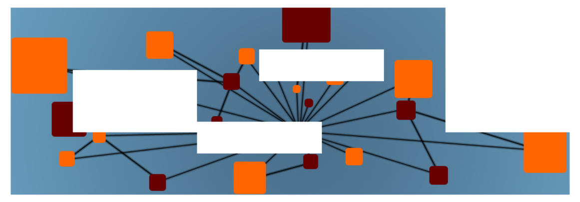

Consensus and synchronization over biologically-inspired networks: from collaboration to antagonism

Subhradeep Roy, Nicole Abaid

Department of Biomedical Engineering and Mechanics, Virginia Tech

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Abstract

We study consensus and synchronization over two types of interaction networks. The first captures both collaborative and antagonistic interactions and the second considers the impact of leaders in purely collaborative interactions. The general network structure in both cases is that of so-called conspecific agents, which is specialized to a numerosity-constrained (NC) network for the second case. NC networks incorporate an upper limit to the number perception of each agent. We establish closed form results for the rate of convergence to consensus over each network type and conditions for stochastic synchronization in the second case.

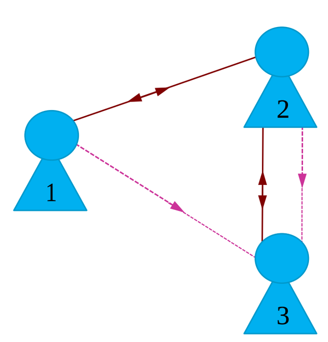
Introduction

- Discrete-time consensus protocol: $x(k+1) = W(k)x(k)$
- Agents' states: $x(k) \in \mathbb{R}^N$ at time k
- State matrix: $W(k) \in \mathbb{R}^{N\times N}$
- Properties: $W(k) = I_N - M(k)$, which implies $W(k)1_N = 1_N$.

Collaborative-antagonistic network considering conspecific agents in terms of adjacency and degree matrices weighted for each agent:

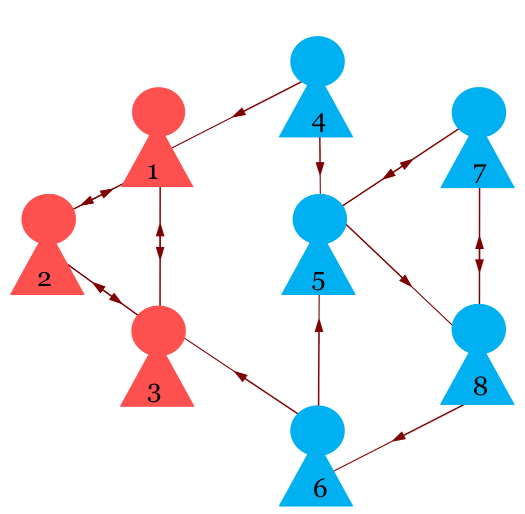
$$A_1(k) = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \\ 0 & 0.3 & 0 \end{bmatrix} \quad D_1(k) = \begin{bmatrix} 0.2 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \quad M_1(k) = D_1(k) - A_1(k) = \begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.3 & 0.3 \end{bmatrix}$$

$$A_2(k) = \begin{bmatrix} 0 & 0 & 0.1 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} \quad D_2(k) = \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad M_2(k) = D_2(k) - A_2(k) = \begin{bmatrix} 0.1 & 0 & -0.1 \\ 0 & 0.2 & -0.2 \\ 0 & 0 & 0 \end{bmatrix}$$



- d_{ji} and ϵ_{ji} are realizations of the random variables $\mathcal{D}_j$ and $\mathcal{E}_j$, respectively, from the bivariate distribution $g_{\mathcal{D}_j \mathcal{E}_j}(d_j, \epsilon_j)$ for the ith agent at kth time step, capturing collaborative and antagonistic interactions when $j = 1, 2$, respectively.
- Realizations of two independent and identically distributed random matrices: $M_i(k) \in \mathbb{R}^{N\times N}$ for $i = 1, 2$
- Realizations of the random matrix: $M(k) = M_1(k) - M_2(k)$
- Property: $M_i(k)1_N = 0_N$, for $i = 1, 2, \implies M(k)1_N = 0_N$

Collaborative NC leader-follower:



- $M(k) = \epsilon L(k)$
- Numerosity-constrained graph Laplacian: $L(k)$
- Fixed cardinality of neighbor set: degree equals n for all agents
- Weight parameter or persuasibility: ϵ
- Leaders have dynamic states and interact only among themselves.
- Followers interact with both the leaders and other followers.

Preliminary results

Consensus:

- We project the state dynamics onto the disagreement space so that stability of disagreement dynamics equals consentability: $\xi(k+1) = \widetilde{W}(k)\xi(k)$
- Disagreement variable: $\xi(k) = Q^T x_k \in \mathbb{R}^{N-1}$, $\widetilde{W} = Q^T W Q$
- Asymptotic convergence factor, $r_a = \rho\left(\mathbf{E}[\widetilde{W} \otimes \widetilde{W}]\right) = \rho(G)$
- $G = (R \otimes R)\mathbf{E}[W \otimes W] = (R \otimes R)(I_{N^2} - (\mathbf{E}[M] \oplus \mathbf{E}[M]) + \mathbf{E}[M \otimes M])$
- Necessary and sufficient condition: $r_a = \rho(G) < 1$

Synchronization:

- Dynamics of a networked oscillator:

$$x_i(k+1) = f(x_i(k)) - \sum_{j=1}^{N} [M]_{ij}(k) f(x_j(k)),$$

- Individual dynamics and nonlinear function for coupling among oscillators: $f(x_i(k))$
- Necessary and sufficient condition: $\ln(\rho(G)) + 2h_{\max} < 0$
- $h_{\max}$ is the largest Lyapunov exponent of the individual dynamics $f(x)$.

Collaborative and antagonistic interactions

- We write $\mathbf{E}[M]$ and $\mathbf{E}[M \otimes M]$ using a counting technique and calculate G.
- We calculate the at most four distinct eigenvalues and associated eigenvectors of G.

$$r_a = \left(1 - \frac{N\eta_1}{N-1}\right)^2 - \frac{N}{N-1}\left(\phi_1{}^2 + \psi_1{}^2\right) + (\phi_2 + \psi_2) + (\phi_3 + \psi_3)$$

- $\phi_1 = \mathbf{E}[\mathcal{E}_1 \mathcal{D}_1]$, $\phi_2 = \mathbf{E}[\mathcal{E}_1{}^2 \mathcal{D}_1{}^2]$, $\phi_3 = \mathbf{E}[\mathcal{E}_1{}^2 \mathcal{D}_1]$
- $\psi_1 = \mathbf{E}[\mathcal{E}_2 \mathcal{D}_2]$, $\psi_2 = \mathbf{E}[\mathcal{E}_2{}^2 \mathcal{D}_2{}^2]$, $\psi_3 = \mathbf{E}[\mathcal{E}_2{}^2 \mathcal{D}_2]$
- $\eta_1 = \phi_1 - \psi_1$
- Bounded interval for synchronization of logistic maps coupled over an NC network: when the solid curve in Figure 1 is less than the dashed line
- Antagonistic interaction enables synchronization at higher $\hat{\epsilon}$.
- These trends are consistent for consensus problems as well, see the examples in Figures 2 and 3 below.

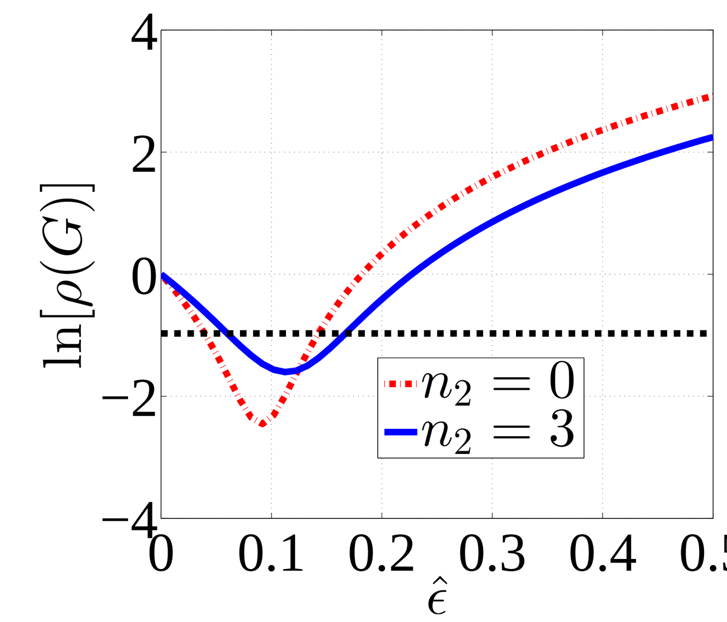


Figure 1: NC network logistic maps with $N = 200$ and $n_1 = 10$. Red dash-dot line: $n_2 = 0$. Blue solid line: $n_2 = 3$. Black dashed line: $-2h_{\max}$ of the logistic maps.

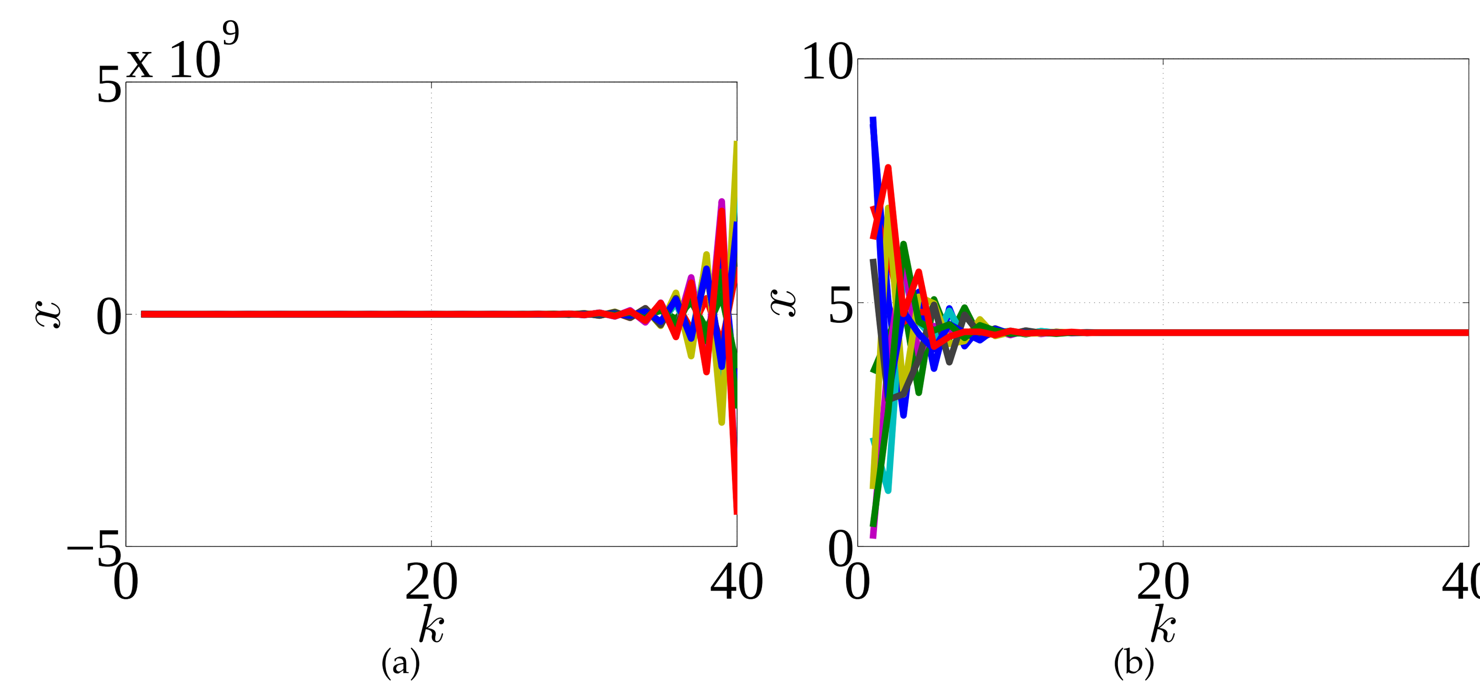


Figure 2: Time series evolution of $N = 10$ agents negotiating consensus over NC network with $n_1 = 8$, $\hat{\epsilon} = 0.3$, and (a) $n_2 = 0$, (b) $n_2 = 4$.

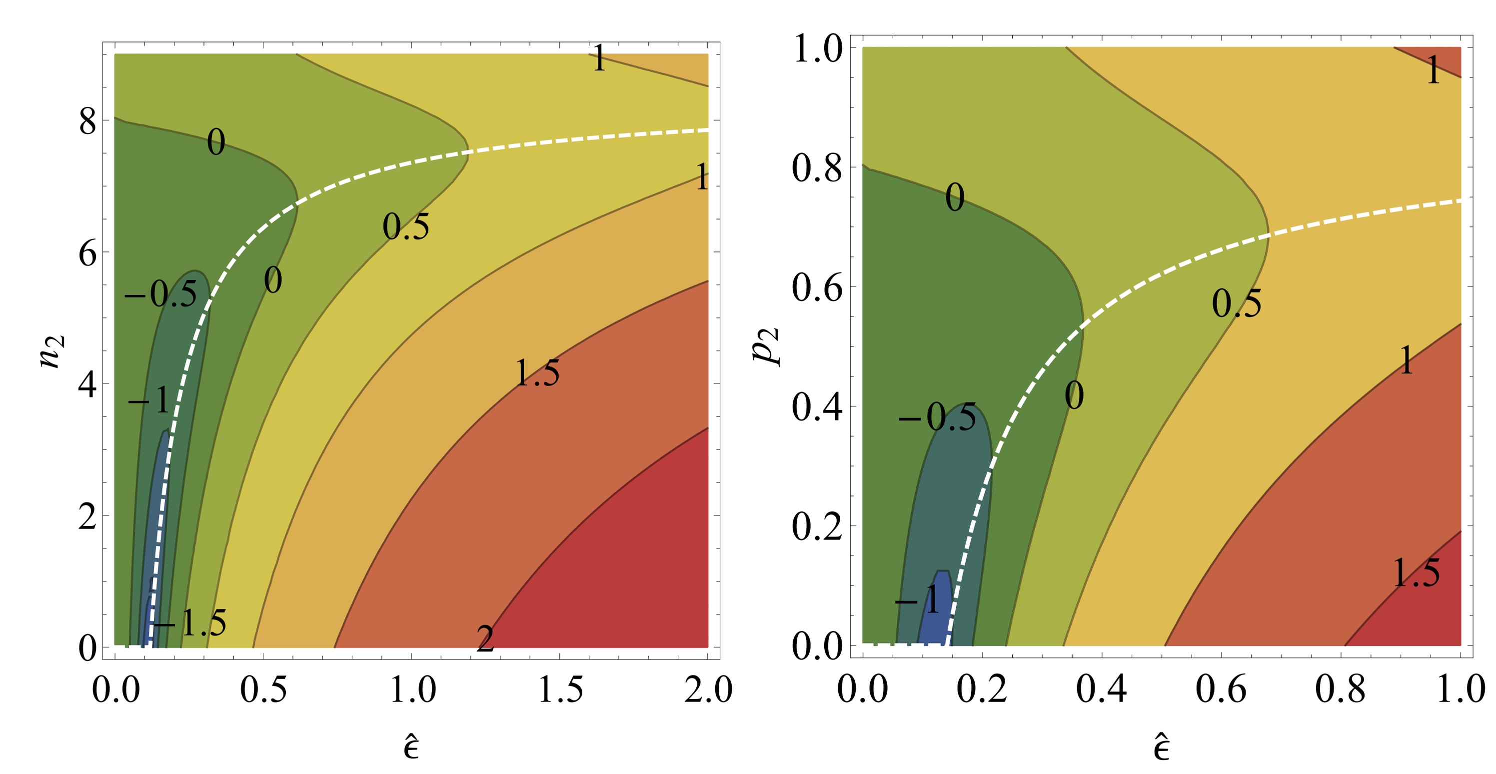


Figure 3: Contour plot of $\log[r_a]$ for $N = 10$ agents coupled over (a) NC networks with $n_1 = 8$, and n_2 and $\hat{\epsilon}$ varying (b) Erdos-Renyi networks with $p_1 = 0.8$, p_2 and $\hat{\epsilon}$ varying. The optimum value of n_2, and the optimum value of p_2, are denoted by the dashed white line.

Summary:

- Antagonistic interactions enable the system to achieve consensus or synchronization which is otherwise not possible for certain values of $\hat{\epsilon}$ and, at times, helps to achieve consensus or synchronization at a relatively faster rate.
- We identify critical values of system parameters that give maxima in convergence speed for two exemplary protocols.

Leader-follower behavior

- Closed-form expressions for the eigensystem of G are computed similarly for this problem.
- G has at most twelve distinct eigenvalues and linearly independent eigenspaces.
- $r_a = \lambda_1$, when $\epsilon \in [0, \epsilon_{cr}]$
- $r_a = \lambda_2$, when $\epsilon \in [\epsilon_{cr}, \infty)$
- Convergence speed increases as r_a decreases.
- Maximum convergence speed: when r_a is minimum for the range of ϵ

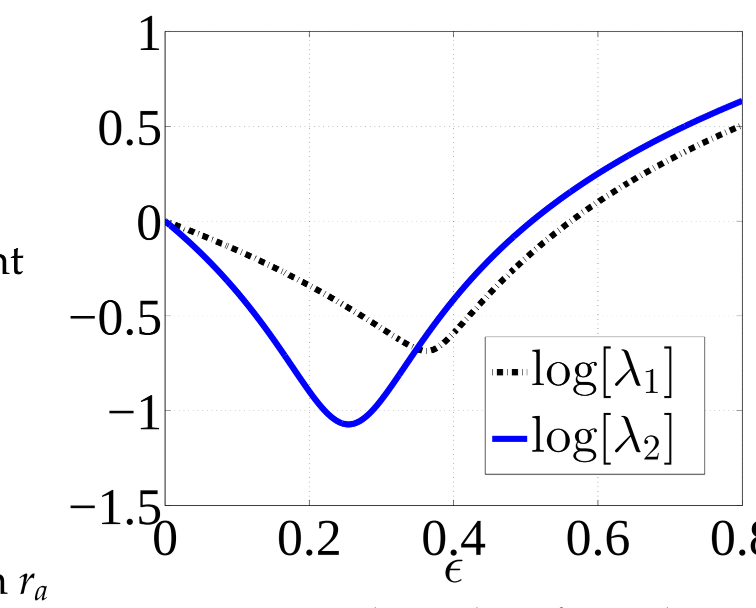


Figure 4: Base-ten logarithm of λ_1 and λ_2 varying with ϵ and $f = l = 6$, $n = 3$.

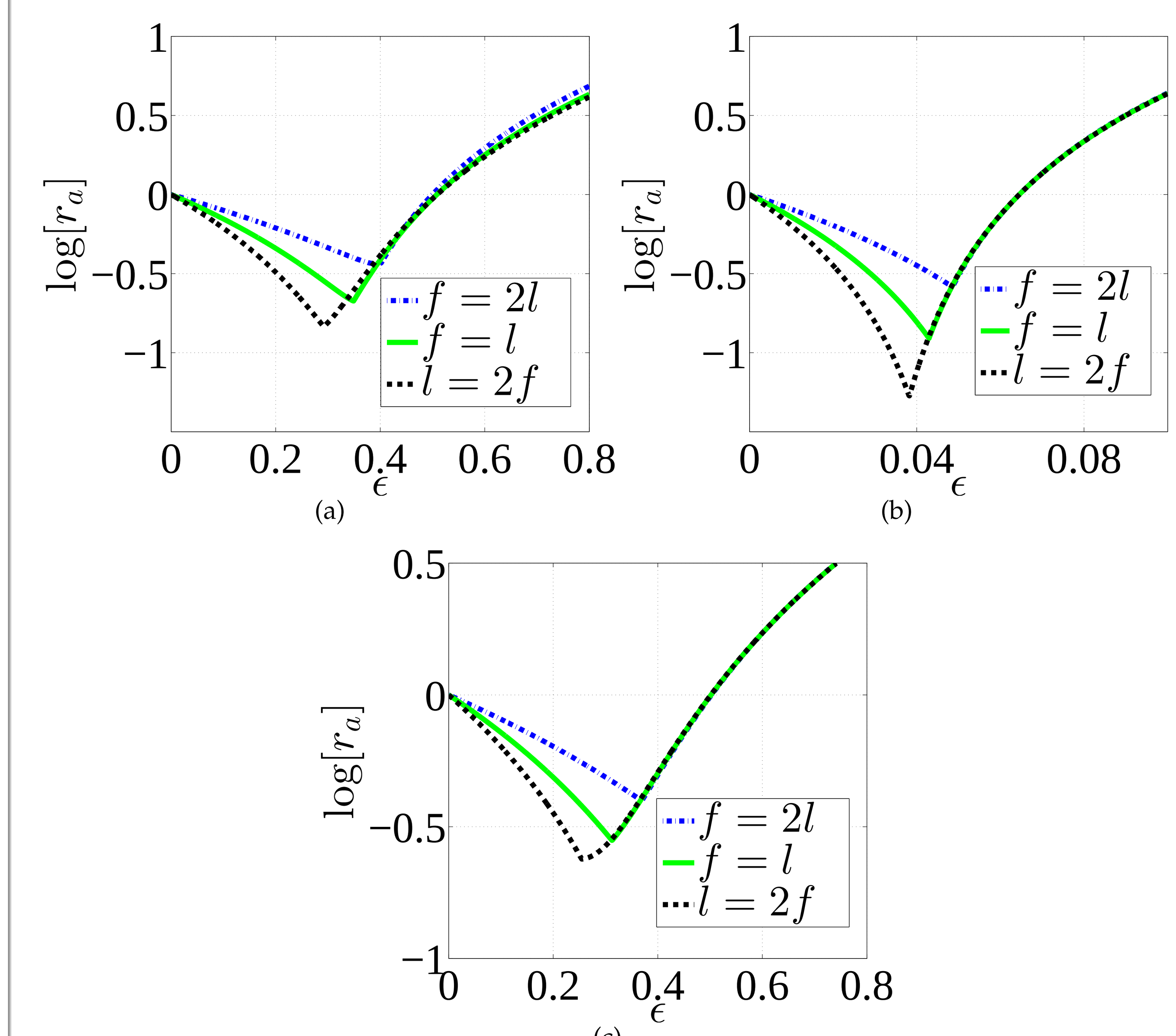


Figure 5: The asymptotic convergence factor for three cases, (a) $N = 12$ and $n = 3$, (b) $N = 120$ and $n = 30$, (c) $N = 120$ and $n = 3$.

Summary:

- *Effect of increasing the proportion of leaders:* Maximum convergence speed increases as the relative number of leaders increases when agents are relatively stubborn.
- *Effect of increasing the group size:* Larger systems may achieve consensus faster if the numerosity scales with the group size and persuasibility is reduced accordingly.
- *Effect of increasing numerosity:* Increasing numerosity results in a faster maximum convergence speed at lower value of persuasibility.

Acknowledgment

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