The random heat equation in $d \geq 3$

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Outline

0. Introduction: a compote of things/buying time

I. KPZ in $d = 1$ vs. $d \geq 3$

II. The random heat equation in $d \geq 3$

III. Indications of the proofs

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KPZ in $d = 1$: $h_t = h_{xx} + h_x^2 + V(t,x)$

$h(t,x)$ – height of a growing interface

$V(t,x)$ – sticky objects falling from the sky

Experimental fact: too many discrete problems have $h_t = h_{xx} + h_x^2 + \dot{W}(t,x)$ as a formal long time-large space continuum limit, $\dot{W}(t,x)$ – Gaussian white noise,

$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)\delta(x-y)$

How regular is $h(t,x)$? Does h_x^2 make sense?

The Hopf-Cole transform: $u(t,x) = \exp(h(t,x))$

Random heat equation: $u_t = \Delta u + V(t,x)u$

Regardless of KPZ, makes sense in $d > 1$: branching Brownian motion, directed polymers...

Simplest random PDE: linear, time-dependent $V(t,x)$, Feynman-Kac can be used

Main issue for RHE: long time/large spatial scale behavior

RHE as linearization of semi-linear PDE with noise

Imagine $\partial_t \Psi = \Delta\Psi + F(\Psi) + V(t,x)$ has a stationary solution $\bar{\Psi}(t,x)$

Stability: linearize $\Psi = \bar{\Psi} + \delta u$

$$u_t = \Delta u + F'(\bar{\Psi}(t,x))u$$

Random potential $V(t,x) = F'(\bar{\Psi}(t,x))$ – is stationary but "more correlated" than $W(t,x)$

Long time behavior?

Interlude. Long time: weak coupling vs. "straight up long time"

Weak coupling problems – microscopic noise is weak (Spencer):

A particle in a random velocity field $\dot{Y}(s) = \varepsilon V(s, Y(t))$

Random heat equation $\partial_s u = \Delta_y u + \varepsilon V(s, y)u$

Random Schrödinger equation $i\partial_s u = \Delta u + \varepsilon V(s, y)u$

Long time: $s \sim \varepsilon^{-m}$ – what is the "right" m?

How long can we control the solutions?

"Straight up long time": strong microscopic noise ("Armstrong")

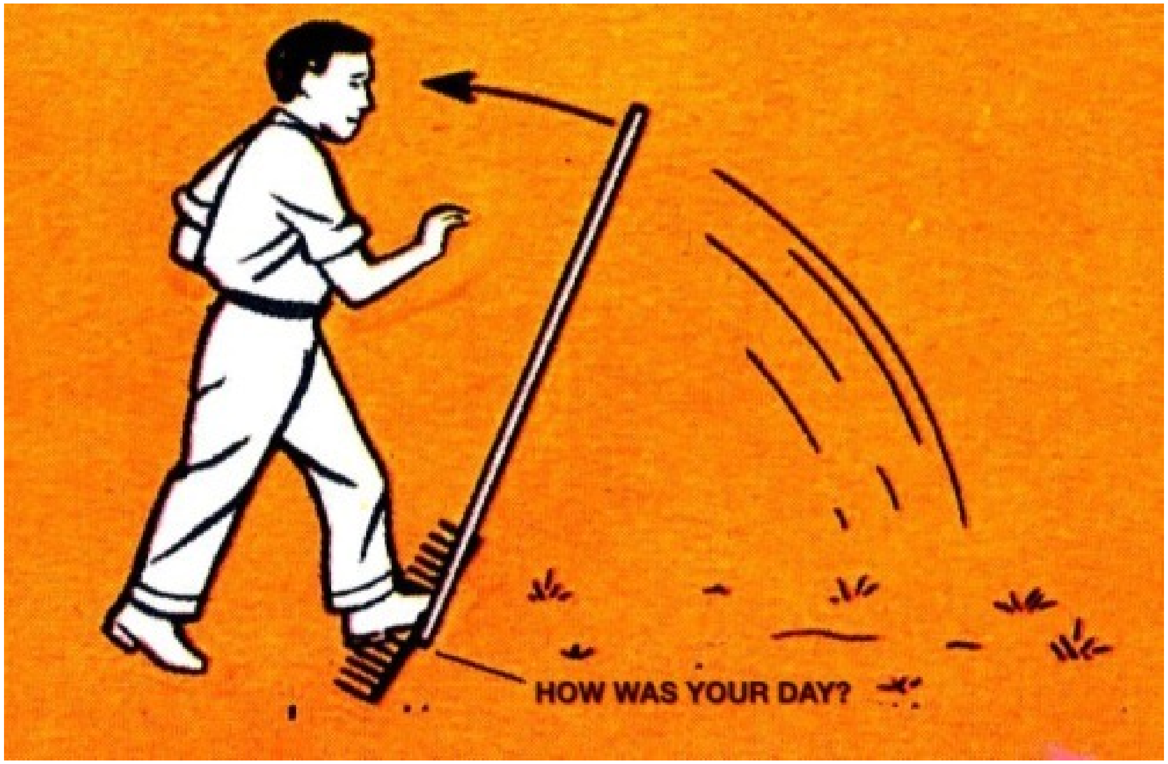
A particle in a random velocity field $\dot{Y}(s) = V(s, Y(t))$

Random heat equation $\partial_s u = \Delta_y u + V(s, y)u$

Random Schrödinger equation $i\partial_s u = \Delta u + V(s, y)u$

Long time: $s \gg 1 =$ correlation time of $V(s, y)$

Weak coupling problems are occasionally harder than they seem to a naive simpleton (see ESY)



Neanderthal weak coupling: central limit theorem

$S_k^\varepsilon := \varepsilon X_1 + \varepsilon X_2 + \cdots + \varepsilon X_k$ is $\approx$ Gaussian if $k \sim \varepsilon^{-2}$

and X_k i.i.d. or rapidly decorrelating

"Randomness of size ε" $\Rightarrow$ time scale ε^{-2}

Typical generalization: $\dot{Y}(s) = \varepsilon V(s, Y)$, then

$Y^\varepsilon(t) = \varepsilon Y(t/\varepsilon^2) \Rightarrow B(t)$ – Brownian motion (Khasminsky, Kesten-Papanicolaou, ...)

Larger times, beyond CLT? \end{interlude}

Back to KPZ in $d = 1$ (still introductory compote)

Can we make sense of

$h_t = h_{xx} + h_x^2 + \dot{W}(t,x)$

$\dot{W}(t,x)$ – Gaussian white noise, a distrbution such that

$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)\delta(x-y)$

How regular is $h(t,x)$? Does h_x^2 make sense?

This is a weak coupling – very long time problem

A toy approximate problem in $d = 1$

Regularize and drop nonlinearity: $\partial_t h^\varepsilon = h^\varepsilon_{xx} + \frac{1}{\varepsilon^{3/2}}\eta(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon})$

$\eta_\varepsilon(t,x) = \varepsilon^{-3/2}\eta(t/\varepsilon^2, x/\varepsilon)$ – regularized white noise

$$\mathbb{E}(\eta_\varepsilon(t,x)\eta_\varepsilon(s,y)) = \varepsilon^{-3}R(\frac{t-s}{\varepsilon^2}, \frac{x-y}{\varepsilon}) \to \delta(t-s)\delta(x-y)$$

Applied math – a multiple scale expansion should be

$$h^\varepsilon(t,x) = \bar{h}(t,x) + \varepsilon^{1/2}h_1(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) + \varepsilon h_2(t,x,\frac{t}{\varepsilon^2},\frac{x}{\varepsilon}) + \dots$$

$\Rightarrow h^\varepsilon(t,x)$ is Hölder $1/2-$ in space and $1/4-$ in time

No way $(h^\varepsilon_x)^2$ has a limit. What can be done?

Hairer'13 KPZ in $d = 1$ (regularity structures)

Formal and non-sensical: $h_t = h_{xx} + h_x^2 + \dot{W}(t,x)$

Regularize: $\partial_t h^\varepsilon = h^\varepsilon_{xx} + (h^\varepsilon_x)^2 + \eta_\varepsilon(t,x)$

$\eta_\varepsilon(t,x) \overset{law}{=} \varepsilon^{-3/2}\eta(t/\varepsilon^2, x/\varepsilon)$ – regularized white noise,

Hopf-Cole: $u^\varepsilon(t,x) = \exp(h^\varepsilon)$, $\partial_t u^\varepsilon = u^\varepsilon_{xx} + \eta_\varepsilon(t,x)u^\varepsilon$

$u^\varepsilon(t,x) \sim e^{-C_\varepsilon t}\bar{u}(t,x)$, $C_\varepsilon = c_1/\varepsilon + c_2$ – renormalization

Multiplicative SHE: $\bar{u}_t = \bar{u}_{xx} + \bar{u}\dot{W}(t,x)$

$$\bar{u}(t,x) = e^{t\Delta}u_0(x) + \int_0^t \int G(t-s, x-y)\bar{u}(s,y)\dot{W}(ds,dy)$$

Microscopic picture in $d = 1$

$\partial_s u = u_{yy} + \sqrt{\varepsilon}\eta(s,y)u$; $\eta(s,y)$ – smooth Gaussian field

Standard CLT time: $t \sim \varepsilon^{-1}$ (noise is $\sqrt{\varepsilon}$)

$\tilde{u}^\varepsilon(t,x) = u(t/\varepsilon, x/\sqrt{\varepsilon}) \to \tilde{u}(t,x)$, $\partial_t \tilde{u} = \tilde{u}_{xx} + c_1 \tilde{u}_\varepsilon$

Hairer – very long time: $t \sim \varepsilon^{-2}$, $u^\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon)$

$\partial_t u^\varepsilon = u^\varepsilon_{xx} + \eta_\varepsilon(t,x)u^\varepsilon \Rightarrow u^\varepsilon(t,x) \sim e^{-(c_1/\varepsilon + c_2)t}\bar{u}(t,x)$,

Multiplicative SHE: $\bar{u}_t = \bar{u}_{xx} + \bar{u}\dot{W}(t,x)$

Correct regularity from the asymptotic expansion

Why can one control $t \sim$ (" noise")$^{-4}$?

The last time naive expansions should work

$$\partial_s u = u_{yy} + \sqrt{\varepsilon}\eta(s,y)u$$

Expand: $u(s,y) = 1 + \sqrt{\varepsilon}\chi(s,y) + \ldots$

$$\partial_s \chi = \chi_{yy} + \eta(s,y) \Rightarrow \mathbb{E}(\chi^2(s,y)) \sim \sqrt{s},$$

so $\sqrt{\varepsilon}\chi(s/\varepsilon^2) \sim O(1)$

Fails for Schrödinger at $t \sim$ (" noise")$^{-2}$

Control on such long time scales is good for the workers

\end{the introduction compote}

Magnen-Unterberger'17 KPZ in $d \geq 3$ (64pp.)

Microscopically: $h_s = \Delta h + |\nabla h|^2 + \eta(s, y)$

$\eta(s, y)$ – microscopic $O(1)$ size smooth noise

Small solutions $h(t/\varepsilon^2, x/\varepsilon) = \varepsilon^{d/2-1} h_\varepsilon(t, x)$:

$h_t^\varepsilon = \Delta h^\varepsilon + \varepsilon^{d/2-1} |\nabla h^\varepsilon|^2 + \varepsilon^{-(1+d/2)} \eta(t/\varepsilon^2, x/\varepsilon)$

Additive SHE : $h_\varepsilon(t, x) - c_\varepsilon t \to \bar{h}(t, x)$, $\bar{h}_t = a_e \Delta \bar{h} + \nu_e \dot{W}(t, x)$

Not a naive linearization. Why $\varepsilon^{d/2-1}$?

"KPZ equation is infra-red super-renormalizable, hence (power-like) asymptotically free at large scales in ≥ 3 dimensions"

An aside: additive and multiplicative SHE in $d \geq 3$

Additive SHE $\partial_t u = \Delta u + \dot{W}(t,x)$

Makes sense, solution is a distribution:

$\int u(t,x)\phi(x)dx$ is defined for $\phi \in C_c^\infty(\mathbb{R}^3)$ but not point-wise: $W_\varepsilon(t,x) = \varepsilon^{-5/2}\eta(t/\varepsilon^2, x/\varepsilon)$ in $d = 3$

Multiplicative SHE $\partial_t u = \Delta u + u\dot{W}(t,x)$

makes no sense in $d > 1$ (multiplying distributions), can not be the long time limit

1D KPZ – 3D KPZ comparison

(1) Small noise, $O(1)$ solutions, "very large" (beyond CLT) time – $O(1)$ noise, small solutions, large time

(2) Make sense of equations with formally non-sensical limits

(3) Multiplicative SHE – additive SHE in the limit

(4) Explicit diffusivity, renormalization constant, and noise – effective parameters

(5) Non-Gaussian fluctuations – Gaussian fluctuations

(6) Interesting – boring

Can this be done as old-fashioned applied math, and without small solutions in $3D$?

Hopf-Cole $\Rightarrow$ the random heat equation

$\partial_t u = \Delta u + \beta V(t,x)u, \quad x \in \mathbb{R}^d, d \geq 3.$

$\beta > 0$ – "coupling", different behavior for β small or large.

$V(t,x)$ – mean-zero space-time stationary Gaussian:

$V(t,x) = \int_{\mathbb{R}^{d+1}} \phi(t-s)\psi(x-y)dW(s,y)$

$\phi \in C_c^\infty(\mathbb{R})$, $\psi \in C_c^\infty(\mathbb{R}^d)$, $\operatorname{supp}\phi \in [0,1]$

I. Large scale spatial averages for $t \gg 1$

$I(T,L) = L^{-d}\int u(T,x)\phi\Big(\frac{x}{L}\Big)dx$, $\phi \in C_c^\infty$, $T \gg 1$, $L \gg 1$

Initial conditions: $u(0,x) = u_0(\varepsilon x)$ or $u(0,x) \equiv 1$.

Standard choice $T \sim \varepsilon^{-2}$, $L \sim \varepsilon^{-1}$ not forced, especially for $u_0(x) \equiv 1$, different for $L^2 \ll T$ and $L^2 \gg T$, need $L \gg 1$.

II. How does $u(T,x)$ look locally for $T \gg 1$?

The rescaled equation with $T \sim L^2$

$u_\varepsilon(t,x) := u(t/\varepsilon^2, x/\varepsilon)$

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{\beta}{\varepsilon^2} V\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) u_\varepsilon, \; u_\varepsilon(0,x) = u_0(x).$$

Formally: $V_\varepsilon(t,x) = \varepsilon^{-2} V(t/\varepsilon^2, x/\varepsilon) \sim \varepsilon^{d/2-1} \nu_0 \dot{W}(t,x)$:

$$\mathbb{E}(V_\varepsilon(s,y) V_\varepsilon(s+t, y+x)) = \frac{1}{\varepsilon^4} R\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right) \sim \varepsilon^{d+2-4} \nu_0^2 \delta(t,x).$$

$\nu_0^2 = \int_{\mathbb{R}^{d+1}} R(s,y)\,ds\,dy.$

$\dot{W}(t,x)$ – space-time white noise.

A very sensible plausible limit

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{\beta}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) u_\varepsilon, \; u_\varepsilon(0,x) = u_0(x)$$

"Approximate" by $\partial_t \phi_\varepsilon = \Delta \phi_\varepsilon + \varepsilon^{d/2-1} \beta \nu_0 \dot{W}(t,x) \phi_\varepsilon$

Makes no sense in $d > 1$ but I am an applied mathematician: a small perturbation of $\partial_t \bar{\phi} = \Delta \bar{\phi}$

Naive guess: $u_\varepsilon(t,x) = \bar{\phi}(t,x) + \varepsilon^{d/2-1} \phi_1(t,x) + \ldots$

The additive stochastic heat equation

$$\partial_t \phi_1 = \Delta \phi_1 + \beta \nu_0 \bar{\phi}(t,x) \dot{W}(t,x)$$

Good things

The limit makes perfect sense: additive stochastic heat equation

Bad things: looks fishy – $\varepsilon^{d/2-1}$ has no role

A related question

$\partial_t u_\varepsilon = \Delta u_\varepsilon + \varepsilon^m \dot{W}_\varepsilon(t,x) u_\varepsilon$

$\dot{W}_\varepsilon(t,x)$ – regularized white noise, what is the "interesting" m?

Superficially: an example of a question in singular stochastic PDEs – need to make sense of the multiplication of distributions (Hairer, Pardoux, Weber, Gubinelli, Otto ...).

Typical result: the solution of the equation with the mollified white noise, after a suitable renormalization, converges to some limit that is sometimes independent of the way in which the noise is mollified, and sometimes depends on the mollification.

Sad reality: our problem is rather simple in the end, no need for fancy machinery.

0. Weak coupling limit: $\beta = \varepsilon$ asymptotically small

$\partial_t u = \Delta u + \varepsilon V(t,x)u$, $u(0,x) = u_0(\varepsilon x)$,

Pardoux, Piatnitskii'12; Bal, Gu'16: $t \sim \varepsilon^{-2}$

$$\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{1}{\varepsilon} V(t/\varepsilon^2, x/\varepsilon) u^\varepsilon$$

Regularization for $\partial_t u = \Delta u + \varepsilon^{d/2} \dot{W}_\varepsilon(t,x) u$

$u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon) e^{-\bar{c}t} \to \bar{u}(t,x)$, $\bar{c} = \int_0^\infty \mathbb{E}_B[R(t,B_t)]dt$

Diffusion equation $\partial_t \bar{u} = \Delta \bar{u}$. Naive guess works here!

Fluctuation is additive SHE with "naive" variance ν_0^2.

Hairer, Pardoux'15, Gu-Tsai'17 ($d = 1$):

weak coupling "very long time" $t \sim \varepsilon^{-4}$

$\partial_t u = u_{xx} + \varepsilon V(t,x)u$,

$u^\varepsilon(t,x) = u(t/\varepsilon^4, x/\varepsilon)$, $\partial_t u^\varepsilon = u^\varepsilon_{xx} + \frac{1}{\varepsilon^3} V(\frac{t}{\varepsilon^4}, \frac{x}{\varepsilon^2}) u^\varepsilon$,

$\varepsilon^{-3} V(t/\varepsilon^4, x/\varepsilon^2) \sim \dot{W}(t,x)$ (no small pre-factor)

Main result: $u_\varepsilon(t,x) \exp\{-(\bar{c} + \varepsilon^2 c_2)t/\varepsilon^2\} \to \bar{u}(t,x)$

The multiplicative stochastic heat equation

$\partial_t \bar{u} = \bar{u}_{xx} + \dot{W}(t,x)\bar{u}$

Back to our problem

$$\partial_t u = \Delta u + \beta V(t,x)u,$$

$t \sim \varepsilon^{-2}$: $\partial_t u^\varepsilon = \Delta u^\varepsilon + \frac{\beta \varepsilon^{d/2-1}}{\varepsilon^{1+d/2}} V(t/\varepsilon^2, x/\varepsilon)u$

Noise is not weak coupling but the formal limit is weaker than the white noise

Naive guess should no longer be true – microscopic dynamics is not "trivial"

1. White in time potentials: small $\beta \in (0, \beta_1)$

Mukherjee, Shamov, Zeitouni'16: V white in time

$V(t,x) = \dot{W}_\psi(t,x) = \int \psi(x-y)dW(t,y)$.

$\partial_t u = \Delta u + \beta \dot{W}_\psi(t,x)u, \ x \in \mathbb{R}^d, \ d \geq 3, \ u(0,x) \equiv 1$.

MSZ'16: $u_\varepsilon(t,x) = u(t/\varepsilon^2, x/\varepsilon) \to \bar{u}(t,x) \equiv 1$ weakly

$\int u_\varepsilon(t,x)g(x)dx \to \int g(x)dx$ for any $g \in \mathcal{C}_c^\infty(\mathbb{R}^d)$.

Pointwise: $u_\varepsilon(t=1,x) \to Z_\infty$ in law, $Z_\infty > 0$ a.s.

The law of Z_∞ – open

White in time potentials: large $\beta > 0$

Mukherjee, Shamov and Zeitouni also show that for all $\beta > \beta_2$ we have $u_\varepsilon(t,x) \to 0$ in probability as $\varepsilon \to 0$, for all $t > 0$ and $x \in \mathbb{R}^d$ fixed. This is the main difference between the weak and strong disorder regimes. Existence of a sharp transition from one regime to the other was also left as an open question.

2. Non-white in time potentials: $\beta \in (0, \beta_1)$ small

Homogenization for spatial averages

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{\beta}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}) u_\varepsilon, \quad t > 0,\ x \in \mathbb{R}^d,\ d \geq 3,$$

Theorem. (Gu, R., Zeitouni'17)

$$\int_{\mathbb{R}^d} u_\varepsilon(t,x) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} g(x) dx \to \int_{\mathbb{R}^d} \bar{u}(t,x) g(x) dx,$$

in probability. The effective diffusion equation:

$$\partial_t \bar{u} = \nabla \cdot a_{\text{eff}} \nabla \bar{u}, \quad \bar{u}(0,x) = u_0(x),\ a_{\text{eff}} \neq \text{Id}\ .$$

Mukherjee'17: $\mathbb{E}(u_\varepsilon(t,x)) \to \bar{u}(t,x)$, directed polymers.

Gaussian fluctuations

Theorem. (Gu, R., Zeitouni'17)

$$\frac{1}{\varepsilon^{d/2-1}}(u_\varepsilon(t,x) - \mathbb{E}[u_\varepsilon(t,x)]) \exp\Big\{ -\frac{c_1 t}{\varepsilon^2} - c_2 \Big\} \Rightarrow \mathcal{U}(t,x)$$

in law. Additive SHE $\nu_{\text{eff}}^2 > 0$, $\nu_{\text{eff}} \neq \nu_0$.

$$\partial_t \mathcal{U} = \nabla \cdot a_{\text{eff}} \nabla \mathcal{U} + \beta \nu_{\text{eff}} \bar{u}(t,x) \dot{W}, \quad \mathcal{U}(0,x) = 0,$$

After integration against a test function $g(x) \in C_c^\infty(\mathbb{R}^d)$.

Why $\varepsilon^{d/2-1}$? What are c_1, c_2, a_{eff} and ν_{eff}?

To a child who does not know how to ask?

What happens locally (no spatial averages)?

$\partial_s \bar\Psi = \Delta \bar\Psi + \beta V(s,y)\bar\Psi - \lambda(\beta)\bar\Psi, \quad \bar\Psi(s,y)$ – stationary

$\partial_s \Psi = \Delta \Psi + \beta V(s,y)\Psi - \lambda(\beta)\Psi, \quad \Psi(0,x) \equiv 1$

Theorem (Dunlap, Gu, R., Zeitouni'18)

(1) For $\beta \in (0,\beta_0)$ there exists a space-time stationary solution $\bar\Psi(t,x)$.

(2) The finite-dimensional distributions of $\Psi(s,\cdot)$ converge as $s \to +\infty$ to those of a multiple of $\overline{\Psi}(s,x)$.

General solutions

$$\partial_t u = \Delta u + \frac{\beta}{\varepsilon^2} V(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}), \quad u(0,x) = u_0(x)$$

Theorem (Dunlap, Gu, R., Zeitouni'18)

For $\beta \in (0, \beta_0)$, we have

$$\lim_{\varepsilon \to 0} \mathbb{E} \left\| u^\varepsilon(t,x) e^{-\lambda(\beta) t/\varepsilon^2} - \overline{u}(t,x) \Psi(t/\varepsilon^2, x/\varepsilon) \right\| = 0$$

Renormalization constants:

$c_1 = \lambda(\beta)$, $e^{c_2} = \mathbb{E}\left[\bar{\Psi}(s,y)\right]$

Old-fashioned applied mathematics

$$u^{\varepsilon}(t,x)=\bar{u}(t,x)\Psi(t/\varepsilon^2,x/\varepsilon)+\varepsilon\chi_j(t/\varepsilon^2,x/\varepsilon)\partial_{x_j}\bar{u}(t,x)+\dots$$

The corrector equation

$$\partial_s\chi_j=\Delta_y\chi_j+(\beta V(s,y)-\lambda(\beta))\,\chi_j+\partial_{y_j}\Psi(s,y)$$

The backward stationary solution

$$\partial_s\Phi+\Delta\Phi+\beta V(s,y)\Phi-\lambda(\beta)\Phi=0$$

Effective diffusivity $a_{\text{eff}}=1+\dfrac{2}{d}\dfrac{\mathbb{E}\left[\Phi(s,y)\text{div}_y\chi(s,y)\right]}{\mathbb{E}\left[\Psi(s,y)\Phi(s,y)\right]}$

Demi-theorems

Why $\varepsilon^{d/2-1}$ weak error – spatial decay of correlations of $\bar{\Psi}(s, y)$

What is ν_{eff} – comes from the coefficient in the spatial decay rate for $R_{\bar{\Psi}}(s, y)$

The renormalization constants (scientifically)

The Feynman-Kac formula: $u(0,x) \equiv 1$ for simplicity

$u(t,x) = \mathbb{E}_B\Big[\Big\{\beta \int_0^t V(t-s, x+B_s)ds\Big\}\Big]$, and

$\mathbb{E}(u(t,x)) = e^{\zeta_t} := \mathbb{E}_B\Big[\exp(I_\beta(B))\Big]$

$I_\beta(B) := \exp\Big\{\beta^2/2 \int_0^t \int_0^t R(s-u, B_s - B_u)dsdu\Big\}\Big]$

With bit of work: $\zeta_t \approx c_1 t + c_2 + o(1)$, as $t \to +\infty$.

This "explains" the renormalization constants.

The tilted Brownian paths as a Markov chain

$\widehat{\mathbb{E}}_{B,t}[f(B)] := \mathbb{E}_B\Big[f(B)\,\exp(I_\beta(B) - \zeta_t)\Big]$

Increments of length 1: $(x_0, \ldots, x_{N+1})$.

The interaction term for $x, y \in \Omega = C([0,1])$:

$$I(x,y) = \beta^2 \int_0^1 \int_0^1 R(s+1-u, y(s) + x(1) - x(u))\,ds\,du.$$

Doob-Krein-Rutman: there exist $\rho > 0$ and $\Psi(y)$

$$\int_\Omega e^{I(x,y)}\Psi(y)\pi(dy) = \rho\Psi(x),\ 0 < c_1 \leq \Psi(y) < +\infty$$

Transition probability $\widehat{\pi}(x, dy) = \dfrac{e^{I(x,y)}\Psi(y)\pi(dy)}{\rho\Psi(x)}$

The Doeblin condition: $\widehat{\pi}(x, A) \geq \gamma\pi(A)$, all $x \in \Omega$, $A \subset \Omega$, with $\gamma \in (0, 1)$.

A coupling argument:

$$\widehat{\pi}(z_1, dz_2) = \gamma\pi(dz_2) + (1-\gamma)\frac{\widehat{\pi}(z_1, dz_2) - \gamma\pi(dz_2)}{1-\gamma},$$

η_k – i.i.d. Bernoulli with the parameter γ: if $\eta_k = 1$, sample Z_k from $\pi(dz)$, and if $\eta_k = 0$, sample Z_k from

$$\frac{\widehat{\pi}(Z_{k-1}, dz) - \gamma\pi(dz)}{1-\gamma}.$$

The invariance principle for the tilted Brownian path

Regeneration times: $T_i = \inf\{j > T_{i-1} : \eta_j = 1\}$.

The path increment in each regeneration block

$$\mathbf{X}_j := \sum_{k=T_j}^{T_{j+1}-1} X_k(1), \quad j = 0, 1, \ldots$$

Proposition. $\varepsilon B_{s/\varepsilon^2} \Rightarrow W_s$, a Brownian motion with the covariance matrix $\boldsymbol{a}_{\text{eff}} := \gamma \mathbb{E}_\pi[\mathbf{X}_1 \mathbf{X}_1^t]$, hence

$\mathbb{E}[u_\varepsilon(t,x)] e^{-\zeta_{t/\varepsilon^2}} \to \bar{u}(t,x)$ as $\varepsilon \to 0$.

Why we need small β and $d \geq 3$

The key quantity:

$$\ell(x, y, X_0, Y_0) = \int_0^\infty 1_{\{|x+\omega_{X_0}(s)-y-\omega_{Y_0}(s)|\leq 1\}} ds,$$

the total "nearby time" of ω_{X_0} and ω_{Y_0}.

Proposition. In $d \geq 3$, $\pi[\ell(x, y, X_0, Y_0) > t] \leq C_1 e^{-C_2 t}$,

hence if $\beta < C_2$, then $\mathbb{E}_\pi[e^{\beta\ell(x,y,X_0,Y_0)}] < \infty$.

This is why $\beta < \beta_1$. Not an artefact of the proof – for large β the solutions should behave differently.

Open questions and work in progress

0. We are back to Joe Keller-like mathematics

1. Local error estimates

2. Long time behavior for large β

3. Turn (a) linear and (b) non-linear (e.g. Φ^4) problems with weak noise/very long time into applied math

4. Very long time for the Schrödinger equation

5. Thanks to Alex Dunlap and Yu Gu