Asymptotic stability of IMEX schemes for stiff hyperbolic PDE's

Sebastian Noelle, RWTH Aachen

joint with

Jochen Schütz, Hamed Zakerzadeh, Klaus Kaiser,

Georgij Bispen, Maria Lukacova, Claus-Dieter Munz, K R Arun

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Isentropic gas dynamics

Dimensionless conservation laws for mass and momentum:

$$\partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0,$$
$$\partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla p(\rho) = 0.$$

$p = p(\rho) = \text{pressure.}$ $c_{\mathrm{ref}} = \sqrt{p'(\rho_{\mathrm{ref}})} = \text{reference sound speed.}$

$$\varepsilon = \frac{u_{\mathrm{ref}}}{c_{\mathrm{ref}}} = \text{Mach number}$$

Zero Mach number limit

Asymptotic expansion (for general $f(\mathbf{x}, t; \varepsilon)$):

$$f(\mathbf{x}, t) = f^{(0)}(\mathbf{x}, t) + \varepsilon f^{(1)}(\mathbf{x}, t) + \varepsilon^2 f^{(2)}(\mathbf{x}, t) \dots$$

gives to leading order

$$\rho = \rho^{(0)}(t) + \varepsilon^2 \rho^{(2)}(\mathbf{x}, t)$$

Leading order equations

Constraints for $\rho^{(0)}$ and $\nabla \cdot \mathbf{u}^{(0)}$:

$$(\nabla \cdot \mathbf{u}^{(0)})(t) = \frac{1}{|\Omega|} \int_{\partial\Omega} \mathbf{u}_{\mathbf{bdry}}^{(0)} \cdot \mathbf{n} dS(\mathbf{x})$$

$$\frac{d}{dt} \rho^{(0)}(t) = -\rho^{(0)}(t) \, (\nabla \cdot \mathbf{u}^{(0)})(t)$$

Newton's law for $\mathbf{u}^{(0)}$:

$$\partial_t \left(\rho^{(0)} \mathbf{u}^{(0)} \right) + \rho^{(0)} \nabla \cdot \left(\mathbf{u}^{(0)} \otimes \mathbf{u}^{(0)} \right) + \nabla p^{(2)} = 0$$

Incompressible equations

Assumption: zero net flux across the boundary.

Consequence: $\rho^{(0)}$ constant, $\mathbf{u}^{(0)}$ divergence free.

Incompressible Euler (Klainerman/Majda 1981)

$$\partial_t \mathbf{u}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} + \frac{\nabla p^{(2)}}{\rho^{(0)}} = 0$$

Elliptic constraint

$$\nabla \cdot \left(\mathbf{u}^{(0)} \cdot \nabla \mathbf{u}^{(0)} \right) + \frac{\Delta p^{(2)}}{\rho^{(0)}} = 0$$

Challenge I: stiffness for small Mach number

Propagation speeds in direction $\mathbf{n}$ ($u_n = \mathbf{u}\cdot\mathbf{n},\ c = \sqrt{p'(\rho)}$):

$$u_n - \frac{c}{\varepsilon},\ u_n,\ u_n + \frac{c}{\varepsilon}.$$

explicit schemes: inefficient ($\Delta t = O(\varepsilon\,\Delta x)$)

implicit schemes: excessively diffusive on advection wave

IMEX schemes: clever mix (Jin, Degond, ...)

Challenge II: asymptotic behavior as $M \to 0$

Challenges:

Asymptotic consistency: for a sequence of well-prepared initial data, the numerical scheme should follow the low Mach number asymptotics

Asymptotic stability: the CFL number should be independent of ε

Admissible Splittings

Definition

A splitting

$$A = \widetilde{A} + \widehat{A}.$$

is **admissible**, if

(i) both $\widetilde{A}$ and $\widehat{A}$ induce a hyperbolic system

(ii)

$$\widetilde{\lambda} := \rho(\widetilde{A}) = O\left(\frac{1}{\varepsilon}\right)$$

$$\widehat{\lambda} := \rho(\widehat{A}) = O(1)$$

CFL Conditions

$$\nu := \lambda_{max} \frac{\Delta t}{\Delta x} \qquad \text{full CFL number}$$

$$\widehat{\nu} := \widehat{\lambda} \frac{\Delta t}{\Delta x} \qquad \text{nonstiff CFL number}$$

$\nu = O(1)$	$\Rightarrow$	$\widehat{\nu} = O(\varepsilon)$	stable	inefficient
$\nu = O\left(\frac{1}{\varepsilon}\right)$	$\Leftarrow$	$\widehat{\nu} = O(1)$	unstable	efficient

Flux-Splitting & IMEX Time-Discretization

Implicit-explicit discretization

Klein 1996
Degond, Tang 2011
Haack, Jin, Liu 2011

$$U^{n+1} = U^n + \widetilde{A}U_x^{n+1} + \widehat{A}U_x^n$$

Plan of the talk

Noelle, Bispen, Arun, Lukacova, Munz — SISC 2014

- splitting A unstable

Bispen, Arun, Lukacova, Noelle — CiCP 2014

- splitting B stable

Schütz, Noelle — JSC 2014

- linear stability theory

Schütz, Kaiser, Noelle, Zakerzadeh — (submitted 2015)

- RS-IMEX splitting

Euler equations

$$U_t + \nabla \cdot F(U) = 0,$$

$$U = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho \mathbf{u}^T \\ \rho \mathbf{u} \otimes \mathbf{u} + \frac{p}{\varepsilon^2} \mathbf{I} \\ (\rho E + p) \mathbf{u}^T \end{pmatrix},$$

Total energy ρE and equation of state:

$$p = (\gamma - 1)\left(E - \frac{\varepsilon^2}{2} \rho |\mathbf{u}|^2 \right),$$

Splitting A (Klein 1995)

$$F(U) = \widetilde{F}(U) + \widehat{F}(U),$$

where

$$\widetilde{F}(U) = \begin{pmatrix} 0 \\ \frac{1-\varepsilon^2}{\varepsilon^2} p\,\mathbf{I} \\ (p - \Pi)\mathbf{u}^T \end{pmatrix}, \quad \widehat{F}(U) = \begin{pmatrix} \rho\mathbf{u}^T \\ \rho\mathbf{u} \otimes \mathbf{u} + p\,\mathbf{I} \\ (\rho E + \Pi)\mathbf{u}^T \end{pmatrix}.$$

Auxiliary pressure

$$\Pi(\mathbf{x}, t) := \varepsilon^2 p(\mathbf{x}, t) + (1 - \varepsilon^2) p_\infty(t),$$

Reference pressure

$$p_\infty(t) = \inf_{\mathbf{x}} p(\mathbf{x}, t)$$

Eigenvalues of subsystems

Eigenvalues of $\widetilde{A} := \widetilde{F}'(U) \cdot \mathbf{n}$

$$\widetilde{\lambda} = 0, \; \pm \frac{1-\varepsilon^2}{\varepsilon} \left(\frac{(\gamma - 1)(p - p_\infty)}{\rho} \right)^{1/2}$$

hyperbolicity. fast and slow waves. implicit timestep.

Eigenvalues of $\widehat{A} := \widehat{F}'(U) \cdot \mathbf{n}$

$$\widehat{\lambda} = u_n, \; u_n \pm c^*$$

hyperbolicity. only slow waves. explicit timestep.

Numerical experiment

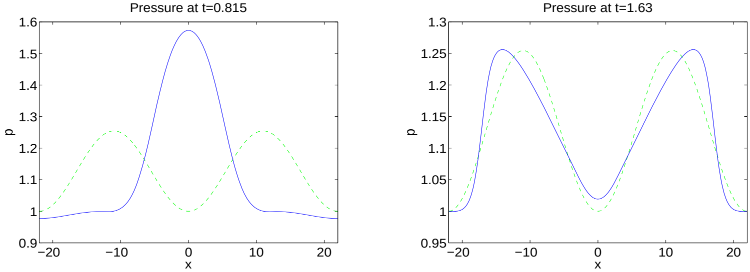
- two colliding acoustic pulses (Klein 1995)
- weakly compressible

$$\rho(x,0) = \rho_0 + \frac{1}{2}\varepsilon\rho_1\left(1-\cos\left(\frac{2\pi x}{L}\right)\right), \ \ \rho_0 = 0.955, \ \ \rho_1 = 2.0,$$

$$u(x,0) = \frac{1}{2}u_0 \ \mathrm{sign}(x)\left(1-\cos\left(\frac{2\pi x}{L}\right)\right), \ \ u_0 = 2\sqrt{\gamma},$$

$$p(x,0) = p_0 + \frac{1}{2}\varepsilon p_1\left(1-\cos\left(\frac{2\pi x}{L}\right)\right), \ \ p_0 = 1.0, \ \ p_1 = 2\gamma.$$

Stability for $\varepsilon = 1/11$



- two colliding pressure pulses
- $\varepsilon = 1/11$, $\widehat{\nu} = 0.9, \nu = 9.9$
- stabilization constant $c_{stab} = 1/12$

Instability for $\varepsilon = 0.01$

difficulty:

- instability for $\varepsilon = 0.01$
- IMEX scheme needs reduced CFL number, $\widehat{\nu} < 0.02$

first fix:

- high order pressure stabilization in elliptic equation
- asymptotic consistency only for $\Delta t = O(\varepsilon^{2/3})$

Shallow water equations

$$U_t + \nabla \cdot F(U) = S(U)$$

$$U = \begin{pmatrix} z \\ h\mathbf{u} \end{pmatrix}, \quad F(U) = \begin{pmatrix} h\mathbf{u}^T \\ h\mathbf{u} \otimes \mathbf{u} \end{pmatrix} + \frac{z^2 - 2zb}{2\varepsilon^2} \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}, \quad S(U) = -\frac{z}{\varepsilon^2} \begin{pmatrix} 0 \\ \nabla^T b \end{pmatrix}$$

with

b	bottom topography
z	water surface
$h = z - b$	water height
$\mathbf{u} = (u, v)$	horizontal velocity
$\varepsilon = \frac{u_{ref}}{\sqrt{gh_{ref}}}$	Froude number

Splitting B (Restelli, Giraldo 2009)

Linearize around $z = 0$, $\mathbf{u} = 0$ (lake at rest):

$$F(U) = \widetilde{F}(U) + \widehat{F}(U),$$
$$S(U) = \widetilde{S}(U) + \widehat{S}(U),$$

where

$$\widetilde{F}(U) = \begin{pmatrix} h\mathbf{u}^T \\ 0 \end{pmatrix} - \frac{bz}{\varepsilon^2}\begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}, \qquad \widetilde{S}(U) = S(U),$$

$$\widehat{F}(U) = \begin{pmatrix} 0 \\ h\mathbf{u} \otimes \mathbf{u} \end{pmatrix} + \frac{z^2}{2\varepsilon^2}\begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix}, \qquad \widehat{S}(U) = 0.$$

Eigenvalues of subsystems

Eigenvalues of $\widetilde{A} := \widetilde{F}'(U)$

$$\widetilde{\lambda} = 0, \pm \frac{1}{\varepsilon} \sqrt{|b|}$$

hyperbolicity. fast and slow waves. implicit timestep.

Eigenvalues of $\widehat{A} := \widehat{F}'(U) \cdot \mathbf{n}$

$$\widehat{\lambda} = 0, u_n, 2u_n$$

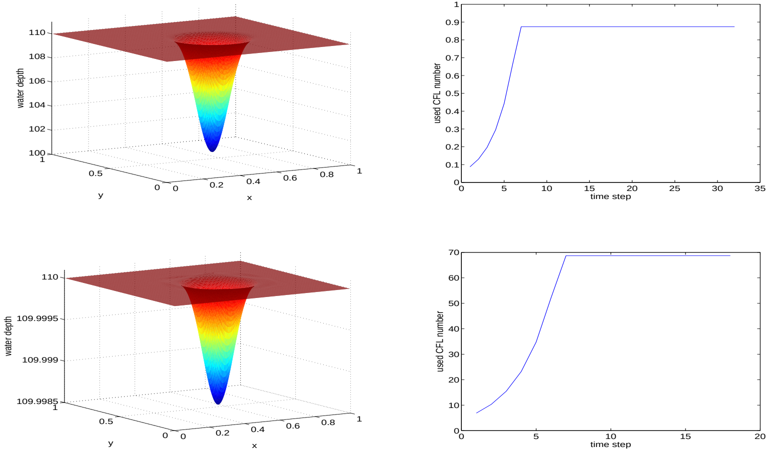
hyperbolicity. only slow waves. explicit timestep.

Numerical experiment

- compactly supported smooth vortex
- transported to the right

$$
\begin{aligned}
h(x,y,0) &= 110 + \left(\frac{\varepsilon\Gamma}{\omega}\right)^2 (k(\omega r_c) - k(\pi)) \\
u(x,y,0) &= 0.6 + \Gamma(1+\cos(\omega r_c))(0.5-y) && \text{if } \omega r_c \le \pi \\
v(x,y,0) &= \Gamma(1+\cos(\omega r_c))(x-0.5) && \text{if } \omega r_c \le \pi
\end{aligned}
$$

Bispen 2014



- vortex, $\varepsilon = 0.8$ (top) and $\varepsilon = 0.01$ (bottom)
- Asymptotic Stability

ε-uniform convergence

Travelling Vortex, L^1-errors and order of convergence in z

	$eps = 0.8$		$eps = 0.05$		$eps = 0.01$	
	error	eoc	error	eoc	error	eoc
20	7.16e-2		1.51e-3		1.35e-4	
40	1.72e-2	2.05	3.07e-4	2.30	4.28e-5	1.65
80	3.68e-3	2.23	5.36e-5	2.51	6.37e-6	2.75
160	9.79e-4	1.91	1.51e-5	1.82	8.20e-7	2.96

- $\hat{\nu} = 0.45$, $\nu = 0.9, 7.2, 35$.

Modified equation, cf. Warming/Hyett 1974

Theorem (Noelle, Schütz 2014)

The modified equation of the IMEX scheme is

$$w_t + A w_x = \frac{\Delta t}{2} C \, w_{xx}$$

with diffusion matrix

$$C := (\widehat{\alpha} + \widetilde{\alpha}) \, \frac{\Delta x}{\Delta t} \, \mathbf{I} - (\widehat{A} - \widetilde{A})(\widehat{A} + \widetilde{A})$$

and numerical viscosities $\widehat{\alpha}$, $\widetilde{\alpha}$.

the crucial commutator

Is C positive definite?

$$
\begin{aligned}
C &= \left((\widehat{\alpha} + \widetilde{\alpha}) \frac{\Delta x}{\Delta t} \mathbf{I} - \widehat{A}^2 \right) &+& \left(\widetilde{A}\widehat{A} - \widehat{A}\widetilde{A} \right) &+& \widetilde{A}^2 \\
&= O(1) &+& O\left(\frac{1}{\varepsilon}\right) &+& O\left(\frac{1}{\varepsilon^2}\right)
\end{aligned}
$$

Yes, if commutator $[\widetilde{A}, \widehat{A}] = 0$

Example

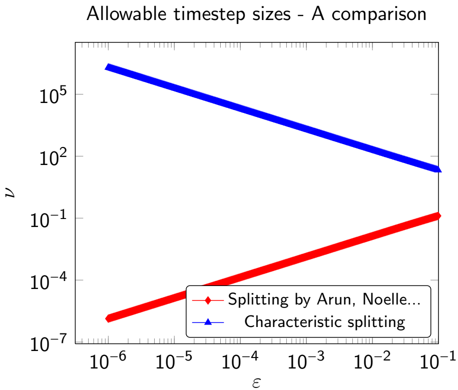
Fourier stability analysis for prototype system

$$A = \begin{pmatrix} a & 1 & 0 \\ \frac{1}{\varepsilon^2} & a & \frac{1}{\varepsilon^2} \\ 0 & 1 & a \end{pmatrix}$$

$a > 0$, eigenvalues

$$\lambda = a, a \pm \frac{\sqrt{2}}{\varepsilon}$$

Euler: classical versus characteristic splitting



Comparison of classical versus characteristic splitting

How to recover stability

- Need e.g.

$$\widetilde{A}\widehat{A} - \widehat{A}\widetilde{A} = O(1)$$

  or

$$\widehat{A} = O(\varepsilon)$$

- Characteristic splitting is not possible in multi-D
- We need a nice piece of luck!!

Reference-Solution IMEX

Nonlinear hyperbolic system of balance laws

$$\partial_t U(x,t;\varepsilon) + \nabla \cdot F(U,x,t;\varepsilon) = S(U,x,t;\varepsilon)$$

with

$$U : \mathbb{R}^d \times \mathbb{R}_+ \times (0,1] \to \mathbb{R}^m, \quad (x,t;\varepsilon) \mapsto U(x,t;\varepsilon)$$

- Challenge: Stiffness as $\varepsilon \to 0$
- Goal: Asymptotic stability

Reference solution and scaled perturbation: $U = \widetilde{U} + D\,V$

$$\begin{array}{lccccc} \widetilde{U}: & \mathbb{R}^d \times \mathbb{R}_+ & \to \mathbb{R}^m, & (x,t) & \mapsto & \widetilde{U}(x,t) \\ V: & \mathbb{R}^d \times \mathbb{R}_+ \times (0,1] & \to \mathbb{R}^m, & (x,t;\varepsilon) & \mapsto & U(x,t;\varepsilon) \end{array}$$

and

$$D = \operatorname{diag}(\varepsilon^{k_1}, \ldots, \varepsilon^{k_m})$$

Taylor expansion with remainder of F and S around $\widetilde{U}$:

$$F = F(\widetilde{U}) + A(\widetilde{U})\; D V + \bar{F}(\widetilde{U}, V) = D\big(\widetilde{G} + \widehat{G} + \bar{G}\big)$$

$$S = S(\widetilde{U}) + \widetilde{S}' V\, D V + \bar{S}(\widetilde{U}, V) = D\big(\widetilde{Z} + \widehat{Z} + \bar{Z}\big)$$

Reference solution and scaled perturbation: $U = \widetilde{U} + D\,V$

$$\begin{array}{llllll} \widetilde{U}: & \mathbb{R}^d \times \mathbb{R}_+ & \to \mathbb{R}^m, & (x,t) & \mapsto & \widetilde{U}(x,t) \\ V: & \mathbb{R}^d \times \mathbb{R}_+ \times (0,1] & \to \mathbb{R}^m, & (x,t;\varepsilon) & \mapsto & U(x,t;\varepsilon) \end{array}$$

and

$$D = \mathrm{diag}(\varepsilon^{k_1}, \dots, \varepsilon^{k_m})$$

Taylor expansion with remainder of F and S around $\widetilde{U}$:

$$F = F(\widetilde{U}) + A(\widetilde{U})\ DV + \bar{F}(\widetilde{U}, V) = D\big(\widetilde{G} + \widehat{G} + \bar{G}\big)$$

$$S = S(\widetilde{U}) + \widetilde{S}'V\,DV + \bar{S}(\widetilde{U}, V) = \underbrace{D\big(\widetilde{Z} + \widehat{Z} + \bar{Z}\big)}_{RS+IM+EX}$$

Theorem (Modified equation for RS-IMEX (N. 2014))

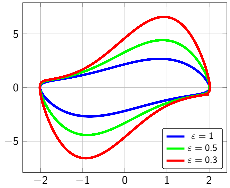
$$B_0 W_t = -\nabla \cdot B_1 + B_2 + \nabla \cdot (B_3 \cdot \nabla W)$$

with

$$B_0 := I - \frac{\Delta t}{2}(\widehat{Z}' - \bar{Z}'),$$
$$B_1 := \widehat{G} + \bar{G} + \frac{\Delta t}{2}\Big((\widehat{G}' - \bar{G}')(\widehat{Z}' + \bar{Z}' - \widehat{G}_x - \bar{G}_x)\Big),$$
$$B_2 := \widehat{Z} + \bar{Z} + \frac{\Delta t}{2}(\widehat{Z}_t - \bar{Z}_t),$$
$$B_3 := \frac{(\widehat{\alpha} + \bar{\alpha})\Delta x}{2} I + \frac{\Delta t}{2}(\widehat{G}' - \bar{G}')(\widehat{G}' + \bar{G}').$$

Study this for each application!

van der Pol and IMEX (Schütz, Kaiser 2015)



- Prototype example

$$\begin{pmatrix} y' \\ z' \end{pmatrix} = \begin{pmatrix} z \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix}.$$

- 'Traditional' splitting: $\begin{pmatrix} 0 \\ \frac{g(y,z)}{\varepsilon} \end{pmatrix} + \begin{pmatrix} z \\ 0 \end{pmatrix}$

van der Pol and IMEX

- 'Reference solution' (RS) $\varepsilon \to 0$:

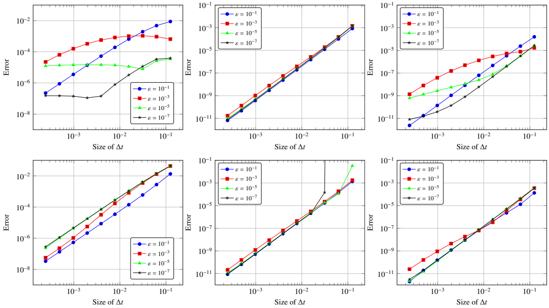
$$\begin{pmatrix} y'_{(0)} \\ 0 \end{pmatrix} = \begin{pmatrix} z_{(0)} \\ g(y_{(0)}, z_{(0)}) \end{pmatrix}.$$

- *RS-IMEX* splitting based on $w_{(0)}$:

$$f(w) = f(w_{(0)}) + f'(w_{(0)})(w - w_{(0)}) + \text{Rest}$$

- Motivation: $w - w_{(0)} = O(\varepsilon)$.

RS-IMEX + Runge-Kutta



(Left to right) DPA-242, BHR-553, BPR-353. (Top to bottom) Standard / RS-IMEX

- IMEX Runge-Kutta (Pareschi, Russo, Boscarino ...)
- standard splitting looses convergence order
- RS-IMEX gives full order of accuracy

Outlook

IMEX

- Examples of uniform CFL stability and stability
- Linearized stability analysis

RS-IMEX

- A natural approach to stiff / non-stiff splitting
- Improves stability of IMEX schemes

To do

- Extend RS-IMEX to many more systems
- Test stability and efficiency
- Do rigorous stability analysis for modified equation
- Higher order accuracy
- Real life applications