Fractional parabolic models arising in flocking dynamics and fluids

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April 18, 2017

Cucker-Smale model (2007)

We study macroscopic versions of systems modeling self-organized collective dynamics of "agents":

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \frac{\lambda}{N} \sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), \end{cases} \qquad (x_i, v_i) \in \Omega \times \mathbb{R}^n \tag{1}$$

Here, ϕ is a positive, bounded influence function which models the binary interactions among agents in Ω. If $\phi(r) \to 0$ slower than $1/r$ then "flocking" occurs in large time: $\max\{|x_i - x_j|\} < D$, and $v_i \to \bar{v}$.

Motsch-Tadmor model (2011)

$$\begin{cases} \dot{x}_i = v_i, \\ \dot{v}_i = \dfrac{\lambda}{\Phi_i} \displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|)(v_j - v_i), \quad \Phi_i = \displaystyle\sum_{j=1}^{N} \phi(|x_i - x_j|) \end{cases}$$

This model is equipped with adaptive normalization to compensate for influence of massive congregation of agents at large distances. Harder to study analytically due to lack of symmetry.

There is a body of literature on both models exploring various aspects: regularity flocking, kinetic, macroscopic descriptions, etc.: S.M. Ahn, H. Choi, Heesun, S.-Y. Ha, H. Lee, E. Carlen, M. Carvalho, P. Degond, B. Wennberg, J.A. Carrillo, Y.-P. Choi, P. Mucha, S. Peszek, E. Tadmor, C. Tan, S. Pérez, M. Fornasier, J. Rosado, G. Toscani, P. Degond, A. Frouvelle, J.-G. Liu, V. Panferov, T. Karper, A. Mellet, K. Trivisa, H. Levine, W.-J. Rappel, I. Cohen, T. Vicsek, A. Zafeiris

From microscopic to kinetic description

S.-Y. Ha, E. Tadmor (2008): derivation of a mean-field model based on BBGKY hierarchy, molecular chaos assumption:

$$\partial_t f + v \cdot \nabla_x f + \lambda \nabla_v \cdot Q(f,f) = 0,$$

where

$$Q(f,f)(x,v,t) = \int_{\mathbb{R}^{2n}} \phi(|x-y|)(v_* - v) f(x, v_*, t) dv_* dy.$$

T. Karper, A. Mellet and K. Trivisa (2015): kinetic version of MT model. Flocking is shown in the sense that

$$S(t) = \sup\{|x-y| : x, y \in \operatorname{supp} f\}$$

remains of bounded diameter, and alignment occurs

$$V(t) = \sup\{|v-v'| : v, v' \in \operatorname{supp} f\} \to 0.$$

From kinetic to macroscopic

Evolution of macroscopic density and momentum

$$\rho(x,t) = \int_{\mathbb{R}^n} f(x,v,t) dv, \quad \rho u(x,t) = \int_{\mathbb{R}^n} v f(x,v,t) dv$$

can be derived from kinetic formulation via a moment closure procedure or by considering a monokinetic ansatz

$$f(x,v,t) = \rho(x,t)\delta(v - u(x,t))$$

or by formal pass to a "large crowds" limit

$$f_N = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}(x) \otimes \delta_{v_i}(v)$$

S.-Y. Ha, E. Tadmor (2008), J. Carrillo, Y.-P. Choi, and S. Perez (2017).

We obtain the following coupled system

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ u_t + u \cdot \nabla u = \int_{\mathbb{R}} \phi(|x-y|)(u(y,t) - u(x,t))\rho(y,t)\, \mathrm{d}y \end{cases}$$

$(x,t) : \mathbb{R}^n \times [0,\infty)$. The velocity equation is "Burgers with commutator forcing":

$$u_t + u \cdot \nabla u = \mathcal{L}(\rho u) - \mathcal{L}(\rho) u$$

where $\mathcal{L} f = \phi * f$ in the L^1-kernel case, or

$$\mathcal{L} f = \int_{\mathbb{R}} \phi(|x-y|)(f(y) - f(x)) dy$$

in the singular case. Clearly if $\phi > 0$ the u-equation is dissipative. We study the system in the context of smooth ϕ, $\phi(r) = \frac{1}{r^{n+\alpha}}$, or local $\mathcal{L} = \Delta$.

1D: the "e" quantity

Due to the commutator structure of the forcing the system in 1D has a special quantity

$$e = u_x + \mathcal{L}\rho,$$

which is transported

$$e_t + (ue)_x = 0.$$

In another form, "e" satisfies the logistic equation

$$\frac{\mathsf{D}}{\mathsf{D}t} e = e(\mathcal{L}\rho - e).$$

Theorem (Carillo, Choi, Tadmor, Tan, 2014, 2016)

Case of smooth $\phi > 0$. If $e_0(x_0) < 0$, then the solution blows up in finite time. If $e_0 \geq 0$, then there exists a classical global solution $(u, \rho) \in W^{1,\infty} \times L^\infty$ and, provided $\int_{\mathbb{R}} \phi(x)\, dx = \infty$, the system flocks with fast alignment:

$$\operatorname{diam}_x \operatorname{supp} \rho(\cdot, t) \leq D_\infty < \infty,$$

$$V(t) = \max u(t) - \min u(t) \leq C e^{-\delta t}.$$

Theorem (Tadmor, RS, 2017)

Let $\inf \phi > 0$ *on* $\mathbb{T}$ *or* $\mathbb{R}$*. For any initial conditions* $(u_0, \rho_0) \in W^{2,\infty} \times (W^{1,\infty} \cap L^1)$ *with* $e_0 > 0$*, the global solution flocks in a strong sense: there exist* $\bar{u} = \mathrm{const}$*, and* $\rho_\infty \in W^{1,\infty} \cap L^1$ *such that*

$$|u(t) - \bar{u}|_\infty + |u_x(t)|_\infty + |u_{xx}(t)|_\infty + |\rho(t) - \bar{\rho}(t)|_{C^\beta} \leq Ce^{-\delta t},$$

for all $t > 0$ *and* $\beta < 1$*, and where* $\bar{\rho}(t) = \rho_\infty(x - t\bar{u})$*.*

Singular kernel, $\phi = \frac{1}{r^{1+\alpha}}$

The case is interesting since it models strong influence of local interactions among agents, and weak but not zero long range interactions.

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ u_t + u u_x = \int_{\mathbb{R}} \rho(y,t) \frac{u(y,t) - u(x,t)}{|x-y|^{1+\alpha}} \, \mathrm{d}y = [\Lambda_\alpha, u]\rho \end{cases}$$

$\Lambda_\alpha = -(-\partial_{xx})^{\alpha/2}$. At $\alpha = 1$ the u-equation becomes critical, like Burgers, but with inhomogeneous dissipation controlled by the density. Such control is impossible on $\mathbb{R}$ due to finite mass

$$M = \int \rho(t,x) \, \mathrm{d}x.$$

So, we restrict ourselves to $\mathbb{T}$ and assume no vacuum $\rho_0 > 0$.

Theorem (E. Tadmor, RS, 2016-2017)

Let $1 \leq \alpha < 2$ *on the periodic torus* $\mathbb{T}$. *For any initial condition* $(u_0, \rho_0) \in H^3 \times H^{2+\alpha}$ *away from the vacuum there exists a unique global solution* $(\rho, u) \in L^\infty([0,\infty); H^3 \times H^{2+\alpha})$. *Moreover, there exists* $C, \delta > 0$ *such that*

$$|u(t) - \bar{u}|_\infty + |u_x(t)|_\infty + |u_{xx}(t)|_\infty + |u_{xxx}(t)|_2 \leq Ce^{-\delta t}, \qquad (2)$$

and there is exponential strong flocking towards $(\bar{u}, \bar{\rho})$, *where* $\bar{u} = Momentum/Mass$ *and* $\bar{\rho} = \rho_\infty(x - t\bar{u}) \in H^3$,

$$|\rho(t) - \bar{\rho}(t)|_{H^s} \leq Ce^{-\delta t}, \qquad t > 0. \qquad (3)$$

Theorem (T. Do, A. Kiselev, L. Ryzhik, and C. Tan, 2017)

Global existence for all $0 < \alpha < 1$ *with fast alignment of velocity:*

$$|u(t) - \bar{u}|_\infty < Ce^{-\delta t}.$$

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Fundamentals

- u satisfies maximum (minimum) principle.
- The e-quantity
$$e = u_x + \Lambda_\alpha \rho,$$
relates higher order terms while itself being of lower order. Indeed,
$$\frac{\mathsf{D}}{\mathsf{D}t}\frac{e}{\rho} = 0.$$
So, $|e| \leq C\rho$. One can lift this to higher order $|e^{(k)}| \leq C|\rho^{(k)}|$.
- If $0 < \alpha < 2$, the density remains bounded above and below uniformly in time.

Replacing $u_x = e - \Lambda_\alpha \rho$ in the mass equation

$$\rho_t + u\rho_x + u_x\rho = 0,$$

we rewrite it as

$$\rho_t + u\rho_x + e\rho = \rho\Lambda_\alpha\rho.$$

Recall, that $|e| \leq \rho$, so the equation is of advection-diffusion type with bounded forcing. Equation for momentum $m = u\rho$ shares similar structure

$$m_t + um_x + em = \rho\Lambda_\alpha m.$$

Both blend into a general class of forced fractional parabolic equations with a rough drift

$$v_t + b \cdot \nabla_x v = \int K(x,h,t)(u(x+h) - u(x))dh + f$$

where

$$K(x,h,t) = \rho(x) \frac{1}{|h|^{1+\alpha}}.$$

Using lower bound on the density the kernel falls under the assumptions of Schwab and Silverstre, 2016, provided $\alpha \geq 1$. Hence, there exists an $\gamma > 0$ such that

$$\begin{aligned} |\rho|_{C^\gamma(\mathbb{T}\times[t+1,t+2))} &\leq C(|\rho|_{L^\infty(t,t+2)} + |\rho e|_{L^\infty(t,t+2)}) \\ |m|_{C^\gamma(\mathbb{T}\times[t+1,t+2))} &\leq C(|m|_{L^\infty(t,t+2)} + |me|_{L^\infty(t,t+2)}) \\ |u|_{C^\gamma(\mathbb{T}\times[t+1,t+2))} &\leq C(|u|_{L^\infty(t,t+2)}, |\rho|_{L^\infty(t,t+2)}), \end{aligned}$$

But, no Schauder estimates are known for these equations! H. Dong, T. Jin: no drift or force; H. Chang: $b = const$; L. Silvestre: bounded drift and force, but pure fractional Laplacian Λ_α; Imbert-Jin-RS: non-symmetric kernel but no drift.

Elements of the proof, $\alpha = 1$

First we establish control over ρ':

$$\partial_t |\rho'|^2 + e'\rho\rho' + 2e|\rho'|^2 = -2|\rho'|^2 \Lambda\rho - \rho\rho'\Lambda\rho'.$$

We can bound the term on the l.h.s. by quadratic

$$|e'\rho\rho' + 2e|\rho'|^2| \leq C|\rho'|^2.$$

So, $\rho\rho'\Lambda\rho'$ fights $|\rho'|^2\Lambda\rho$ (both originating from dissipation!)

$$\rho\rho'\Lambda\rho' \geq c\mathsf{D}\rho'(x) = c\int_{\mathbb{R}} \frac{|\rho'(x) - \rho'(x+z)|^2}{|z|^2} dz.$$

$$\Lambda\rho(x) = \int_{|z|<r} \frac{\rho'(x+z) - \rho'(x)}{z} dz - \int_{r<|z|<2\pi} \frac{\rho(x+z) - \rho(x)}{|z|^2} dz$$
$$- \int_{2\pi<|z|} \frac{\rho(x+z) - \rho(x)}{|z|^2} dz.$$

Optimizing we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho'|^2 \leq c_1 + c_2|\rho'|^{2+\gamma} - c_3 \mathrm{D}\rho'(x),$$

Using the nonlinear maximum bound from Constantin-Vicol, 2012:

$$\mathrm{D}\rho'(x) \geq c_4 \frac{|\rho'(x)|^3}{|\rho|_\infty} \geq c_5 |\rho'(x)|^3,$$

we can further hide the quadratic term into dissipation to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}|\rho'|^2 \leq c_1 + c_2|\rho'|^{2+\gamma} - c_3|\rho'|^3, \tag{4}$$

This implies control over u', then u'', then u''', then e'', then ρ'''. Hence global existence.

Strong Flocking

The velocity alignment goes to its natural limit $\bar{u} = P/M$. Denote $\widetilde{\rho}(x,t) := \rho(x + t\bar{u}, t)$. Then $\widetilde{\rho}$ satisfies

$$\widetilde{\rho}_t + (u - \bar{u})\widetilde{\rho}_x + u_x \widetilde{\rho} = 0.$$

We have $|u - \bar{u}|_\infty < e^{-\delta t}$, and $|\rho| < C$. We need

$$|u_x|_\infty < e^{-\delta t}.$$

If we have that, then $|\widetilde{\rho}_t|_\infty < Ce^{-\delta t}$. This proves that $\widetilde{\rho}(t)$ is Cauchy as $t \to \infty$, and hence there exists a unique limiting state, $\rho_\infty(x)$, such that

$$|\widetilde{\rho}(\cdot,t) - \rho_\infty(\cdot)|_\infty < C_1 e^{-\delta t}.$$

Shifting x this can be expressed in terms of ρ and $\bar{\rho}(\cdot,t) = \rho_\infty(x - t\bar{u})$

$$|\rho(\cdot,t) - \bar{\rho}(\cdot,t)|_\infty < C_1 e^{-\delta t}.$$

$$\frac{\mathsf{d}}{\mathsf{d}t}|u'|^2 \leq c_2|u'|^3 + c_6|u'|^2 - c_7\mathsf{D}u'(x).$$

Lemma (Enhancement of dissipation by small amplitudes)

Let $u \in C^1(\mathbb{T})$ be a given function with variation

$$V = \max u - \min u.$$

There is an absolute constant $c_1 > 0$ such that the following pointwise estimate holds

$$Du'(x) \geq c_1 \frac{|u'(x)|^3}{V}. \tag{5}$$

In addition, there is an absolute constant $c_2 > 0$ such that for all $B > 0$ one has

$$Du'(x) \geq B|u'(x)|^2 - c_2 B^3 V^2. \tag{6}$$

Done.

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Case $0 < \alpha < 1$

Recall that e is of lower order. Let us assume that $e = 0$. Then

$$u_x = -\Lambda_\alpha \rho$$

So, the drift is more regular

$$u \sim \partial_x^{\alpha-1} \rho \in C^{1-\alpha}.$$

Hence, the density equation

$$\rho_t + u\rho_x = \rho \Lambda_\alpha \rho$$

is critical for all $0 < \alpha < 1$! DKRT result is based on construction of modulus of continuity as previously for other critical equations such as Burgers, SQG. Our argument is based on nonlinear maximum principle of Constantin-Vicol adopted to nonlinear dissipation. The latter gives quantitative estimates on long time behavior $\rightarrow$ strong flocking. Strong flocking remains *open* in the range $0 < \alpha < 1$.

P.G. Lemarié & F. Lelievre model

$$u_t + u \cdot \nabla u + \nabla p = 0.$$

P.G. Lemarié & F. Lelievre looked into scalar model replacing ∇ with Λ and p with u:

$$u_t - u|\nabla|u + |\nabla|(u^2) = \nu \Delta u.$$

Even the inviscid model is dissipative "at small scales"

$$u_t = u|\nabla|u - |\nabla|(u^2) = \int_{\mathbb{R}} u(y,t) \frac{u(y,t) - u(x,t)}{|x-y|^2} \, \mathrm{d}y.$$

Theorem (C.Imbert, F.Vigneron, RS, 2015)

For $u_0 > 0$ with $u_0 \in L^\infty(\mathbb{T})$ there exists a global self-regularizing solution $u \in C^\infty_{t,x}((\varepsilon, \infty) \times \mathbb{T})$. For $u_0 < 0$ there is a finite time blowup.

Symmetrization of the kernel $\rightarrow$ DiGiorgi via Caffarelli-Chang-Vasseur $\rightarrow$ Schauder (Imbert-Jin-RS) $\rightarrow$ Bootstrap.

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Thank you!