$$\frac{d}{dt}\mathcal{H}(t) = \frac{d}{dt}\int f(v,t)\log f(v,t)\mathrm{d}v = \int Q(f,f)(v,t)\log f(v,t)\mathrm{d}v \leq 0,$$

and equality holds if and only if $f(v,t)$ is Maxwellian in v.

We will always assume that $f_0 = f(\cdot,0) \in L^1_2 \cap L\log L(\mathbb{R}^3)$, i.e.

$$\int f_0(v)\log f_0(v)\mathrm{d}v < \infty \qquad \int f_0(v)(1+|v|^2)\mathrm{d}v < \infty$$

The grazing collisions limit: truncation

How does the Boltzmann equation become the Landau equation in the Coulomb case? Choose a suitable truncation of $B(|u|^{\gamma}, \hat{u} \cdot \sigma)$. Examples:

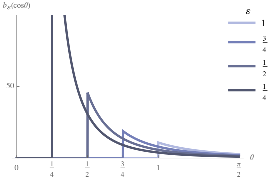
Classical Rutherford truncation:

[Villani, '98] and [Gamba, Haack '15]

$$b_{\varepsilon}(\cos\theta)\sin\theta \mathrm{d}\theta =$$

$$-\frac{I_{\sin(\theta/2)>\varepsilon}(\theta)}{2\pi\log\sin(\varepsilon/2)}\sin^{-4}(\theta/2)\sin\theta \mathrm{d}\theta.$$

* [Gamba, Haack]: also works for $4 < m < 6$ (different ε - coefficient)



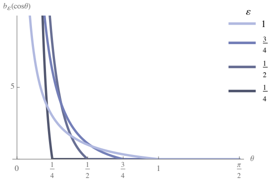
Approximation of $\theta-$singularity:

[Lingbing He, '14]

$$b_{\varepsilon}(\cos\theta)\sin\theta \mathrm{d}\theta =$$

$$k\varepsilon^{1+2\varepsilon}\theta^{2\varepsilon-3}\zeta\left(\frac{\theta}{\varepsilon}\right)\mathrm{d}\theta,$$

where $\zeta \in C([0,\pi/2])$, $supp(\zeta) \in [0,1]$.



The grazing collisions limit: the convergence $Q_{B_\varepsilon} \to Q_L$

Many possibilities for b_ε, but in order for $Q_{B_\varepsilon}(f,f) \to Q_L(f,f)$ need

$$\begin{cases} \lim_{\varepsilon\to 0} \beta_2^\varepsilon = \lim_{\varepsilon\to 0} \int_0^\pi b_\varepsilon(\cos\theta)\sin^2(\theta/2)\sin\theta \mathrm{d}\theta = \frac{2}{\pi} < \infty, \\ \lim_{\varepsilon\to 0} \beta_k^\varepsilon = \lim_{\varepsilon\longrightarrow 0} \int_0^\pi b_\varepsilon(\cos\theta)\sin^k(\theta/2)\sin\theta \mathrm{d}\theta = 0 \quad \forall k > 2, \\ \forall \theta_0 \in \left(0, \frac{\pi}{2}\right), \lim_{\varepsilon\longrightarrow 0} b_\varepsilon(\cos\theta) = 0 \text{ uniformly on } \{\theta > \theta_0\}. \end{cases}$$

Proposition 1.

Consider a sequence of nonnegative collision kernels,
$B_\varepsilon = B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma) = |u|^\gamma b_\varepsilon(\hat{u}\cdot\theta)$, $-3 \le \gamma \le -1$, *satisfying the properties above, and let* $0 \le f \in L^1_2 \cap L^p$, *where*

$$p > \frac{6}{8+\gamma} \text{ if } \gamma \le -2,$$
$$p > 1 \text{ if } \gamma > -2.$$

Then for any test function $\varphi \in C_0^\infty(\mathbb{R}^3)$ *and for any* $t > 0$,

$$\lim_{\varepsilon\to 0} \left| \int \left(Q_{B_\varepsilon}(f,f)(v,t) - Q_L(f,f)(v,t)\right)\varphi(v)\mathrm{d}v \right| = 0.$$

Proof

One can check that

$$2\int Q_L(f,f)\varphi \mathrm{d}v = \iint ff_*|u|^\gamma \left[\boldsymbol{G_L^1(v,v_*) + G_L^2(v,v_*)}\right]\mathrm{d}v_*\mathrm{d}v$$

$$= \iint ff_*|u|^\gamma \big[\boldsymbol{-4(\partial_{v_i}\varphi(v) - \partial_{v_{*i}}\varphi(v_*))u_i}$$

$$\boldsymbol{+ |u|^2(\partial_{v_iv_j}\varphi(v) + \partial_{v_{*i}v_{*j}}\varphi(v_*))\Pi(u)_{ij}}\big]\mathrm{d}v_*\mathrm{d}v,$$

$$2\int Q_{B_\varepsilon}(f,f)\varphi \mathrm{d}v = \iint ff_*\boldsymbol{G[B_\varepsilon]}(v,v_*)\mathrm{d}v_*\mathrm{d}v$$

$$= \iint ff_*|u|^\gamma \left[\int_0^{\pi/2} \boldsymbol{b_\varepsilon(\cos)}\int (\varphi' + \varphi'_* - \varphi - \varphi_*)\, \mathbf{d}\boldsymbol{\phi}\mathbf{d}\boldsymbol{\theta}\right]\mathrm{d}v_*\mathrm{d}v.$$

Taylor expansion of φ:

$$(\varphi' - \varphi) + (\varphi'_* - \varphi_*) = \partial_{v_i}\varphi(v)(v'_i - v_i) + \partial v_{*i}\varphi(v_*)({v_*}'_i - v_{*i})$$

$$+ \frac{1}{2}\partial_{v_iv_j}\varphi(v)(v'_i - v_i)(v'_j - v_j) + \frac{1}{2}\partial_{v_{*i}v_{*j}}\varphi(v_*)({v_*}'_i - v_{*i})({v_*}'_j - v_{*j})$$

$$+\frac{1}{6}\partial_{v_iv_jv_k}\varphi(\zeta)(v'_i - v_i)(v'_j - v_j)(v'_k - v_k) + \frac{1}{6}\partial_{v_{*i}v_{*j}v_{*k}}\varphi(\xi)({v_*}'_i - v_{*i})({v_*}'_j - v_{*j})({v_*}'_k - v_{*k})$$

$$\boxed{= (\partial_{v_i}\varphi - \partial_{v_{*i}}\varphi_*)(v'_i - v_i) + \frac{1}{2}(\partial_{v_iv_j}\varphi + \partial v_{*i}v_{*j}\varphi_*)(v'_i - v_i)(v'_j - v_j) + Err.}$$

Proof (II)

Then,

$$\boldsymbol{G[B_\varepsilon]} = G_1[B_\varepsilon] + G_2[B_\varepsilon] + G_3[B_\varepsilon]$$

$$= (\partial_{v_i}\varphi - \partial_{v_{*i}}\varphi_*) \int_0^{\pi/2} b_\varepsilon(\cos\theta)\sin\theta \int_{-\pi}^{\pi} (v_i' - v_i)\mathrm{d}\phi\mathrm{d}\theta$$

$$+ \frac{1}{2}(\partial_{v_iv_j}\varphi(v) + \partial_{v_{*i}v_{*j}}\varphi(v_*)) \int_0^{\pi/2} b_\varepsilon(\cos\theta)\sin\theta \int_{-\pi}^{\pi} (v_i' - v_i)(v_j' - v_j)]\mathrm{d}\phi\mathrm{d}\theta$$

$$+ \int_0^{\pi/2} b_\varepsilon(\cos\theta) \int_{-\pi}^{\pi} (Err)\mathrm{d}\phi\mathrm{d}\theta.$$

Using the geometry of collisions,

$$\int_{-\pi}^{\pi} (v_i' - v_i)\mathrm{d}\phi = -2\pi u_i \sin^2(\theta/2)$$

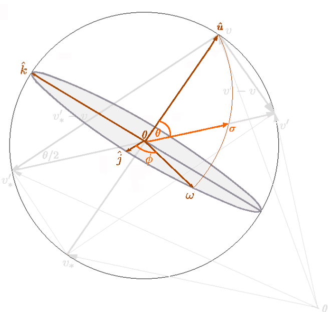
$$\int_{-\pi}^{\pi} (v_i' - v_i)(v_j' - v_j)\mathrm{d}\phi$$

$$= \pi\sin^4(\theta/2)\Big(2u_iu_j - |u|^2\Pi(u)_{ij}\Big)$$

$$+\pi|u|^2\Pi(u)_{ij}\sin^2(\theta/2)$$

$$\int_{-\pi}^{\pi} |v' - v|^3\mathrm{d}\phi = |u|^3 \int_{-\pi}^{\pi} \sin^3(\theta/2)\mathrm{d}\phi$$

$$= 2\pi|u|^3\sin^3(\theta/2)$$



Proof (III)

So,

$$\int_0^{\pi/2} b_\varepsilon(\cos\theta)\sin\theta \int_{-\pi}^{\pi} (v_i' - v_i)\mathrm{d}\phi\mathrm{d}\theta = \boldsymbol{-2\pi u_i \beta_2^\varepsilon},$$

$$\int_0^{\pi/2} b_\varepsilon(\cos\theta)\sin\theta \int_{-\pi}^{\pi} (v_i' - v_i)(v_j' - v_j)]\mathrm{d}\phi\mathrm{d}\theta = \boldsymbol{\pi\left(2u_iu_j - |u|^2\Pi(u)_{ij}\right)\beta_4^\varepsilon} \boldsymbol{+ \pi|u|^2\Pi(u)_{ij}\beta_2^\varepsilon}$$

$$\int_0^{\pi/2} b_\varepsilon(\cos\theta) \int_{-\pi}^{\pi} (Err)\mathrm{d}\phi\mathrm{d}\theta \le \boldsymbol{\frac{1}{3}\|D^2\varphi\|_{L^\infty}|u|^3\beta_3^\varepsilon}.$$

Thanks to the assumption that $f \in L^p$, the integrals below are finite and we can use the Dominated Convergence Theorem to get

$$\begin{aligned}\int Q_{B_\varepsilon}(f,f)\varphi\mathrm{d}v &= \iint ff_*|u|^\gamma G[B_\varepsilon]\mathrm{d}v_*\mathrm{d}v \\ &\to \iint ff_*|u|^\gamma \left[\boldsymbol{-4(\partial_{v_i}\varphi - \partial_{v_{*i}}\varphi_*)u_i}\right]\mathrm{d}v_*\mathrm{d}v \\ &+ \iint ff_*|u|^\gamma \left[\boldsymbol{|u|^2(\partial_{v_iv_j}\varphi + \partial_{v_{*i}v_{*j}}\varphi_*)\Pi(u)_{ij}}\right] \\ &= \iint ff_*|u|^\gamma \boldsymbol{G_L^1}\mathrm{d}v_*\mathrm{d}v + \iint ff_*|u|^\gamma \boldsymbol{G_L^2}\mathrm{d}v_*\mathrm{d}v = \int Q_L(f,f)\varphi\mathrm{d}v.\end{aligned}$$

Weak formulations (I)

Two types of weak solutions to the Boltzmann and Landau equations:

1. "classical" weak solutions ('weak solutions') - exist only when $\gamma > -2$.
2. weak-H solutions ('H-solutions') - they are *weaker* than weak solutions and exist for $\gamma \leq -2$.

The difference: "definition" of $\int Q\varphi$.

<u>Boltzmann</u>: Formally, if $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\int Q_B(f,f)(v)\varphi(v)\mathrm{d}v = \iint_{\mathbb{R}^3\times\mathbb{R}^3}\int_{S^2} B(|u|^\gamma, \hat{u}\cdot\sigma)\left(f' f'_* - f f_*\right)\varphi(v)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v$$

$$= \frac{1}{4}\iint\int B(|u|^\gamma, \hat{u}\cdot\sigma)\left(f' f'_* - f f_*\right)\cdot$$

(1) $$\cdot\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v \leftarrow \textit{H form}$$

$$= \frac{1}{4}\iint\int B(|u|^\gamma, \hat{u}\cdot\sigma) f(v') f(v'_*)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v$$

$$- \frac{1}{4}\iint\int B(|u|^\gamma, \hat{u}\cdot\sigma) f(v) f(v_*)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v$$

(2) $$= \frac{1}{2}\iint f(v)f(v_*)|u|^\gamma\int b(\hat{u}\cdot\sigma)\left(\varphi + \varphi_* - \varphi' - \varphi'_*\right)\mathrm{d}\sigma\mathrm{d}v_*\mathrm{d}v. \leftarrow \textit{weak form}$$

Landau: Similarly for $\varphi \in C_0^\infty(\mathbb{R}^3)$, formally

$$\int Q_L(f,f)(v)\varphi(v)\mathrm{d}v = \int_{\mathbb{R}^3} \varphi(v)\partial_{v_i} \int_{\mathbb{R}^3} |u|^{\gamma+2}\Pi(u)_{ij} \left(f_*\partial_{v_j} f - f\partial_{v_{*j}} f_*\right) \mathrm{d}v_*\mathrm{d}v$$

$$= -\frac{1}{2} \iint |u|^{\gamma+2} \left(\partial_{v_i}\varphi - \partial_{v_{*i}}\varphi_*\right) \Pi(u)_{ij} \cdot$$

(3) $$\cdot \left(f_*\partial_{v_j} f - f\partial_{v_{*j}} f_*\right) \mathrm{d}v_*\mathrm{d}v \leftarrow \textit{H form}$$

$$= -2 \iint f(v)f(v_*)|u|^\gamma (\partial_{v_i}\varphi(v) - \partial_{v_{*i}}\varphi(v_*))u_i \mathrm{d}v_*\mathrm{d}v$$

(4) $$+\frac{1}{2} \iint f(v)f(v_*)|u|^{\gamma+2}(\partial_{v_i v_j}\varphi(v) + \partial_{v_{*i}v_{*j}}\varphi(v_*))\Pi(u)_{ij}\mathrm{d}v_*\mathrm{d}v \leftarrow \textit{weak form.}$$

- If $\gamma > -2$ and $f \in L^1_2 \cap L\log L$, the H and weak forms are equivalent and well defined $\Rightarrow$ weak and H solutions are the same.
- If $\boldsymbol{\gamma \le -2}$ and $f \in L^1_2 \cap L\log L$, the weak forms blow up. H forms could blow up too, unless

$$\left|\frac{d}{dt}\mathcal{H}(t)\right| = \left|\frac{d}{dt}\int f(v,t)\log f(v,t)\mathrm{d}v\right| = \left|\int Q_{B,L}(f,f)(v,t)\log f(v,t)\mathrm{d}v\right| < \infty.$$

Villani, 1998: existence of global weak solutions for $\gamma > -2$, global H solutions for $\gamma \le -2$, for both Boltzmann and Landau equations.

From [Bobylev, Potapenko, 2013]:
Consider

$$B_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma) = g_\varepsilon(|u|^\gamma, \cos\theta) = \frac{4}{\pi\varepsilon}\delta_0\left(1 - \cos\theta - \min\{2, \varepsilon|u|^\gamma\}\right).$$

This kernel does not separate its variables, $\cos\theta$ and $|u|^\gamma$. As a consequence,

- takes care of the singularity at $u = 0$ *as well as* the one at $\theta = 0$.
- allows for the splitting of Q_{g_ε} into its *gain* and *loss* terms:

$$\begin{aligned} Q_{g_\varepsilon}(f,f)(v) &= Q^+_{g_\varepsilon}(f,f)(v) - Q^-_{g_\varepsilon}(f,f)(v) \\ &= \int_{\mathbb{R}^3}\int_{S^2} g_\varepsilon(|u|^\gamma, \hat{u}\cdot\sigma) f(v')f(v'_*), \mathrm{d}\sigma \mathrm{d}v_* \mathrm{d}v - \frac{8}{\varepsilon} f(v), \end{aligned}$$

so that

$$\partial_t f(v,t) + \frac{8}{\varepsilon} f(v,t) = Q^+_{g_\varepsilon}(f,f)(v,t).$$

Connection with Q_L

Theorem 2.
Let $p > \frac{6}{8+\gamma}$ if $\gamma \in [-3,-2]$ and $p > 1$ if $\gamma \in (-2,-1)$, and let $f_\varepsilon \in L^1_2 \cap L^p(\mathbb{R}^3)$. Then for all time, $|Q_{g_\varepsilon}(f,f) - Q_L(f,f)| \longrightarrow 0$ in the distributional sense as $\varepsilon \to 0$. That is, for any $\varphi \in C_0^\infty(\mathbb{R}^3)$,

$$\lim_{\varepsilon \longrightarrow 0} \left| \int_{\mathbb{R}^3} \left(Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) - Q_L(f_\varepsilon, f_\varepsilon)(v,t)\right) \varphi(v) \mathrm{d}v \right| = 0.$$

Formal proof.
Compute $\lim_{\varepsilon \to 0} \beta_k^\varepsilon$ for $k = 2$ and $k > 2$:

$$\beta_2^\varepsilon = \int_0^{\frac{\pi}{2}} g_\varepsilon(|u|^\gamma, \cos\theta) \sin^2(\theta/2) \sin\theta \mathrm{d}\theta = \frac{1}{2} \int_0^1 g_\varepsilon(|u|^\gamma, \mu)(1-\mu)\mathrm{d}\mu$$
$$= \frac{2}{\pi\varepsilon}(1 - \mu_\varepsilon(|u|)) I_{\varepsilon|u|^\gamma \le 1} = \frac{2}{\pi\varepsilon} m_\varepsilon(|u|) I_{\varepsilon|u|^\gamma \le 1} = \frac{2}{\pi}|u|^\gamma I_{\varepsilon|u|^\gamma \le 1}$$
$$\to \frac{2}{\pi}|u|^\gamma \text{ as } \varepsilon \to 0.$$

Proof (II)

And for $k > 2$,

$$\beta_k^\varepsilon = \int_0^{\frac{\pi}{2}} g_\varepsilon(|u|^\gamma, \mu) \sin^k(\theta/2) \sin\theta \mathrm{d}\theta$$

$$= \int_0^1 g_\varepsilon(|u|^\gamma, \mu) \left(\frac{1}{2}(1-\mu)\right)^{\frac{k}{2}} \mathrm{d}\mu = 2^{-\frac{k}{2}} \frac{4}{\pi\varepsilon}(1-\mu_\varepsilon(|u|))^{\frac{k}{2}} I_{\varepsilon|u|^\gamma \le 1}$$

$$= 2^{-\frac{k}{2}} \frac{4}{\pi\varepsilon} m_\varepsilon(|u|)^{\frac{k}{2}} I_{\varepsilon|u|^\gamma \le 1} = \frac{2^{2-\frac{k}{2}}}{\pi} \varepsilon^{\frac{k}{2}-1} |u|^{\gamma k/2} I_{\varepsilon|u|^\gamma \le 1}$$

$$\longrightarrow 0 \text{ as } \varepsilon \longrightarrow 0.$$

□

Note:
After integrating over μ, g_ε behaves in the same way as the usual B_ε does.

The Cauchy problem

Theorem 3.
For $1 \leq p \leq \infty$, *let* $E := L^1_2 \cap L^p(\mathbb{R}^3)$ *be a Banach space with the norm* $\|f\|_E := \|f\|_{L^1} + \|f\|_{L^p}$, *and let* $0 \leq f_0 \in F \cap L\log L$ *with* $\|f_0\|_{L^1} = 1$, *where*

$$F := \left\{ f \in E : f \geq 0, \|f\|_{L^1} = \|f_0\|_{L^1} = 1, \|f\|_{L^p(\mathbb{R}^3)} \leq C \right\}$$

for some $C > 0$. *Then, there exists a unique weak solution,* f_ε, *to the Boltzmann equation*

$$\begin{cases} \partial_t f_\varepsilon(v,t) = Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(v,t) \\ f_\varepsilon(v,0) = f_0(v) \end{cases}$$

which lies in $C^1((0,\infty), E) \cap C^1([0,\infty), F)$ *that preserves mass, momentum, energy, and whose entropy is bounded by the initial entropy.*

Proof(I)

Part I: Bressan's theorem

Theorem 4 (Bressan, '06).

Let E be a Banach space, F a bounded, convex and closed subset of E, and $Q : F \longrightarrow E$ an operator such that the following holds:

(i) *Holder continuity: for all $f, h \in F$,*

$$\|Q[f] - Q[h]\|_E \leq C\|f - h\|_E^\beta \quad \text{for some } \beta \in (0,1)$$

(ii) *the subtangent condition: for all $f \in F$,*

$$\liminf_{\delta \to 0+} \frac{1}{\delta} dist_E(f + \delta Q[f], F) = 0$$

(iii) *the one-sided Lipschitz condition: for all $f, h \in F$,*

$$[Q[f] - Q[h], f - h] \leq C\|f - h\|_E \quad \text{for all } f, h \in F$$

where $[\phi, \psi] := \lim_{\delta \to 0+} \delta^{-1}(\|\psi + \delta\phi\|_E - \|\psi\|_E)$.

Then, the equation

$$\begin{cases} \partial_t f = Q[f] \text{ on } [0,\infty) \times E \\ f(0) = f_0 \geq 0 \in F \text{ on } \{0\} \times E \end{cases}$$

has a unique solution, $f \in C^1((0,\infty), E)$.

Proof (II)

Continuity of $Q^+_{g_\varepsilon}$

Theorem 5.
Let $1 \le p, q, r \le \infty$, $1/p + 1/q = 1 + 1/r$. Then the bilinear operator $Q^+_{g_\varepsilon}$ extends to a bounded operator from $L^p(\mathbb{R}^3) \times L^q(\mathbb{R}^3)$ to $L^r(\mathbb{R}^3)$, and

$$\|Q^+_{g_\varepsilon}(f,h)\|_{L^r(\mathbb{R}^3)} \le \frac{16}{\varepsilon}\|f\|_{L^p(\mathbb{R}^3)}\|h\|_{L^q(\mathbb{R}^3)}.$$

For any $f, h \in F$,

$$\begin{aligned}
\|Q_{g_\varepsilon}(f,f) - Q_{g_\varepsilon}(h,h)\|_E &\le \|Q_{g_\varepsilon}(f,f-h))\|_E + \|Q_{g_\varepsilon}(f-h,h)\|_E \\
&= \|Q_{g_\varepsilon}(f,f-h)\|_{L^1} + \|Q_{g_\varepsilon}(f,f-h)\|_{L^p} \\
&+ \|Q_{g_\varepsilon}(f-h,h)\|_{L^1} + \|Q_{g_\varepsilon}(f-h,h)\|_{L^p} \\
\le \frac{16}{\varepsilon}\|f\|_{L^1}\|f-h\|_{L^1} &+ \frac{16}{\varepsilon}\|f\|_{L^p}\|f-h\|_{L^1} + \frac{16}{\varepsilon}\|f-h\|_{L^1}\|h\|_{L^1} \\
+ \frac{16}{\varepsilon}\|f-h\|_{L^p}\|h\|_{L^1} &= \frac{16}{\varepsilon}\|f-h\|_{L^1}(2 + \|f\|_{L^p}) + \frac{16}{\varepsilon}\|f-h\|_{L^p} \\
&\le \frac{16}{\varepsilon}(2 + C)\|f-h\|_E.
\end{aligned}$$

Proof (III)

f_ε are *weak* solutions:

If $f \in L^p$ for $p = 1 + r$, then the weak form of Q_{g_ε} is well defined.

Lemma 6.
Let $p > 1$, $k > 0$ and $\alpha < 0$ such that $\alpha p' > -6$. If $f \in L^1 \cap L^p(\mathbb{R}^3)$ and $h(v) := |v|^\alpha$, then there exists $C = C(p) > 0$ such that

$$\iint f(v)f(v_*)|v-v_*|^\alpha \mathrm{d}v_*\mathrm{d}v \le \|f\|^2_{L^1(\mathbb{R}^3)} + C\|f\|^2_{L^p(\mathbb{R}^3)} \text{ if } \alpha < 0.$$

Proof.
Let $h_1(v) := |v|^\alpha I_{|v|\le 1}$ and $h_2(v) := |v|^\alpha I_{|v|>1}$, so that $h_1 + h_2 = h$. Then

$$\iint f(v)f(v_*)h(v-v_*)\mathrm{d}v_*\mathrm{d}v = \int f(v) f * h_1(v)\mathrm{d}v + \int f(v) f * h_2(v)\mathrm{d}v.$$

By Holder's and Young's inequalities,

$$\int f f * h_1 \mathrm{d}v \le \|f\|_{L^p}\|f * h_1\|_{L^{p'}} \le \|f\|^2_{L^p}\|h_1\|_{L^{p'/2}},$$

$$\int f f * h_2 \mathrm{d}v \le \|f\|_{L^1}\|f * h_2\|_{L^\infty} \le \|f\|^2_{L^1}\|h_2\|_{L^\infty} = \|f\|^2_{L^1}.$$

Note that $\|h_1\|_{L^{p'/2}}$ depends only on $|S^2|$ and p'. □

A uniform bound on f_ε(I)

Lemma 7.
For any $p \in [1,\infty]$, $K > 1$ and $0 \le f_\varepsilon \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$ a weak solution to the Boltzmann equation, the following holds: if $1 < p < \infty$,

$$\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)\|_{L^p(\mathbb{R}^3)} \le K^{\frac{1}{2p'}} \frac{16}{\varepsilon} \|f_\varepsilon\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} + \frac{16}{\varepsilon \log K} \|f_0\|_{L\log L(\mathbb{R}^3)} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \text{ for } p < \infty,$$

and if $p = \infty$,

$$\|Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)\|_{L^\infty(\mathbb{R}^3)} \le \frac{16K}{\varepsilon} + \frac{16}{\varepsilon \log K} \|f_0\|_{L\log L(\mathbb{R}^3)} \|f_\varepsilon\|_{L^\infty(\mathbb{R}^3)}$$

.

Proof. The idea:

$$\begin{aligned} Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon) = A + B &:= Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon I_{f_\varepsilon \le K}) + Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon I_{f_\varepsilon > K}) \\ &\le K^{\frac{1}{2p'}} Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon^{\frac{p+1}{2p}}) + \frac{1}{\log K} Q^+_{g_\varepsilon}(f_\varepsilon, f_\varepsilon \log f_\varepsilon) \\ &\le K^{\frac{1}{2p'}} \frac{16}{\varepsilon} \|f_\varepsilon\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} + \frac{16}{\varepsilon \log K} \|f_\varepsilon\|_{L^p(\mathbb{R}^3)} \|f_0\|_{L\log L(\mathbb{R}^3)}. \end{aligned}$$

A uniform bound on f_ε (II)

Theorem 8.
Let $f_\varepsilon = f_\varepsilon(v,t) \geq 0 \in L^1(\mathbb{R}^3)$ be a weak solution to the Boltzmann equation with nonnegative initial data
$0 \leq f_\varepsilon(v,0) := f_0 \in L^1_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3) \cap L\log L(\mathbb{R}^3)$, $1 \leq p \leq \infty$. *Then f_ε remains in $L^p(\mathbb{R}^3)$, uniformly in ε and time. More specifically,*

$$\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} \leq \max\left\{16e^{8\|f_0\|_{L\log L}}, \|f_0\|_{L^p}\right\} \text{ for all } t > 0.$$

Proof.(for $p \neq \infty$)
The idea: let $K := e^{4\|f_0\|_{L\log L}} > 1$.

$$\begin{aligned}
&\varepsilon\|Q_{g_\varepsilon}(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^p(\mathbb{R}^3)} \\
&\quad \leq 16K^{\frac{1}{2p'}}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} + 8\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}\left(\frac{2}{\log K}\|f_0\|_{L\log L(\mathbb{R}^3)} - 1\right) \\
&\quad = 16K^{\frac{1}{2p'}}\|f_\varepsilon(\cdot,t)\|^{\frac{1}{2}}_{L^p(\mathbb{R}^3)} - 4\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}.
\end{aligned}$$

Then

$$
\begin{aligned}
&= \varepsilon p \|f_\varepsilon(\cdot,t)\|_{L^p}^{p-1} \frac{d}{dt} \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} \\
&\quad = \varepsilon \frac{d}{dt} \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}^p = \varepsilon \int \frac{d}{dt} f_\varepsilon(v,t) f_\varepsilon^{p-1}(v,t) \mathrm{d}v \\
&\quad = \varepsilon \int_{\mathbb{R}^3} Q_{g_\varepsilon}^+(f_\varepsilon, f_\varepsilon)(v,t) f_\varepsilon^{p-1}(v,t) \mathrm{d}v - 8 \int f_\varepsilon^p(v,t) \mathrm{d}v \\
&\quad \leq \varepsilon \|Q_{g_\varepsilon}^+(f_\varepsilon, f_\varepsilon)(\cdot,t)\|_{L^p(\mathbb{R}^3)} \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}^{p-1} - 8 \|f_\varepsilon\|_{L^p}^p \\
&\quad \leq 16 K^{\frac{1}{2p'}} \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}^{p-\frac{1}{2}} - 4 \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}^p .
\end{aligned}
$$

Multiply both sides by $\frac{1}{2} \|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)}^{\frac{1}{2}-p}$:

$$
\varepsilon p \frac{d}{dt} \|f_\varepsilon(\cdot,t)\|_{L^p}^{\frac{1}{2}} \leq 8K - 2 \|f_\varepsilon(\cdot,t)\|_{L^p}^{\frac{1}{2}} .
$$

Let $u(t) := \|f_\varepsilon(\cdot,t)\|_{L^p}^{\frac{1}{2}}$:

$$
\begin{cases}
u'(t) \leq \dfrac{8K}{\varepsilon p} - \dfrac{2}{\varepsilon p} u(t) \\[2ex]
u(0) = \|f_0\|_{L^p}^{\frac{1}{2}} .
\end{cases}
$$

Calculus:

$$\frac{d}{dt}\left(u(t)e^{\frac{2t}{\varepsilon p}}\right) \leq \frac{8K}{\varepsilon p}e^{\frac{2t}{\varepsilon p}}$$

$$u(t)e^{\frac{2t}{\varepsilon p}} - u(0) = \frac{4K}{p}\left(e^{\frac{2t}{\varepsilon p}} - 1\right)$$

$$u(t) \leq e^{-\frac{2t}{\varepsilon p}}\left(u(0) - \frac{4K}{p}\right) + \frac{4K}{p}.$$

Then

$$\|f_\varepsilon(\cdot,t)\|_{L^p(\mathbb{R}^3)} = u^2(t) \leq \left(4K + (u(0) - 4K)e^{-\frac{2t}{\varepsilon p}}\right)^2$$

$$\leq \max\left\{4K, \|f_0\|_{L^p(\mathbb{R}^3)}^{\frac{1}{2}}\right\}^2 \leq \max\left\{16e^{8\|f_0\|_{L\log L}}, \|f_0\|_{L^p(\mathbb{R}^3)}\right\}.$$

□

Thank you for your attention!