Macroscopic limits and asymptotic behavior of some kinetic models in Astrophysics and Biology

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Outline

- Overview of kinetic models in Astrophysics
- Overview of kinetic theory in Special Relativity
- A BGK model in Special Relativity; asymptotic regimes
- Asymptotics for kinetic equations under non-standard scaling relations
- A two-species kinetic coagulation model; asymptotic behavior

Kinetic models in Astrophysics

There are a number of kinetic descriptions in Astrophysics that stem from the "Liouville equation".

The number $\int_\Omega \int_V f(t,x,v)\, dxdv$ gives the quantity of particles that at time t have a position in $\Omega \subset \mathbb{R}^N$ and velocity in $V \subset \mathbb{R}^N$.

Under the action of a force field F and no short range interactions, we have that

$$\partial_t f + \nabla_x(vf) + \nabla_v(Ff) = 0.$$

The trajectories of the particles are given by

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= F. \end{aligned}$$

Some examples

1) The attractive Vlasov-Poisson system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0, \\ \Delta \phi = \rho(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\, dv, \quad \lim_{|x|\to\infty} \phi(t,x) = 0 \ \forall t \in \mathbb{R}^+ \end{cases}$$

2) The Einstein–Vlasov system:

$$\begin{cases} p^\alpha \partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \partial_{p^\alpha} f = 0, \\ G_{\alpha\beta} = 8\pi T_{\alpha\beta} = 8\pi \int p_\alpha p_\beta |g|^{1/2} f \frac{dp^1 dp^2 dp^3}{-p^0}. \end{cases}$$

What about dissipation?

- These models allow for plenty of steady states and a very rich dynamics. But note that these descriptions neglect close encounters.
- It is important to be able to introduce dissipative effects. For instance, close encounters explain the loss of memory in astrophysical systems on the long time run.

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- It is important to be able to introduce dissipative effects. For instance, close encounters explain the loss of memory in astrophysical systems on the long time run.
- A way to introduce such dissipation effects is to add a Fokker–Planck term (introducing stochasticity).

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathrm{div}_v(\beta v f + \sigma \nabla_v f), \\ \rho(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\, dv, \\ \Delta \phi = \rho(t,x), \quad \lim_{|x| \to \infty} \phi(t,x) = 0 \ \forall t \in \mathbb{R}^+. \end{cases}$$

What about dissipation?

- These models allow for plenty of steady states and a very rich dynamics. But note that these descriptions neglect close encounters.
- It is important to be able to introduce dissipative effects. For instance, close encounters explain the loss of memory in astrophysical systems on the long time run.
- If we reverse the sign of the interactions, another way to introduce dissipation is to consider the Boltzmann equation or model equations for it like the BGK equation.

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \begin{cases} C(f,f) \\ \nu \left(M(f) - f \right) \end{cases}, \\ \rho(t,x) = \int_{\mathbb{R}^3_v} f(t,x,v)\, dv, \\ \Delta \phi = -\rho(t,x), \quad \lim_{|x|\to\infty} \phi(t,x) = 0 \ \forall t \in \mathbb{R}^+. \end{cases}$$

What about dissipation?

- These models allow for plenty of steady states and a very rich dynamics. But note that these descriptions neglect close encounters.
- It is important to be able to introduce dissipative effects. For instance, close encounters explain the loss of memory in astrophysical systems on the long time run.

This is far more delicate in relativistic settings. In fact, most of the literature refers to situations in which gravity is not taken into account (that is, Special Relativity). Even in this simplified setting there are many interesting problems to consider. One of these topics is the study of hydrodynamical limits of kinetic models in special relativity.

To sum up...

Kinetic models (and their hydrodynamic limits) can be used in Astrophysics to describe:

- galaxies
- interstellar plasma
- (relativistic) jets
- stars themselves
- gases of neutrinos
- gases of photons
- ...

Kinetic theory in SR: Some notation conventions

All the gas particles have the same rest mass m.

Space-time coordinates x^μ in four-dimensional Minkowsky's space: $(x^0 = ct, x^1, x^2, x^3)$

Energy-momentum four-vector:

$$q^\mu = (q^0, c\mathbf{q}), \quad q^0 := c\sqrt{(mc)^2 + |\mathbf{q}|^2}.$$

$g_{\mu\nu} = g^{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$.

Greek indices run from zero to three. Einstein's summation convention is used.

The proper volume element $d\mathbf{q}/q^0$ is invariant with respect to Lorentz transformations.

Macroscopic quantities

1. Particle-density four-vector

$$N^\mu(t,x) = \int_{\mathbb{R}^3} q^\mu f(t,x,\mathbf{q}) \frac{d\mathbf{q}}{q^0},$$

2. Energy-momentum tensor

$$T^{\mu\nu}(t,x) = \frac{1}{m} \int_{\mathbb{R}^3} q^\mu q^\nu f(t,x,\mathbf{q}) \frac{d\mathbf{q}}{q^0},$$

3. Entropy four-vector

$$S^\mu(t,x) = -\frac{k_B}{m} \int_{\mathbb{R}^3} q^\mu f(t,x,\mathbf{q}) \log\left(\frac{f(t,x,\mathbf{q})}{\eta}\right) \frac{d\mathbf{q}}{q^0},$$

being k_B the Boltzmann's constant and $\eta = m/h^3$, with h the Planck constant.

Macroscopic quantities

1. The proper particle density n_f

$$n_f = \sqrt{N^\mu N_\mu}.$$

2. The dimensionless velocity four-vector u_f

$$n_f u_f^\mu = N^\mu.$$

3. The proper energy density e_f

$$e_f = (u_f)_\mu (u_f)_\nu T^{\mu\nu}.$$

4. The proper pressure p_f

$$p_f = \frac{1}{3}((u_f)_\mu (u_f)_\nu - g_{\mu\nu}) T^{\mu\nu}.$$

5. The proper entropy density σ_f

$$\sigma_f = S^\mu (u_f)_\mu.$$

The relativistic Boltzmann equation

It can be written in compact form as

$$q^\mu \partial_\mu f^\epsilon = \frac{1}{\epsilon} C(f^\epsilon, f^\epsilon)$$

Here ϵ is the Knudsen number.
The collision kernel satisfies the following identities:

$$\int_{\mathbb{R}^3} (1, q^\mu) C(f, f) \frac{d\mathbf{q}}{q^0} = 0.$$

As a consequence, the following conservation laws hold (particle number, energy and momentum):

$$\frac{\partial N^\mu}{\partial x^\mu} = 0, \quad \frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0.$$

The Jüttner equilibria

The Jüttner distribution is a function describing the state of a gas in equilibrium, depending on five parameters: $n \geq 0$, $\beta > 0$ and $\mathbf{u} \in \mathbb{R}^3$:

$$J(n, \beta, \mathbf{u}; \mathbf{q}) = \frac{n}{(mc)^3 M(\beta)} \exp\left\{ -\frac{\beta}{mc^2} u_\mu q^\mu \right\}.$$

Here $M(\beta)$ is a normalization constant.
Since $J(n, \beta, \mathbf{u}; \mathbf{q})$ is thought of as an equilibrium distribution, we understand that

- n is regarded as its particle density,
- $\mathbf{u}$ as the spatial part of the four-velocity u
- $mc^2/(k_B\beta)$ as the equilibrium temperature.

The relativistic Euler fluid equations

Five relations encoding conservation of the number of particles, energy and momentum. In compact form they read

$$\partial_\mu N^\mu = 0, \quad \partial_\nu T^{\mu\nu} = 0$$

where for a perfect fluid the matter quantities are defined as

$$N^\mu = n\, u^\mu$$

and

$$T^{\mu\nu} = (e + p) u^\mu u^\nu + p (g^{-1})^{\mu\nu}.$$

The relativistic Boltzmann equation: hydrodynamic limit

Theorem

Let $(n, \mathbf{u}, p)$ be a smooth solution to the relativistic Euler equations in some interval $[0, T]$. Under some technical assumptions on the collision kernel and the global Jüttner equilibrium, there exists some $\epsilon_0 > 0$ such that: For each $0 < \epsilon \leq \epsilon_0$ there exists a unique classical solution on $[0, T]$ of the relativistic Boltzmann equation having the form of a Hilbert expansion:

$$f^\epsilon(t, x, \mathbf{q}) = J(n(t, x), \mathbf{u}(t, x), \beta(t, x); \mathbf{q}) + \sum_{n=1}^{6} \epsilon^n f_n + \epsilon^3 f_{R;\epsilon} \geq 0.$$

J. SPECK, R.M. STRAIN, Hilbert expansions from the Boltzmann equation to relativistic fluids, Comm. Math. Phys. 304, (2011) 229–280.

The BGK-Marle model

If no forces are involved, any relativistic phase density should obey a kinetic equation of the form

$$q^\mu \frac{\partial f}{\partial x^\mu} = C(f).$$

The collision term should be determined in such a way that the conservation laws for particle number, energy and momentum hold. Let us construct a simpler model obeying these restrictions, namely, a model having the following form:

$$\partial_t f + \frac{c\mathbf{q}}{\sqrt{(mc)^2 + |\mathbf{q}|^2}} \cdot \nabla_x f = \frac{mc^2\omega}{q^0}(J_f - f).$$

C. Marle, Sur l'établissement des equations de l'hydrodynamique des fluides relativistes dissipatifs, I. L'equation de Boltzmann relativiste, Ann. Inst. Henri Poicaré 10, (1969) 67–127.

A. Bellouquid, J. Calvo, J. Nieto, J. Soler, On the relativistic BGK-Boltzmann model: Asymptotics and Hydrodynamics, Journal of Statistical Physics 149, (2012) 284–316.

Constructing a BGK-Marle model

Proposition:

Let $f = f(\mathbf{q}) \geq 0$ such that N^μ, $T^{\mu\nu}$ exist. Then, we can find a unique set of quantities n, $\mathbf{u}$, β and a function $J_f := J(n, \beta, \mathbf{u}; \mathbf{q})$ such that

$$\int_{\mathbb{R}^3} (1, q^\mu) J_f \frac{d\mathbf{q}}{q^0} = \int_{\mathbb{R}^3} (1, q^\mu) f \frac{d\mathbf{q}}{q^0}.$$

Moreover, for this choice of J_f, a relativistic phase density f verifying

$$\partial_t f + \frac{c\mathbf{q}}{\sqrt{(mc)^2 + |\mathbf{q}|^2}} \cdot \nabla_x f = \frac{mc^2\omega}{q^0}(J_f - f)$$

fulfills the conservation laws $\partial_{x^\mu} N^\mu = 0, \quad \partial_{x^\nu} T^{\mu\nu} = 0$ and an H-theorem:

$$-\frac{\partial S^\mu}{\partial x^\mu} \leq 0.$$

Distinghised limits

Let us choose a scaling such that:

$$\mathbf{q} = \bar{\mathbf{q}}\mu, \quad x = \bar{x}L, \quad t = \bar{t}\tau,$$

where "$\bar{\ }$" stands for dimensionless numbers and μ, L and τ stand for typical microscopic momentum, typical length and typical time, respectively. Also, we perform the change of unknown

$$f(t, x, \mathbf{q}) = \frac{\mathcal{N}}{\mu^3}\bar{f}(\bar{t}, \bar{x}, \bar{\mathbf{q}}),$$

where $\mathcal{N}$ is the typical density. Then, the BGK-Marle model becomes

$$\frac{\partial}{\partial \bar{t}}\bar{f} + \left(\frac{\tau\mu}{mL}\right)\frac{\bar{\mathbf{q}} \cdot \nabla_{\bar{x}}\bar{f}}{\sqrt{1 + \left|\frac{\mu}{mc}\bar{\mathbf{q}}\right|^2}} = (\omega\tau)\frac{1}{\sqrt{1 + \left|\frac{\mu}{mc}\bar{\mathbf{q}}\right|^2}}\left(\bar{J}_{\bar{f}} - \bar{f}\right).$$

Distinghised limits

Limit	Scaling relations			Asymptotics
Non-relativistic	$\mu = \frac{mL}{\tau}$	$\omega\tau = \text{cst}$	$\frac{\eta\mu^3}{\mathcal{N}} = \text{cst}$	$\mu \ll mc$
Hydrodynamic	$\mu = \frac{mL}{\tau}$	$c = \frac{L}{\tau}$	$\frac{\eta\mu^3}{\mathcal{N}} = \text{cst}$	$\frac{1}{\omega} \ll \tau$
Ultra-relativistic	$\omega\tau = \text{cst}$	$c = \frac{L}{\tau}$	$\frac{\eta\mu^3}{\mathcal{N}} = \text{cst}$	$mc \ll \mu$

$$\frac{\partial}{\partial \bar{t}}\bar{f} + \left(\frac{\tau\mu}{mL}\right)\frac{\bar{\mathbf{q}} \cdot \nabla_{\bar{x}}\bar{f}}{\sqrt{1 + \left|\frac{\mu}{mc}\bar{\mathbf{q}}\right|^2}} = (\omega\tau)\frac{1}{\sqrt{1 + \left|\frac{\mu}{mc}\bar{\mathbf{q}}\right|^2}}\left(\bar{J}_{\bar{f}} - \bar{f}\right).$$

The non-relativistic limit

Let $\epsilon = \mu/(mc) \ll 1$. We have that:

- The Jüttner function coincides with the classical Maxwellian up to order $O(\epsilon^2)$.
- The BGK-Marle model coincides with the classical BGK equation
$$\partial_t f + v \cdot \nabla_x f = \nu \left(M(f) - f \right)$$
up to order $O(\epsilon^2)$.
- The entropy σ_f coincides with the non-relativistic one up to order $O(\epsilon^2)$.

The hydrodynamic limit

Let $\epsilon = \omega\tau \ll 1$. Let $f^{\epsilon}_{Eu}(t, x, \mathbf{q})$ be a sequence of solutions of the equation

$$\partial_t f^{\epsilon}_{Eu} + \frac{\mathbf{q}}{\sqrt{1+\mathbf{q}^2}} \cdot \nabla_x f^{\epsilon}_{Eu} = \frac{1}{\epsilon\sqrt{1+\mathbf{q}^2}}(J_{f^{\epsilon}_{Eu}} - f^{\epsilon}_{Eu})$$

with nonnegative initial condition $0 \le f^{\epsilon}_{Eu}(0, x, \mathbf{q}) = f^{\epsilon,0}_{Eu}(x, \mathbf{q})$. Assume that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \left(1+\sqrt{1+|\mathbf{q}|^2}+|x|+|\log\left(f^{\epsilon,0}_{Eu}(x,\mathbf{q})\right)|\right) f^{\epsilon,0}_{Eu}(x,\mathbf{q})d\mathbf{q}dx < \infty.$$

Assume also that $f^{\epsilon}_{Eu}(t, x, \mathbf{q})$ converges almost everywhere to $f_{Eu}(t, x, \mathbf{q})$, as ϵ goes to zero.

The hydrodynamic limit

Let $\epsilon = \omega\tau \ll 1$. Let $f^{\epsilon}_{Eu}(t, x, \mathbf{q})$ be a sequence of solutions of the equation

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with nonnegative initial condition $0 \leq f^{\epsilon}_{Eu}(0, x, \mathbf{q}) = f^{\epsilon,0}_{Eu}(x, \mathbf{q})$.

Theorem

Let the assumptions on the previous slide hold. Then,

- the pointwise limit $f_{Eu}(t, x, \mathbf{q})$ is a Jüttner distribution
- its moments are the corresponding limits of those of f^{ϵ}_{Eu}
- the functions $n_{f_{Eu}}, \mathbf{u}_{f_{Eu}}$ and $\beta_{f_{Eu}}$ associated with f_{Eu} solve the relativistic Euler equations.

The kinetic equation of state

To close the relativistic Euler system we add an equation of state. If we are performing the hydrodynamic limit of a kinetic model having Jüttner functions as equilibria, the right choice for the equation of state is obtained extrapolating the relations that hold between the thermodynamic quantities of these Jüttner functions ("kinetic equation of state"):

$$p = mc^2 \frac{n}{\beta}, \tag{1}$$

$$e = mc^2 n \frac{K_1(\beta)}{K_2(\beta)} + 3p, \tag{2}$$

$$n = 4\pi e^4 m^3 c^3 h^{-3} \exp\left(\frac{-\sigma}{k_B}\right) \frac{K_2(\beta)}{\beta} \exp\left(\beta \frac{K_1(\beta)}{K_2(\beta)}\right). \tag{3}$$

(Speck, Strain, Synge,...)

The kinetic equation of state

Let the speed of sound be defined as

$$c_S := c\sqrt{\frac{\partial p}{\partial e}}.$$

Theorem

Under the kinetic equation of state (1)–(3), there holds that:

1. The relativistic Euler system is hyperbolic.
2. The relativistic Euler system is causal (the speed of sound is real and less than the speed of light).
3. The speed of sound verifies $0 < c_S < c/\sqrt{3}$.

J. SPECK, R.M. STRAIN, Hilbert expansions from the Boltzmann equation to relativistic fluids, Comm. Math. Phys. 304, (2011) 229–280.

J. CALVO, On the hyperbolicity and causality of the relativistic Euler system under the kinetic equation of state, Comm. Pure Appl. Anal. 12, (2013) 1341–1347.

High and low field limits: The case of Vlasov–Poisson–Fokker–Planck

Let us consider the Vlasov–Poisson–Fokker–Planck model:

$$\partial_t f + \alpha v \partial_x f - \frac{1}{\beta}\partial_x \phi \, \partial_v f = \frac{\alpha}{\beta}\,\partial_v (v\,f + \partial_v f), \quad -\partial^2_{xx}\phi \; = \rho.$$

Scaling parameters:

$$\beta = \frac{\text{mean free path}}{\text{typical lengthscale}}$$

$$\alpha = \frac{\text{thermal velocity}}{\text{macroscopic mean velocity}}$$

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Low field limit (parabolic regime): $\beta = \epsilon \to 0$ and $\alpha = 1/\epsilon$. The limiting density ρ verifies a drift–diffusion equation

$$\partial_t \rho - \partial_x (\rho \, \partial_x \phi) + \partial_{xx}^2 \rho = 0, \qquad -\partial_{xx}^2 \phi = \rho,$$

while the microscopic distribution function behaves like a Gaussian function in the velocity variable

$$f_\epsilon(t, x, v) \to \rho(t, x) \frac{e^{-\frac{|v|^2}{2}}}{\sqrt{2\pi}} \in Ker[L_{FP}].$$

F. Poupaud, J. Soler, Parabolic Limit and stability of the Vlasov-Poisson-Fokker-Planck system, Math. Mod. Meth. Appl. Sci. 10 (2000) 1027-1045.

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$$\partial_t f + \alpha v \partial_x f - \frac{1}{\beta}\partial_x \phi \, \partial_v f = \frac{\alpha}{\beta}\, \partial_v (v\, f + \partial_v f), \quad -\partial^2_{xx}\phi \; = \rho.$$

High field limit (hyperbolic regime): $\beta = \epsilon \to 0$ and $\alpha = 1$. The limiting equation for the density ρ incorporates no diffusion term

$$\partial_t \rho - \partial_x(\rho\, \partial_x \phi) = 0, \qquad -\partial^2_{xx}\phi = \rho,$$

while the microscopic distribution function resembles a Gaussian centered at $-\partial_x\phi$:

$$f_\epsilon(t,x,v) \to \rho(t,x)\, \frac{e^{-\frac{|v+\partial_x\phi|^2}{2}}}{\sqrt{2\pi}}.$$

J. NIETO, F. POUPAUD, J. SOLER, High-Field Limit for the Vlasov-Poisson-Fokker-Planck system, Arch. Rat. Mech. Anal. 158 (2001) 29-59.

More about macroscopic limits: intermediate scalings

Given a kinetic model, we introduce a general intermediate scaling depending on a parameter $\gamma \in [0, 1]$, which covers the parabolic and the hyperbolic behaviors as particular cases:

$$\epsilon^{1+\gamma}\,\partial_t f_\epsilon + \epsilon\, v \partial_x f_\epsilon - \epsilon^\gamma\,\partial_x \phi_\epsilon\,\partial_v f_\epsilon = L[f_\epsilon]\,,$$

$$-\partial^2_{xx}\phi_\epsilon \;= \rho_\epsilon, \quad \rho_\epsilon(t,x) := \int_{\mathbb{R}} f_\epsilon(t,x,v)\,dv.$$

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$$-\partial^2_{xx}\phi_\epsilon \;= \rho_\epsilon, \quad \rho_\epsilon(t,x) := \int_{\mathbb{R}} f_\epsilon(t,x,v)\,dv.$$

We assume that L satisfies, for $d\mu = (1+v^2)dv$:

- $L : L^1(\mathbb{R}, d\mu) \to W^{-2,\infty}(\mathbb{R})$ is a continuous linear map.
- Mass conservation: $\int_{\mathbb{R}} L[f]dv = 0$, for all $f \in L^1(\mathbb{R}, d\mu)$.
- Friction: $\int_{\mathbb{R}} vL[f]dv = -k_1 \int_{\mathbb{R}} vfdv$, for all $f \in L^1(\mathbb{R}, d\mu)$.
- Energy dissipation: $\int_{\mathbb{R}} v^2 L[f]dv = k_2 \int_{\mathbb{R}} fdv - k_3 \int_{\mathbb{R}} v^2 fdv$, for all $f \in L^1(\mathbb{R}, d\mu)$.

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$$\epsilon^{1+\gamma}\, \partial_t f_\epsilon + \epsilon\, v \partial_x f_\epsilon - \epsilon^{\gamma}\, \partial_x \phi_\epsilon\, \partial_v f_\epsilon = L[f_\epsilon]\,,$$

$$-\partial^2_{xx}\phi_\epsilon \;= \rho_\epsilon, \quad \rho_\epsilon(t,x) := \int_{\mathbb{R}} f_\epsilon(t,x,v)\, dv.$$

A similar idea was considered by Bardos-Golse-Levermore for the Boltzmann equation:

$$\epsilon \partial_t f_\epsilon + v \partial_x f_\epsilon = \frac{1}{\epsilon^q} C(f_\epsilon), \quad q \geq 1.$$

For $q = 1$ and $\epsilon \to 0$, Navier-Stokes is obtained.
For $q > 1$ and $\epsilon \to 0$, incompressible Euler is obtained.

C. BARDOS, F. GOLSE, D. LEVERMORE, Fluid dynamic limits of kinetic equations I. Formal derivations, J. Stat. Phys. 63 (1991) 323–344.

More about macroscopic limits: intermediate scalings

Theorem

Let f_ϵ be a solution of

$$\epsilon^{1+\gamma}\,\partial_t f_\epsilon + \epsilon\, v\partial_x f_\epsilon - \epsilon^\gamma\,\partial_x\phi_\epsilon\,\partial_v f_\epsilon = L[f_\epsilon] \qquad -\partial^2_{xx}\phi_\epsilon \; = \rho_\epsilon,$$

such that some mild boundedness and continuity assumptions hold. Then, the associated density ρ_ϵ converges to some density ρ in $C^0(0,T;\mathcal{M}(\mathbb{R})-\text{weak}^*)$ verifying

$$\partial_t\rho - \partial_x(\rho\,\partial_x\phi) = 0, \qquad -\partial^2_{xx}\phi = \rho,$$

and the distribution function f_ϵ converges in a distributional sense to $f_0 \in Ker(L)$.

A. BELLOUQUID, J. CALVO, J. NIETO, J. SOLER, Hyperbolic vs parabolic asymptotics in kinetic theory towards fluid dynamic models, SIAM J. Appl. Math. 73 (2013) 1327–1346.

Vlasov–Poisson–Fokker–Planck revisited

Consider $\gamma \in (0,1)$ and a sequence of solutions to

$$\varepsilon^{1+\gamma}\,\partial_t f_\epsilon + \epsilon\, v\partial_x f_\epsilon - \varepsilon^{\gamma}\,\partial_x\phi_\epsilon\,\partial_v f_\epsilon = \partial_v(v\,f + \partial_v f), \qquad -\partial^2_{xx}\phi_\epsilon \; = \rho_\epsilon .$$

The limit density satisfies the equation

$$\partial_t\rho - \partial_x(\rho\,\partial_x\phi) = 0, \qquad -\partial^2_{xx}\phi = \rho,$$

and there holds that

$$f_\epsilon(t,x,v) - \rho(t,x)\,\frac{e^{-\frac{|v+\varepsilon^{\gamma}\partial_x\phi|^2}{2}}}{\sqrt{2\pi}} \to 0.$$

Coagulation and aggregation phenomena

Many physical phenomena consist of many small particles that can stick together to form aggregates or new particles of larger size. To name a few:

- polymerization
- coalescence of aerosols (liquid or solid particles suspended in a gas)
- colloidal crystallization

Such processes do also show up in biological contexts (adhesion of cells, bacteria, etc)

Coagulation and aggregation phenomena

Some ways in which coagulation phenomena can be described:

- discrete models: Smoluchowski, Becker–Döring
- continuous models (may include also the spatial distribution): Lifshitz–Slyozov,...
- *kinetic models*

Departure from the standard viewpoint: Biological “particles” already exchange momentum with the environment. Most of the times the total momentum after adhesion is not preserved.

In fact, particles may completely stop moving after adhesion, and the resulting aggregate may behave in a functionally different way than that of the free particles (i.e. a different “state”).

A two-species kinetic model

There is real need for models that do not impose conservation of momentum!

We propose a mesoscopic kinetic description that tries to codify the common features to all these biological processes.

- $f : [0, T] \times \mathbb{R}^d_x \times \mathbb{R}^d_v \to \mathbb{R}$ free particles.
- $\rho : [0, T] \times \mathbb{R}^d_x \to \mathbb{R}$ coagulated or stuck particles.
- Supplied with initial data $0 \leq f_0(x, v) \in (L^1 \cap L^\infty)(\mathbb{R}^{2d})$ and $0 \leq \rho_0(x) \in (L^1 \cap L^\infty)(\mathbb{R}^d_x)$.

J. CALVO, P.-E. JABIN, Large time asymptotics for a modified coagulation model, J. Diff. Eq. 250 (2011) 2807–2837.

The equation for the free species

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' - \beta(v)\rho(t,x) f(t,x,v)$$

- $\alpha(v,v')$ gives the probability for two free particles with velocities v and v' to coagulate.
- $\beta(v)$ gives the probability for one free particle with velocity v to coagulate with a stuck particle.

The equation for the stuck species

$$\frac{\partial \rho}{\partial t} = \int_{\mathbb{R}^{2d}} \alpha(v, v') f(t, x, v') f(t, x, v)\, dv' dv$$

$$+\rho(t, x) \int_{\mathbb{R}^d_v} \beta(v) f(t, x, v)\, dv,$$

- $\alpha(v, v')$ gives the probability for two free particles with velocities v and v' to coagulate.
- $\beta(v)$ gives the probability for one free particle with velocity v to coagulate with a stuck particle.

The kinetic coagulation model

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \displaystyle\int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' \\ \qquad\qquad -\beta(v)\rho(t,x) f(t,x,v), \\ \dfrac{\partial \rho}{\partial t} = \displaystyle\int_{\mathbb{R}^{2d}} \alpha(v,v') f(t,x,v') f(t,x,v)\, dv' dv \\ \qquad\qquad + \rho(t,x) \displaystyle\int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv. \end{cases}$$

For most of the problems of interest additional terms would be needed.

We wish to study the common aggregation structure to all these models (in any spatial dimension $d \geq 1$).

About the kernels

Kernels are usually too complicated in real applications. Their exact expression is however not needed for our purposes, but only the way in which they behave.

- In most physical situations coagulation kernels behave polynomially.

- We assume then, for some $a \in \mathbb{R}$:
  - $c_1 |v - v'|^a \leq \alpha(v, v') \leq C_1 |v - v'|^a$,
  - $c_2 |v|^a \leq \beta(v) \leq C_2 |v|^a$.

Long time behavior: Mass transfer

$$
\begin{cases}
\dfrac{\partial f}{\partial t} + v \cdot \nabla_x f = -f(t,x,v) \displaystyle\int_{\mathbb{R}^d} \alpha(v,v') f(t,x,v')\, dv' \\
\qquad\qquad -\beta(v)\rho(t,x) f(t,x,v), \\
\dfrac{\partial \rho}{\partial t} = \displaystyle\int_{\mathbb{R}^{2d}} \alpha(v,v') f(t,x,v') f(t,x,v)\, dv' dv \\
\qquad\qquad + \rho(t,x) \displaystyle\int_{\mathbb{R}^d_v} \beta(v) f(t,x,v)\, dv.
\end{cases}
$$

The total mass of the system is preserved.

Mass is transferred from f to ρ.

Sketch of the asymptotic behavior

It is essentially determined by collisions at low relative velocity (escaping speedy particles)

It is completely determined solely in terms of d and a.

- If $-d < a \leq 1 - d$ then the whole free species will eventually become stuck.
- If $a > 1 - d$ then a fraction of free particles are able to escape interactions with the stuck species. The long time behavior of f is self-similar.

J. CALVO, P.-E. JABIN, Large time asymptotics for a modified coagulation model, J. Diff. Eq. 250 (2011) 2807–2837.

Recap

- A BGK-type model in Special Relativity was constructed and its hydrodynamic limit was analyzed.
- Macroscopic limits of kinetic models under non-standard scaling relations were discussed.
- A two-species kinetic coagulation model was proposed and its long time behavior was analyzed.