# Mean Field Limit for Stochastic Particle Systems With Singular Forces

Zhenfu Wang
joint work with Pierre-Emmanuel Jabin

Department of Mathematics and CSCAMM

University of Maryland

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# Outline

# Setting

Consider the classical Newton dynamics for N particles in the mean filed scaling in the classical regime. Denote $(X_i, V_i)$ the position and the velocity of particle number $i$. Then

$$\begin{cases} \mathrm{d}X_i = V_i \,\mathrm{d}t, \\ \mathrm{d}V_i = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma_N}\,\mathrm{d}W_i^t, \end{cases} \tag{1}$$

where $X_i \in \Omega$ ($\mathbb{T}^d$ or $\mathbb{R}^d$) and $V_i \in \mathbb{R}^d$, and $W_i$ are $N$ independent Brownian motions which may model random collisions on particles with rate $\sqrt{2\sigma_N}$. In particular, if $\sigma_N = 0$, the system (1) is deterministic. The interaction kernels $K$ model 2−body interaction force between particles.
As a companion, we also consider the 1st order systems

$$\mathrm{d}X_i = \frac{1}{N}\sum_{j\neq i} K(X_i - X_j)\,\mathrm{d}t + \sqrt{2\sigma_N}\,\mathrm{d}W_i \quad (i = 1, 2, \cdots, N.) \tag{2}$$

where $X_i \in \Omega = \mathbb{R}^d$ or $\mathbb{T}^d$.

# Complexity of Particle Systems

Particle Systems:

- Given by individual based models: conceptually simple;
- The number $N$ of particles are usually very large: Analytically and computationally complicated. $N \sim 10^{25}$ in typical physical settings and $N$ can be $10^8 \sim 10^9$ in Bio-Science settings.
- Ubiquitous: Physics (particles can represent ions and electrons in plasmas or molecules in a fluid and even galaxies in some cosmological models), Bio-sciences (modeling the collective behavior of animals or micro-organisms); Economics or Social Science (Opinion dynamics, consensus model, Mean field games)...

Understanding *how this complexity can be reduced* is a challenging but critical problem with potentially deep impact in various fields.

# Complexity Reduction: Mean Field Equations

The classical strategy to reduce the complexity is to derive a mesoscopic or macroscopic systems: We no longer trace the exact trajectory of each particle. Instead we try to capture the statistical or averaged information which is embedded in the densities typically solving a non-linear PDEs.

In our case, for very large $N$, one expects to replace the systems (1) and (2) by the (McKean-)Vlasov equations

$$\partial_t f + v \cdot \nabla_x f + K \star \rho \cdot \nabla_v f = \sigma \Delta_v f, \quad \rho(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,\mathrm{d}v, \tag{3}$$

and

$$\partial_t \rho + \operatorname{div}_x(\rho(K \star \rho)) = \sigma \Delta_x \rho. \tag{4}$$

Our central goal is to **show the mean field limit of the systems** (1) **or** (2) **towards McKean-Vlasov equations** (3) **or** (4) **respectively as** $N \to \infty$ **and** $\sigma_N \to \sigma \geq 0$ and in particular **quantify the distance between the particle systems and the limit PDEs for fixed** $N$.

# History Remarks

- Maxwell, Boltzmann: Deriving Boltzmann equation; Still Open. Recent Progress by Lanford 1975 and more recent by Gallagher, Saint-Raymond and Texier 2014.
- Vlasov-Poisson system, that is (3) with $K(x) = \gamma x/|x|^d$. Still open. Recent progress by Hauray and Jabin 2015, and D. Lazarovici and Pickl 2015.
- Classical (McKean-)Vlasov type PDEs: $K \in W^{1,\infty}_{loc}$. Dobrusin 1979. McKean 1967.
- 1st order system with singular kernels. The Biot-Savart kernel $K(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2}$. 2D Navier-Stokes equations: Osada 1987, Fournier, Hauray and Mischer 2014. 2D Euler: Goodman, Hou and Lowengrub 1990.

# More Examples of Kernels and Applications

- Biological systems modeling the collective motion of micro-organisms. The 1st order system with the Poisson kernel $K$ as the particle methods to approximate the Keller-Segel equation of chemotaxis. Fournier and Jourdain, 2015, Godinh and Quininao 2015, Liu and Yang 2016.
- Aggregation models given by the 1st order system (2), typically with $K = -\nabla W$ and an extra potential term.
- Newton dynamics with $K = -\nabla U$ and with more friction and self-propulsion terms, modeling the short-range repulsion and long-range attraction mechanism in Bio-science and in Physics.
- Alignment Model: Cucker-Smale model, Motsch-Tadmor model, Krause model...
- $K$ can be step functions in some rating models, $K$ can be highly oscillatory or even a Dirac mass in certain direction for instance $K(x) = (\varphi(x_2), \delta_0(x_1))$ (2D collisional model with $\mathrm{div}_x K = 0$.)

# Outline

# Main Ideas

We developed a new statistical approach to attack the mean field limit problems for systems with with realistic and hence more singular or rough interaction kernels: for instance by studying the $N-$body Liouville equation, relative entropy estimates, identifying the critical scales and regularity, combinatorics results in the spirit of Law of Large Numbers...
In the following, we will use the Newton Dynamics (1) as an example.
**Main Idea: Directly compare the joint distribution $f_N$ and the $f^{\otimes N}$ where $f$ solves the limit PDE through the scaled relative entropy** (First time applied in the mean field limit context)

$$H_N(t) := H_N(f_N|f^{\otimes N}) = \frac{1}{N}\int_{\mathbb{R}^{2dN}} f_N \log \frac{f_N}{f^{\otimes N}}\, \mathrm{d}x_1\, \mathrm{d}v_1 \cdots \mathrm{d}x_N\, \mathrm{d}v_N.$$

# The Liouville Equation

The key object now is the joint distribution of $N-$particle $f_N(t, x_1, v_1, \cdots, x_N, v_N)$ governed by the Liouville equation

$$\partial_t f_N + \sum_{i=1}^{N} v_i \cdot \nabla_{x_i} f_N + \frac{1}{N} \sum_{i=1}^{N} \sum_{j \neq i} K(x_i - x_j) \cdot \nabla_{v_i} f_N = \sigma_N \sum_{i=1}^{N} \Delta_{v_i} f_N. \quad (5)$$

Define the marginals $f_{N,k}$ of $f_N$ as

$$f_{N,k}(t, x_1, v_1, \cdots, x_k, v_k) = \int_{(\mathbb{R}^{2d})^{N-k}} f_N(t, x_1, v_1, \cdots, x_N, v_N) \, \mathrm{d}x_{k+1} v_{k+1} \cdots \mathrm{d}x_N \, \mathrm{d}v_N,$$

for $k = 1, 2, \cdots, N-1$.

# The Liouville Equation

The reason to study $f_N$ and its marginal $f_{N,k}$: only possible and sufficient for practical purposes.
One has the existence of weak solutions which dissipate the entropy

$$\int_{(\mathbb{R}^{2d})^N} f_N(t,X,V)\log f_N(t,X,V)\,\mathrm{d}X\,\mathrm{d}V + \varepsilon_N \int_0^t \int_{(\mathbb{R}^{2d})^N} \frac{|\nabla_V f_N(s,X,V)|^2}{f_N(s,X,V)}\,\mathrm{d}X\,\mathrm{d}V$$
$$\leq \int_{(\mathbb{R}^{2d})^N} f_N(0)\log f_N(0)\,\mathrm{d}X\,\mathrm{d}V, \quad \text{for } a.e.\ t,$$

given initial data with finite entropy or moments for instance, where $X = (x_1, \cdots, x_N)$ and $V = (v_1, \cdots, v_N)$ for simplicity.
**In our framework, the existence of weak solutions is sufficient in contrast to the existence of well-defined flows in other results.**

# A Intermediate Scale

Indeed, **the scaled relative entropy indicates a critical intermediate scale under which one can approximate the marginal $f_{N,k}$ by $f^{\otimes k}$.**

Can we show that $f_N$ is close to $f^{\otimes N}$? The answer is NO. For instance by taking $f_N = \tilde{f}^{\otimes N}$, the relative entropy (not scaled)

$$\int_{\mathbb{R}^{2dN}} \tilde{f}^{\otimes N} \log \frac{\tilde{f}^{\otimes N}}{f^{\otimes N}} = N \int_{\mathbb{R}^{2d}} \tilde{f} \log \frac{\tilde{f}}{f}$$

might goes to infinity even though $\int \tilde{f} \log \frac{\tilde{f}}{f}$ is relatively small.

Consequently, we do not have a proper object to approximate $f_N$, but instead we now do have a proper way to approximate the marginal $f_{N,k}$, *i.e.* through $f^{\otimes k}$.
**Good Enough!**

# A Intermediate Scale

The argument goes as the following: once we can know that the scaled relative entropy $H_N(t)$ is in the order of $1/N$, we can show that the marginals $f_{N,k}$ are indeed very close to $f^{\otimes k}$ for any fixed $k$, even though $f_N$ might be still far away from $f^{\otimes N}$.

Technical reasons for the above critical observation are the monotonicity of the scaled relative entropy

$$H_k(f_{N,k}|f^{\otimes k}) := \frac{1}{k}\int_{\mathbb{R}^{2dk}} f_{N,k} \log \frac{f_{N,k}}{f^{\otimes k}} \, \mathrm{d}x_1 \, \mathrm{d}v_1 \cdots \mathrm{d}x_k \, \mathrm{d}v_k \leq H_N(f_N|f^{\otimes N})$$

and the classical Csiszár-Kullback-Pinsker inequality

$$\|f_{N,k} - f^{\otimes k}\|_{L^1} \leq \sqrt{2kH_k(f_{N,k}|f^{\otimes k})}.$$

# Quantitative Version of Propagation of chaos

The results we obtained are in the flavor of propagation of chaos: **If initially $H_N(0)$ is in the order of $1/N$, then $H_N(t) \lesssim \frac{1}{N}$ up to certain time $T$.**

Indeed, $H_N(t) \to 0$ as $N \to \infty$ gives a strong version of (Kac's ) chaos and our results give a strong version of propagation of chaos and hence implies mean field limit for *a. e.* initial data.

In particular, if initially $f_N(0) = \Pi_{i=1}^N f_0(x_i, v_i)$ (i.i.d. ), then $H_N(0) = 0$.

We can provide the precise estimates quantifying the distance between particle systems and the limit PDE model by studying the evolution of the relative entropy.

# Outline

# Mean Field Limit for Vlasov Systems with Bounded Forces

In the article [1], the main result with $\sigma > 0$ can be formulated as follows

**Theorem 3.1 (Jabin and Wang [1])**

*Assume that $K \in L^\infty$ and that there exists $f_t \in L^\infty([0,\ T],\ L^1(\mathbb{R}^{2d}) \cap W^{1,p}(\mathbb{R}^{2d}))$ for every $1 \le p \le \infty$ which solves the limiting equation (3) with in addition*

$$\theta_f = \sup_{t \in [0,\ T]} \int_{\mathbb{R}^{2d}} e^{\lambda_f \, |\nabla_v \log f|} f \,\mathrm{d}x\,\mathrm{d}v < \infty,$$

*for some $\lambda_f > 0$. Furthermore assume initially that*

$$\sup_{N \ge 2} H_N(f_N(0)) < \infty, \quad H_N(f_N(0)|f_0^{\otimes N}) \to 0,\ \text{as } N \to \infty.$$

*and*

$$\sup_{N\ge 2} \frac{1}{N} \int_{E^N} \sum_{i=1}^{N} \left(1 + |x_i|^2 + |v_i|^2\right) f_N(0, x_1, v_1, \cdots, x_N, v_N) \,\mathrm{d}x_1\,\mathrm{d}v_1 \cdots \mathrm{d}x_N\,\mathrm{d}v_N < \infty.$$

Then there exists a universal constant $C > 0$ s.t. for any $t \in [0, T]$,

$$H_N(f_N(t)|f_t^{\otimes N}) \leq e^{C\,\|K\|_{L^\infty}\,\theta_f\,t/\lambda_f}\,\left(H_N(f_0^N|f_0^{\otimes N}) + \frac{C}{N}\right).$$

## Remark

- By comparing $f_N$ solving the Liouville equation and $f^{\otimes N}$ solving a modified Liouville equation, one has the relative entropy estimate

$$H_N(t) \leq H_N(0) - \frac{1}{N}\int_0^t \int_{\mathbb{R}^{2dN}} f_N R_N \,\mathrm{d}Z\,\mathrm{d}s, \tag{6}$$

where

$$R_N = \frac{1}{N}\sum_{i,j=1}^{N} \nabla_{v_i} \log f(x_i, v_i)\{K(x_i - x_j) - K \star \rho(x_i)\}.$$

Trivial bound $\int_0^t \int \cdot \in O(N)$, but we can show it is in the order 1. **Laws of Large Numbers, Cancellation rules**.

## Remark

- In [1], we can deal with the deterministic case with $\sigma_N \equiv 0$, the nonvanishing viscosity case with $\sigma_N \to \sigma > 0$ as $N \to \infty$ and the vanishing viscosity case with $\sigma_N \to 0$ as $N \to \infty$ at the same time with proper modified conditions.
- **This is the only result so far to our knowledge for non Lipschitz kernels $K$ in the stochastic case with vanishing or degenerate diffusion.**
- The presence of noise does not play any significant role in [1]. Even so we gain one more derivative compared to classical theory which requires that $K \in W^{1,\infty}$.
- By carefully exploiting the diffusion or **the dissipation of the relative entropy** in particular, we then push our results to the case with $K \in W^{-1,\infty}$, gaining two more derivatives compared to the classical results.

# Proof of the Main Theorem

Main Step of the Proof: Denote that $\bar{f}_N = f^{\otimes N}$,

- the tensor product of $f$ solves

$$\partial_t \bar{f}_N + L_N \bar{f}_N = \bar{f}_N R_N.$$

- Since $f_N$ is a weak solution to the Liouville equation,

$$H_N(t) = \frac{1}{N} \int_{(\Omega \times \mathbb{R}^d)^N} f_N \log(\frac{f_N}{\bar{f}_N}) \, \mathrm{d}Z = \frac{1}{N} \int f_N \log f_N - \frac{1}{N} \int f_N \log \bar{f}_N$$
$$\leq \frac{1}{N} \int f_N^0 \log f_N^0 - \frac{\varepsilon_N}{N} \int_0^t \int \frac{|\nabla_V f_N|^2}{f_N} - \frac{1}{N} \int f_N \log \bar{f}_N.$$

- Since $\bar{f}_N$ is smooth, $\log \bar{f}_N$ can be used as a test function against $f_N$, weak solution to the Liouville Equation

$$\int f_N \log \bar{f}_N = \int f_N^0 \log \bar{f}_N^0 + \int_0^t \int f_N(s, X, V) \, (\partial_t \log \bar{f}_N + L_N^* \log \bar{f}_N) \, \mathrm{d}Z \, \mathrm{d}s.$$

# Main Steps of the Proof -continued

- Substitute the equation of $\bar{f}_N$,

$$\int f_N \log \bar{f}_N = \int f_N^0 \log \bar{f}_N^0 + \int_0^t \int f_N R_N + \varepsilon_N \int_0^t \int f_N \left( \frac{\Delta_V \bar{f}_N}{\bar{f}_N} + \Delta_V \log \bar{f}_N \right).$$

- Classical entropy estimates shows that

$$\int \frac{|\nabla_V f_N|^2}{f_N} + \int f_N \left( \frac{\Delta_V \bar{f}_N}{\bar{f}_N} + \Delta_V \log \bar{f}_N \right) \mathrm{d}Z \geq 0,$$

which gives

$$H_N(t) \leq H_N(0) - \frac{1}{N} \int_0^t \int f_N R_N \, \mathrm{d}Z \, \mathrm{d}s. \tag{7}$$

# Main Steps of the Proof-continued

- Frenchel's inequality for function $u(x) = x \log x$, i.e.for $x, y \geq 0$,

$$xy \leq x \log x + \exp(y-1).$$

Hence,

$$-f_N R_N \leq \frac{\bar{f}_N}{\nu}\left(\frac{f_N}{\bar{f}_N}\nu\left|R_N\right|\right) \leq \frac{\bar{f}_N}{\nu}\left(\frac{f_N}{\bar{f}_N}\log(\frac{f_N}{\bar{f}_N}) + \exp(\nu\left|R_N\right|)\right),$$

which gives

$$\begin{aligned} H_N(t) \leq & H_N(0) + \frac{1}{\nu}\int_0^t H_N(s)\,\mathrm{d}s \\ & + \frac{1}{\nu}\frac{1}{N}\int_0^t \int \bar{f}_N \exp(\nu\left|R_N\right|)\,\mathrm{d}Z\,\mathrm{d}s. \end{aligned} \tag{8}$$

# Main Steps of the Proof-continued

We may apply the main estimate to $\tilde{R}_N = \nu\, R_N$. This implies that

$$L = \sup_N \sup_{t\in[0,\ T]} \int \bar{f}_N \exp(\nu\,|R_N|)\,\mathrm{d}Z < \infty.$$

Inserting this in (8) gives

$$H(f_N|\bar{f}_N)(t) \leq H(f_N|\bar{f}_N)(0) + \frac{1}{\nu}\int_0^t H(f_N|\bar{f}_N)(s)\ \mathrm{d}s + \frac{Lt}{\nu N},$$

and up to time $T > 0$, by Gronwall's inequality

$$H_N(t) \leq \left(H_N(0) + \frac{LT}{\nu N}\right)\ \exp(t/\nu). \tag{9}$$

# The scale of $R_N$

Recall that

$$R_N = \frac{1}{N}\sum_{i,j=1}^{N} \nabla_{v_i} \log f(x_i, v_i) \cdot \{K(x_i - x_j) - K * \rho(x_i)\},$$

with the convention that $K(0) = 0$. A trivial bound for $|R_N|$ is simply

$$|R_N| \leq (2\|K\|_{L^\infty} \|\nabla_v \log f\|_{L^\infty})\, N. \tag{10}$$

However inserting this bound in the inequality of $H_N(t)$ would only give that $H_N(t) = O(1)$ without any chance of converging. Instead the main estimate essentially proves that $R_N$ is of order 1 and not of order $N$.

**Goal**: To get

$$\int_{(\Omega\times\mathbb{R}^d)^N} \bar{f}_N \exp(|R_N|)\, \mathrm{d}Z \leq C < \infty,$$

where $C$ doesn't depend on $N$.

# Idea of the Proof of the Main Estimates

By Taylor expansion,

$$\int \bar{f}_N \exp(|R_N|) \, \mathrm{d}Z \leq 3 \sum_{k=0}^{\infty} \frac{1}{(2k)!} \int |R_N|^{2k} \bar{f}_N \, \mathrm{d}Z.$$

It is now sufficient to bound the summation in the right hand side.
**IDEA**: We divide the summation in two different cases: $k$ is small compared to $N$ or $k$ is comparable or larger than $N$.
The first part, $3k \leq N$, is more delicate and requires some preparatory combinatorics work.
The second part, $3k > N$, is almost trivial since now the coefficients $\frac{1}{(2k)!}$ dominates. The trivial bound for $|R_N|$ is good enough in this case.

# Mean Field Limit for 1st Order Systems

**Notation:** Now $\rho_N$ is the joint distribution of $(X_1, \cdots, X_N)$.

Theorem 3.2 (Jabin and Wang [3])

*Assume* $K \in W^{-1,\infty}$ *and* $\mathrm{div}_x K \in L^\infty$ *and* $\rho(t,x) \in L^\infty([0,\ T],\ L^1(\Omega) \cap W^{2,p}_{loc})$ *for every* $1 \le p \le \infty$ *solves the macroscopic equation* (4) *with*

$$\sup_{t\in[0,T]} \|\nabla_x \log \rho\|_{L^\infty} < \infty, \quad \sup_{t\in[0,T]} \left( \sup_{p\ge 1} \frac{\|R\|_{L^p(\rho\,\mathrm{d}x)}}{p} \right) < \infty,$$

*where we define that*

$$R_{hl}(x) = \frac{1}{\rho(x)} \partial_l \partial_h \rho(x), \quad R(x) = \sum_{h,l=1}^{d} |R_{hl}(x)|$$

*Assume that the initial data* $\rho_N^0$ *of the Liouville equation satisfies the entropy and moment bounds and*

$$H_N(\rho_N^0|\rho_0^{\otimes N}) = \frac{1}{N}\int_{\Omega^N} \rho_N^0 \log(\frac{\rho_N^0}{\rho_0^{\otimes N}})\, \mathrm{d}X \to 0, \quad \text{as} \quad N \to \infty.$$

Consider a corresponding weak solution $\rho_N$ to the Liouville equation then

$$\sup_{t\in[0,\ T]} H_N(\rho_N|\rho_t^{\otimes N})(t) \longrightarrow 0, \quad \text{as} \quad N \to \infty.$$

## Remark

In this case, the presence of noise is significant. Indeed, now the dissipation of the relative entropy reads (for simplicity assume that $\sigma_N = \sigma > 0$),

$$H_N(t) \leq H_N(0) - \int_0^t D_N \,\mathrm{d}s - \frac{1}{N}\int_0^t \int_{\Omega^N} \rho_N R_N \,\mathrm{d}X\,\mathrm{d}s - \frac{1}{N}\int_0^t \int_{\Omega^N} \rho_N Q_N \,\mathrm{d}X\,\mathrm{d}s,$$

where

$$D_N = \sigma \frac{1}{N}\int_{\Omega^N} \rho_N |\nabla_X \log \frac{\rho_N}{\rho^{\otimes N}}|^2 \,\mathrm{d}X.$$

### Remark

- The assumption can be relaxed to
  **The kernel $K$ permits a decomposition $K = K_1 + K_2$, where $K_1 \in W^{-1,\infty}$, $\mathrm{div} K_1 \in L^\infty$ and $K_2 \in L^\infty$.**
- The above result applies to the case $K(x) = (\phi(x_2), \delta_0(x_1))$ for certain 2D collision model.
- To complete the control on $H_N(t)$, more delicate combinatorics arguments are needed.

# Further Discussion

1. Quantitative result for Stochastic Vortex Model.
2. The 2nd order Systems with more singular kernels: Couple the relative entropy estimates with the dispersion estimates.
3. The 1st order system with homogeneous type kernel: Couple the interaction energy and the Monge-Kantorovich-Wasserstein distance.
4. Deriving Stochastic Euler Equations...
5. Fluctuation around the limit law: second order correction, large derivation type results...
6. More general particle models, Boltzmann type...
7. Weak-Weak type stability results...

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# Thank you!