Suppression of blow-up in Chemotaxis through fluid flow

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Outline

Introduction

- **Chemotaxis** is the movement of cells in response to chemical stimulus.

Figure: Chemotaxis[1]

We focus on the specific case when bacteria emit chemical signals to attract others of the same kind. (E.coli)

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Figure: Chemotaxis[1]

We focus on the specific case when bacteria emit chemical signals to attract others of the same kind. (E.coli)

- The **Patlak-Keller-Segel (PKS) equation** is designed to analyse this phenomena.

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- Consider the two-dimensional PKS equation with additional advection, which models the chemotaxis in moving fluid:

$$\begin{cases} \partial_t n + \overbrace{\nabla\cdot(n\nabla c)}^{\text{Aggregation}} + \overbrace{Au\cdot\nabla n}^{\text{Fluid Transport}} = \overbrace{\Delta n}^{\text{Diffusion}}, \\ c = (-\Delta)^{-1} n, \\ n(x,y,t=0) = n_{in}(x,y). \end{cases} \tag{mPKS}$$

Here n and c denote the bacteria density and the chemo-attractant density, respectively. The divergence free vector field u represents the underlying fluid velocity. $A \in \mathbb{R}_+$ denotes its magnitude. If $Au \equiv 0$, the equation is the classical PKS equation.

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- The equation (mPKS) is a *nonlocal* equation.

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- If $||n_{in}||_1 < 8\pi$, the diffusion dominates the aggregation, and we have global well-posedness of solutions.([5],[6])

  If $||n_{in}||_1 > 8\pi$, the aggregation dominates the diffusion, which yields finite time blow-up. Dirac mass appears.

  If $||n_{in}||_1 = 8\pi$, the solution will form Dirac mass when time approaches infinity. ([3])

Theorem

Consider the PKS equation (mPKS) *subject to C^∞ initial data. If $Au = 0$, $M := ||n_{in}||_1 > 8\pi$ and $\int n_{in}|x|^2 dx < \infty$, the solution n blows up in finite time.*

Proof.

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- It is straightforward to get the evolution equation for second moment:

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2 n(x,t)dx = 4M\left(1 - \frac{M}{8\pi}\right)$$

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- If the $M > 8\pi$, the second moment decreases at a constant rate. Suppose the solution remains smooth for all time, the second moment will reach zero at a finite time T^*, which is a contradiction.

$\square$

Introduction: Goal

- By adding an extra fluid transport term in the classical PKS equation, we hope to answer the following question:

Is it possible to find **simple** vector fields $A\mathbf{u}$ such that for any smooth enough initial data, the solutions n do not blow up for any finite time?

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Idea of the proof: Apply L^2 energy estimate to obtain:

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Since mixing flows enhance the negative term, the L^2 norm of n is bounded for all time and suppression of blow-up follows.

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- Recall the equation

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Suppression of blow-up through mixing

Theorem ([2] 2D case, Jacob Bedrossian and H.)

Consider the equation (mPKS) *on a torus* $\mathbb{T}^2$*. Let* $\mathbf{u}(x,y) = (u(y), 0)$ *be smooth non-degenerate shear flow and let* $n_{in} \in C^\infty(\mathbb{T}^2)$ *be arbitrary. There exists an* A_0 *such that if* $A > A_0$*, then the solution to* (mPKS) *is global in time.*

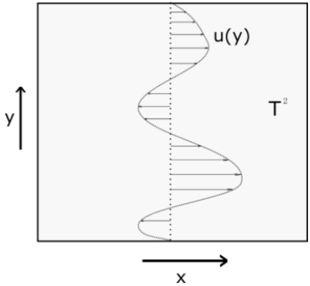

Figure: Nondegenerate shear flow

Theorem (3D case, Jacob Bedrossian and H.)

(a) *Let* $\mathbf{u} = (u(y_1), 0, 0)$ *be smooth non-degenerate shear flow and let* $n_{in} \in C^\infty(\mathbb{T}^3)$ *be arbitrary such that* $\|n_{in}\|_{L^1} < 8\pi$ *and* $\min_{x\in\mathbb{T}^3} n_{in}(x) > 0$*. Then there exists an* A_0 *such that if* $A > A_0$ *then the solution to* (mPKS) *is global in time.*

(b) *Suppose* $\mathbf{u} = (u(y_1), 0, 0)$ *is smooth non-degenerate shear flow. Let* $n_{in} \in C^\infty(\mathbb{T} \times \mathbb{R}^2)$ *be arbitrary such that* $\|n_{in}\|_{L^1} < 8\pi$ *and* $\int n_{in}(x, y)\, |y|^2\, dxdy < \infty$*. Then, there exists an* A_0*, such that if* $A > A_0$ *then the solution to* (mPKS) *is global in time.*

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- It is clear that $||n_{in}||_1 < 8\pi$ is essential in 3D. Indeed, consider any solution to the 3D problem which is constant in the x direction: $n(t, x, y_1, y_2) = n(t, y_1, y_2)$. This solution will solve (mPKS) on $\mathbb{T}^2$ with $A = 0$ and hence the 8π critical mass will still apply.

Idea of the proof: 1. A Different Time Scale

- We can divide both side of (mPKS) by A and rescale in time to get the following equation:

$$\partial_t n + u(y)\partial_x n = A^{-1}\Delta n - A^{-1}\nabla\cdot(n\nabla(-\Delta)^{-1}n). \tag{1}$$

When A is large, equation (1) can be regarded as a perturbation to the passive scalar equation with small viscosity A^{-1}:

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- Direct energy method yields that the solution decay like $e^{-A^{-1}t}$. This is not enough for our analysis. To prove suppression of blow-up, we need to study the enhanced diffusion effect of the passive scalar equation (PS).

Idea of the proof: 2. Passive Scalar Equation Revisited

- When stirring a cup of coffee, we see that the information in the angular direction diffuses faster than the one in the radial direction does. This is the enhanced diffusion effect of shear flow.

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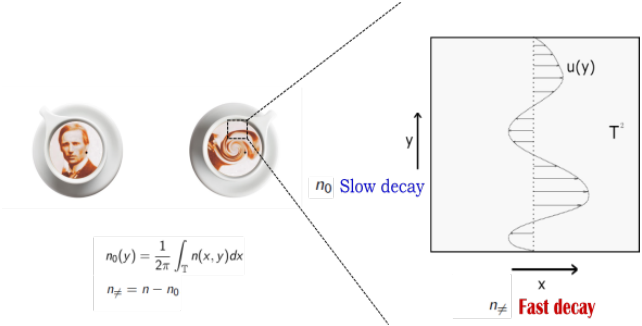

Figure: Stirring a cup of coffee

Idea of the proof: 2. Passive Scalar Equation Revisited

- In the paper [1], Jacob Bedrossian and Michele Coti Zelati proved that if $u(y)$ is nondegenerate shear flow, the x dependent part of the solution $\rho_{\neq}$ to the passive scalar equation satisfies the estimate

$$||\rho_{\neq}(t)||_{L^2}^2 \lesssim ||\rho_{\neq}(0)||_{H^1}^2 exp\{-\frac{ct}{A^{1/2}\log A}\}. \tag{2}$$

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- The proof is based on analyzing the hypocoercivity functional introduced in C. Villani's work [11],

$$\begin{aligned}\Phi[\rho_{\neq}] =& ||\rho_{\neq}||_{L^2}^2 + ||\sqrt{\alpha}(\partial_x)\partial_y\rho_{\neq}||_2^2 \\ &+ 2\langle\beta u'\partial_x\rho_{\neq}, \partial_y\rho_{\neq}\rangle + ||\sqrt{\gamma}(\partial_x)u'\partial_x\rho_{\neq}||_{L^2}^2,\end{aligned} \tag{3}$$

and showing that $\Phi[\rho_{\neq}(t)] \leq \Phi[\rho_{\neq}(0)]\exp\{-c\frac{t}{A^{1/2}}\}$.

Idea Of The Proof: 3. Energy Estimates

- In order to apply the enhanced diffusion estimate of the passive scalar equation, it is natural to separate the solution to the PKS equation (1) into x independent part and x dependent part:

$$\partial_t n_0 = \frac{1}{A}\Delta n_0 + \frac{1}{A}\nabla_y \cdot (\nabla_y c_0 n_0) + \text{Interaction}, \tag{4}$$

$$\partial_t n_{\neq} + u(y)\partial_x n_{\neq} = \frac{1}{A}\Delta n_{\neq} + \text{Interaction}. \tag{5}$$

$$n_0(y) = \frac{1}{2\pi}\int_{\mathbb{T}} n(x,y)dx, \quad n_{\neq} = n - n_0. \tag{6}$$

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- Since the dimension of the n_0 equation is *one dimension lower* than the full problem, we can use the classical PKS technique to show that the H^1 norm of the solution is bounded uniformly in time. For the second equation, we use the functional Φ to prove an enhanced diffusion estimate $\Phi[n_{\neq}] \le \Phi[n_{\neq}(0)]e^{-\frac{ct}{A^{1/2}}}$.

Idea Of The Proof: 2D case

- Nonzero modes: By taking the time derivative of $\Phi[n_{\neq}]$, we obtain:

$$\begin{aligned}\frac{d}{dt}\Phi[n_{\neq}] =& -\frac{c}{A^{1/2}}||n_{\neq}||_2^2 - \frac{1}{A}||\nabla n_{\neq}||_2^2 - ... + \frac{1}{A}\langle \nabla n_{\neq}, \nabla c_0 n_{\neq}\rangle + ...\\ \leq& -\frac{c}{A^{1/2}}||n_{\neq}||_2^2 - \frac{1}{2A}||\nabla n_{\neq}||_2^2 + ... + \frac{1}{2A^{1/2}}\frac{||\nabla c_0||_\infty^2}{A^{1/2}}||n_{\neq}||_2^2\end{aligned}$$

By choosing A large, we can absorb the last term in the first term. Do the same for the other terms, we can estimate $\frac{d}{dt}\Phi$.

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- Zero mode: standard energy estimate.
- For the three-dimensional case, the n_0 equation becomes critical. By a modified free energy approach, we can still propagate the regularity of the solutions.

Suppression of blow-up through fast splitting scenario

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- First, the shear flow induced enhanced diffusion effect on $\mathbb{R}^2$ can be extremely slow. This poses difficulties when we adapt the previous approach to the (mPKS) on the plane.

- Second, the plane $\mathbb{R}^2$ is unbounded, whereas the Torus is compact. Therefore, we have the freedom to send masses to infinity on the plane $\mathbb{R}^2$.

Suppression of blow-up through fast splitting scenario

- Now we exploit yet another mechanism to suppress the blow-up on $\underline{\mathbb{R}^2}$, which we called the *fast splitting scenario*.

[2] Picture from 'One-dimensional model equations for hyperbolic fluid flow', Tam Do, V. Hoang, Maria Radosz, Xiaoqian Xu

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- Now we exploit yet another mechanism to suppress the blow-up on $\underline{\mathbb{R}^2}$, which we called the *fast splitting scenario*.
- The flow we considered is the Hyperbolic flow $u(x_1, x_2) = A(-x_1, x_2)$. This flow splits cell density into upper and lower part.

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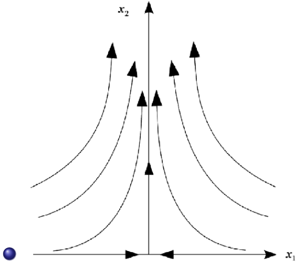

Figure: Hyperbolic flow. [2]

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$$y_+(t) := \frac{1}{M_+}\int_{x_2\geq 0} n(x,t)x_2 dx, \quad M_+ := \int_{x_2\geq 0} n(x,t)dx \equiv \frac{M}{2}.$$

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- We introduce the dimensionless quantity

$$\eta := \frac{y_+}{\sqrt{V_+/M_+}} \approx \frac{\text{Average distance to the boundary}}{\text{Standard deviation}}.$$

Suppression of blow-up through fast splitting scenario

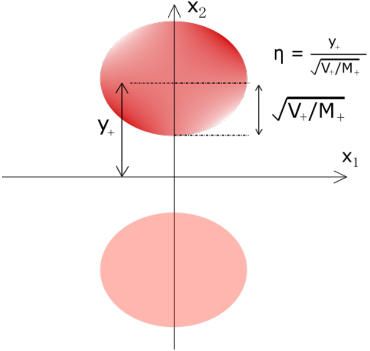

Figure: Definition of the dimensionless number η

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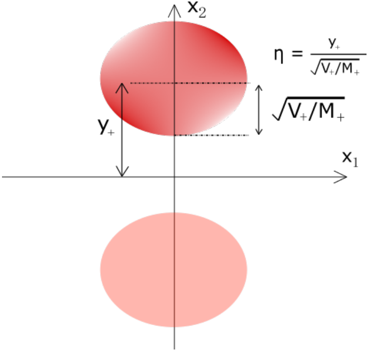

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- Recall the equation (mPKS)

$$\begin{cases} \partial_t n + \nabla \cdot (\nabla c n) + A(-x_1, x_2) \cdot \nabla n = \Delta n; \\ -\Delta c = n, \quad n(x, t = 0) = n_{in}(x), \quad (x_1, x_2) \in \mathbb{R}^2. \end{cases}$$

Suppression of blow-up through fast splitting scenario

Theorem (E. Tadmor and H., [8], 17)

Consider the PKS equation (mPKS) *subject to regular initial data* n_{in} *with total mass* $M = ||n_{in}||_1 < 2 \times 8\pi$. *Assume* n_{in} *is symmetric about the* x_1*-axis, and the dimensionless number*

$$\eta(0) = \frac{y_+(0)}{\sqrt{V_+(0)/M_+}} > \sqrt{2}. \tag{7}$$

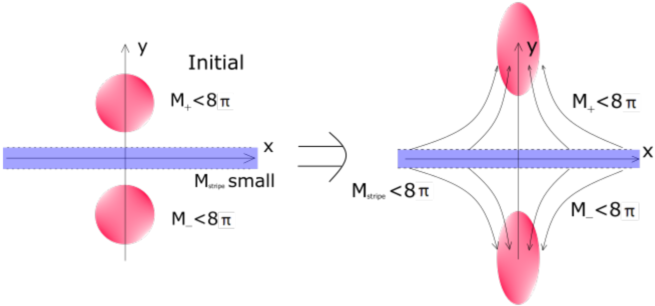
Then there exists a large enough amplitude A such that the free energy solution exists for all time.

Remark

Since the hyperbolic flow $A(-x_1, x_2)$ *is the gradient of a harmonic potential* $H = \frac{A}{2}(-x_1^2 + x_2^2)$, *the* (mPKS) *has a decreasing free energy*

$$E_H[n] = \int n \log n dx + \frac{1}{4\pi} \iint n(x) \log |x - y| n(y) dx dy - \int H n dx.$$

Ingredients of the proof: Heuristics



- Figure: Suppression of blow-up through a fast splitting scenario

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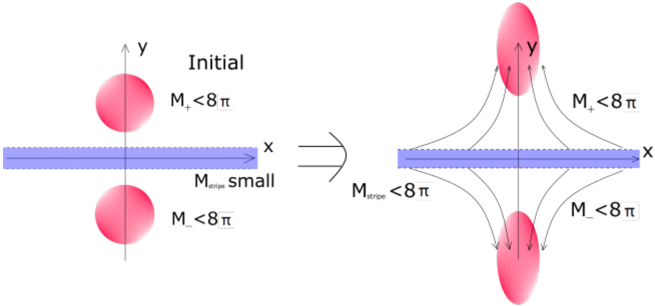

Figure: Suppression of blow-up through a fast splitting scenario

- Due to symmetry, the only possible blow-up position is on the x_1 axis. Therefore, we use the quantity η to control the mass inside the critical strip $S_\delta := \{x||x_2| \leq \delta\}$. Once the mass inside the strip is shown to be smaller than 8π, we can prove global in time well-posedness.

Ingredients of the proof: Review of the classical PKS

- It is standard to derive global existence once we control the entropy

$$S[n] = \int n \log n dx.$$

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- The $(\text{mPKS})_{Au\equiv 0}$ equation is a gradient flow of the free energy:

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- This concludes the proof. The $M = ||n||_1 < 8\pi$ condition is essential.

Message: Consider the log-HLS inequality

$$K\int_{\mathbb{R}^2} n\log^+ n dx + 2\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\log|x-y|n(y)dxdy \geq C(M).$$

If one can the improve the constant K for $M > 8\pi$, then the critical mass M can be improved. It is because in general,

$$S[n] \leq \frac{E[n_{in}]+C}{1-\frac{K}{8\pi}}.$$

Gluing reconstruction of the log-HLS inequality

- To prove suppression of blow-up, the following log-HLS inequality is essential:

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- To obtain the inequality, we heuristically glue the following two log-HLS inequality together (following [4]):

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- Next we need to show that $K < 8\pi$.

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$$||n||_{L^1(|x_2|\leq\delta)} = 2\int_{S_\delta=\{0\leq x_2\leq\delta\}} ndx \leq 2\int_{|x_2-y_+|\geq\eta\sqrt{V_+/M_+}} ndx \leq \frac{2M_+}{\eta^2}.$$

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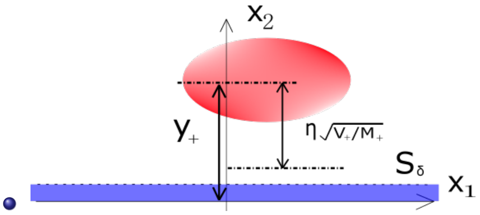

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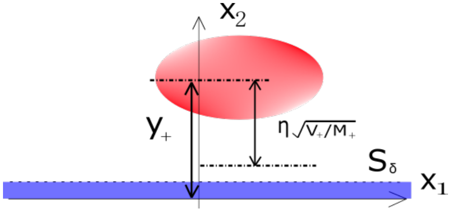

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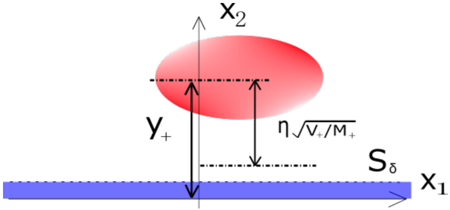
- 

- Next we use moment estimates to show that $\eta(t)$ is approximately constant along (mPKS).
- Therefore, if $\eta(0)$ is large enough, $||n(t)||_{L^1(|x_2|\le\delta)} \ll 8\pi - M/2$, and the suppression of blow-up follows.

Ingredient of the proof: Proof of the theorem

- In the $\eta(0) > \sqrt{2}$ case, the key is to get the log-HLS

$$K \int_{\mathbb{R}^2} n \log^+ n dx + 2 \iint_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \log |x - y| n(y) dx dy \geq C, \quad K < 8\pi.$$

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- The idea is to glue the following three log-HLS inequalities together:

$$M_\pm \int_{\mathbb{R}^2_\pm \backslash S_\delta} n \log^+ n dx + 2 \iint_{(\mathbb{R}^2_\pm \backslash S_\delta)^2} n(x) \log|x-y| n(y) dx dy \geq C,$$

$$\frac{M}{2} \int_{S_\delta} n \log^+ n dx + 2 \iint_{S_\delta \times S_\delta} n(x) \log|x-y| n(y) dx dy \geq C,$$

where $S_\delta := \{x \| |x_2| \leq \delta\}$. Here we use the fact that $\eta(0) > \sqrt{2}$ when we derive the last log-HLS.

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- We dynamically determine the boundaries of these three domains so that the error created during the gluing process is small. Once the gluing process is completed, existence follows.

Summary

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Summary

- Classical PKS equation exhibits blow-up phenomena when the total mass is large $||n_{in}||_{L^1} > 8\pi$;
- We use simple mixing shear flow to suppress the blow-up of (mPKS) on $\mathbb{T}^2$ and $\mathbb{T}^3$;
- We use a fast splitting hyperbolic flow to suppress the blow-up of (mPKS) on the plane $\mathbb{R}^2$.

Thank you!

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